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ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in $|z|<1, p(0)=1, p(z)\neq 0$ in $|z|<1$ and $|\arg p'(z)|<\pi(\alpha-1)/4$ in $|z|<1$ where $1<\alpha<2$. Then we have

$$|\arg p(z)|<\frac{\pi}{2}\alpha \quad \text{in } |z|<1.$$

1. INTRODUCTION.

Let $\mathcal{N}$ be the class of all functions $p(z)$ which are analytic in the unit disc $\mathbb{E}=\{z:|z|<1\}$ and equal to 1 at $z=0$. We say $p(z)\in\mathcal{N}$ a Carathéodory function if it satisfies the condition $\mathrm{Re}\, p(z)>0$ in $\mathbb{E}$.

If $F(z)$ and $G(z)$ are analytic in $\mathbb{E}$, then $F(z)$ is subordinate to $G(z)$, written by $F(z)\prec G(z)$, if $G(z)$ is univalent in $\mathbb{E}$, $F(0)=G(0)$ and $F(z)\subset G(z)$.

In [1, Theorem 5], Miller and Mocanu proved the following theorem.

Theorem A. *Let $p(z)\in\mathcal{N}$ and suppose that*

$$p(z)+zp'(z)\prec\left[\frac{1+z}{1-z}\right]^{\alpha}\Longrightarrow p(z)\prec\left[\frac{1+z}{1-z}\right]^{\beta}$$

where $\alpha=\alpha(\beta)=\beta+(2/\pi)\mathrm{Tan}^{-1}\beta$, $0<\beta<\beta_0=1.21872\cdots$ and β_0 is the root of the equation

$$\beta\pi=\frac{3}{2}\pi-\mathrm{Tan}^{-1}\beta.$$

On the other hand, Nunokawa [2] proved the following lemma.

Lemma 1. *Let $p(z)\in\mathcal{N}$, $p(z)\neq 0$ in $\mathbb{E}$ and suppose that there exists a point $z_0\in\mathbb{E}$ such that*

$$|\arg p(z)|<\frac{\pi}{2}\alpha \quad \textit{in } |z|<|z_0|$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}\alpha$$

where $0<\alpha$. Then we have

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$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2}\alpha$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad \text{and} \quad a > 0.$$

Applying Lemma 1, we can easily obtain the following result.

Theorem B. *Let $p(z) \in \mathcal{N}$, $p(z) \neq \mathbb{E}$ and suppose that*

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2}\left(\beta + \frac{2}{\pi}\mathrm{Tan}^{-1}\beta\right) \quad \text{in } \mathbb{E}$$

where $0 < \beta$. Then we have

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

Remark 1. For the case $0 < \beta < \beta_0$, Theorem B is obtained from Theorem A but Theorem B holds to be true for all the case $0 < \beta$ if we consider the function $p(z)$ on the infinitely many sheeted Riemann surfaces which are cut along the negative half of real axis.

Applying Lemma 1, Nunokawa [2] obtained Theorem C.

Theorem C. *Let $p(z) \in \mathcal{N}$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that*

$$\left|\arg\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right| < \frac{\pi}{2}\alpha(\beta) \quad \text{in } \mathbb{E}$$

where $0 < \beta \leq 1$,

$$\alpha(\beta) = \beta + \frac{2}{\pi}\mathrm{Tan}^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta)\cos\frac{\pi}{2}(1-\beta)},$$

$$p(\beta) = (1+\beta)^{(1+\beta)/2} \quad \text{and} \quad q(\beta) = (1-\beta)^{(\beta-1)/2}.$$

Then we have

$$|\arg p(z)| < \frac{\pi}{2}\beta \quad \text{in } \mathbb{E}.$$

Remark 2. Theorem C holds to be true for all the case $0 < \beta$ if we also consider it like as Remark 1.

ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

In the distortion theorem of analytic function theory, if we suppose some assumptions for $|f'(z)|$, then we can easiy get some results for $|f(z)|$ by applying integral inequality

$$|f(z) - f(0)| \leq \int_0^z |f'(t)||dt|.$$

On the other hand, we can not find out any results for the rotation theorem of analytic functions between $|\arg p'(z)|$ and $|\arg p(z)|$.

2. Main result.

Theorem. *Let $p(z) \in \mathcal{N}$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that*

$$|\arg p'(z)| < \frac{\pi}{4}(\alpha - 1) \quad \text{in } \mathbb{E},$$

where $1 < \alpha < 2$. Then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } \mathbb{E}.$$

Proof. Let us suppose that if there exits a point $z_0 \in \mathbb{E}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha.$$

Applying Lemma 1, let us suppose $\arg p(z_0) = \pi\alpha/2$, then we have

$$\begin{aligned} p'(z_0) &= \frac{p(z_0)}{z_0} i\alpha k \\ &= \left(\frac{p(z_0) - 1}{z_0}\right)\left(\frac{i\alpha k p(z_0)}{p(z_0) - 1}\right) \\ &= \left(\frac{1}{z_0}\int_0^{z_0} p'(t)\,dt\right)\left(\frac{i\alpha k p(z_0)}{p(z_0) - 1}\right) \\ &= \left(\frac{1}{r}\int_0^{r} p'(\rho e^{i\theta})\,d\rho\right)\left(\frac{i\alpha k p(z_0)}{p(z_0) - 1}\right) \end{aligned}$$

where $z_0 = re^{i\theta}$, $t = \rho e^{i\theta}$ and $0 \leq \rho \leq r$. Therefore we have

$$\arg p'(z_0) = \frac{\pi}{2} + \frac{\pi}{2}\alpha + \arg\left(\frac{1}{r}\int_0^r p'(\rho e^{i\theta})\,d\rho\right) + \arg\left(\frac{\overline{p(z_0)} - 1}{|p(z_0) - 1|^2}\right).$$

Applying the property of integral mean of the integral (See Pommerenke [3, Lemma 1]), we have

$$\begin{aligned} \arg p'(z_0) &\geq \frac{\pi}{2} + \frac{\pi}{2}\alpha - \frac{\pi}{4}(\alpha - 1) - \pi \\ &= \frac{\pi}{4}(\alpha - 1). \end{aligned}$$

This contradicts the assumption.

For the case $\arg p(z_0) = -\pi\alpha/2$, we also have the following

$$\arg p'(z_0) = \arg \frac{p(z_0)}{z_0}(-i\alpha k) = \arg\left(\frac{1}{z_0}\int_0^{z_0} p'(t)\,dt\right) + \arg\left(\frac{-i\alpha k p(z_0)}{p(z_0)-1}\right)$$

where $1 \leq k$. Therefore, we have

$$\begin{aligned}\arg p'(z_0) &= -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \arg\left(\frac{1}{r}\int_0^{r} p'(\rho e^{i\theta})\,d\rho\right) + \arg\left(\frac{\overline{p(z_0)}-1}{|p(z_0)-1|^2}\right)\\ &\geq -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \frac{\pi}{4}(\alpha-1) + \pi\\ &= -\frac{\pi}{4}(\alpha-1).\end{aligned}$$

This contradicts the assumption and so this completes the proof. □

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