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THE SUBORDINATION THEOREM FOR λ-SPIRALLIKE FUNCTIONS OF ORDER α

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ABSTRACT. We proved a subordination relation for a subclass of the class of λ-spirallike functions of order α.

1. Introduction

Let A denote the class of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. And let S denote the subclass of A consisting of analytic and univalent function $f(z)$ in unit disk U.

A function $f(z)$ in S is said to be convex if

$$\text{(1.1)} \qquad Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in U).$$

And we denote by K the class of all convex functions.

Definition 1.1. A function $f(z)$ in S is said to be λ-spirallike of order α, $(0 \leq \alpha < 1)$, if

$$\text{(1.2)} \qquad Re\left\{e^{i\lambda} z \frac{f'(z)}{f(z)}\right\} > \alpha \cos\lambda \quad (z \in U),$$

for some real λ $\left(|\lambda| < \frac{\pi}{2}\right)$. The class of the functions is denoted by $S_p^{\alpha}(\lambda)$.

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Definition 1.2. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are analytic in U, then their Hadamard product, $f * g$ is function defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.3}$$

The function $f * g$ is also analytic in U.

Definition 1.3. Let f be analytic in U, g analytic and univalent in U and $f(0) = g(0)$. Then by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U, we shall mean that $f(U) \subset g(U)$.

Definition 1.4. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z) = \sum_{k=1}^{\infty} a_k z^k$, $a_1 = 1$ is regular, univalent and convex in U, we have

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad \text{in } U. \tag{1.4}$$

Lemma 1.5. *The sequence $\{b_n\}_{n=1}^{\infty}$ is subordinating factor sequence if and only if*

$$Re\left[1 + 2\sum_{n=1}^{\infty} b_n z^n\right] > 0 \quad (z \in U). \tag{1.5}$$

The above lemma is due to (Wilf [2]).

In this paper, we prove a subordination relation for a subclass of the class of λ-spirallike functions of order α.

2. Main results

Before proving our next results, we need the following the Lemmas.

Lemma 2.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic with $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \beta$ for $0 \leq \beta < 1$ and $z \in U$. Then $f(z) \in S_p^{\alpha}(\lambda)$ for $|\lambda| \leq \cos^{-1}\left(\frac{1-\beta}{1-\alpha}\right)$.*

Proof. We may write $\dfrac{zf'(z)}{f(z)} - 1 = (1-\beta)w(z)$, where $|w(z)| < 1$ for $z \in U$. Thus

$$\begin{aligned} Re\left[e^{i\lambda}\frac{zf'(z)}{f(z)}\right] &= Re[e^{i\lambda}(1 + (1-\beta)w(z))] \\ &= \cos\lambda + (1-\beta)Re\{e^{i\lambda}w(z)\} \\ &\geq \cos\lambda - (1-\beta)|e^{i\lambda}w(z)| \\ &> \cos\lambda - (1-\beta) \geq \alpha\cos\lambda \end{aligned}$$

for $|\lambda| \leq \cos^{-1}\dfrac{1-\beta}{1-\alpha}$, and the proof is complete.

Lemma 2.2. *If* $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < (1-\alpha)\cos\lambda$, *then* $f \in S_p^{\alpha}(\lambda)$.

Proof. Set $\beta = 1 - (1-\alpha)\cos\lambda$ in Lemma 2.1.

Theorem 2.3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z_n$. *If*

$$\sum_{n=2}^{\infty}\left\{1 + \frac{n-1}{1-\alpha}\sec\lambda\right\}|a_n| < 1, \tag{2.1}$$

then $f(z) \in S_p^{\alpha}(\lambda)$.

Proof. By Lemma 2.2, it suffices to show that $\left|\dfrac{zf'(z)}{f(z)} - 1\right| < (1-\alpha)\cos\lambda$. We have

$$\begin{aligned} \left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z + \sum_{n=2}^{\infty} a_n z^n}\right| &< \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z^n|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty}(n-1)|a_n|}{1 - \sum_{n=2}^{\infty}|a_n|}. \end{aligned}$$

Thus last expression is bounded above by $(1-\alpha)\cos\lambda$, if

$$\sum_{n=2}^{\infty}(n-1)|a_n| \leq (1-\alpha)\cos\lambda\left(1 - \sum_{n=2}^{\infty}|a_n|\right). \tag{2.2}$$

which is equivalent to

$$\sum_{n=2}^{\infty}\left\{1+\frac{n-1}{1-\alpha}\sec\lambda\right\}|a_n|\le 1. \tag{2.3}$$

Remark 1. Taking $\lambda=0$ in (2.1), we obtain a sufficient condition for $f(z)$ to be starlike of order α (H. Silberman [1]).

Let us denote by $G(\lambda,\alpha)$, the class of functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ whose coefficients satisfy the condition (2.1).

Theorem 2.4. *Let $f\in G(\lambda,\alpha)$. Then*

$$\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}(f*g)(z)\prec g(z)\quad \text{for } z\in U \tag{2.4}$$

for every function $g(z)$ in the class K.

In purticular

$$Re f(z)>-\frac{2(1-\alpha)+\sec\lambda}{(1-\alpha)+\sec\lambda}\quad \text{for } z\in U. \tag{2.5}$$

The constant $\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}$ *connot be replace by any larger one.*

Proof. Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in $G(\lambda,\alpha)$ and let $g(z)=z+\sum_{n=2}^{\infty}c_nz^n$ be in K. Then

$$\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}(f*g)(z)=\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}\left(z+\sum_{n=2}^{\infty}a_nc_nz^n\right). \tag{2.6}$$

Thus, by definition 1.4, the assertion of our theroem will hold if the sequence

$$\left(\frac{\{(1-\alpha)+\sec\lambda\}a_n}{2(2(1-\alpha)+\sec\lambda)}\right)_{n=1}^{\infty} \tag{2.7}$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 1.5, this will be the case if and only if

$$Re\left[1+2\sum_{n=1}^{\infty}\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}a_nz^n\right]>0\quad \text{for } z\in U. \tag{2.8}$$

$$
\begin{aligned}
&Re\left[1+\sum_{n=1}^{\infty}\frac{(1-\alpha)+\sec\lambda}{2(1-\alpha)+\sec\lambda}a_nz^n\right]\\
&= Re\left[1+\frac{(1-\alpha)+\sec\lambda}{2(1-\alpha)+\sec\lambda}z+\frac{1-\alpha}{2(1-\alpha)+\sec\lambda}\sum_{n=2}^{\infty}\left(1+\frac{\sec\lambda}{1-\alpha}\right)a_nz^n\right]\\
&\geq 1-\frac{(1-\alpha)+\sec\lambda}{2(1-\alpha)+\sec\lambda}r-\frac{1-\alpha}{2(1-\alpha)+\sec\lambda}\sum_{n=2}^{\infty}\left(1+\frac{(n-1)\sec\lambda}{1-\alpha}\right)|a_n|r^n \quad (|z|=r)\\
&\geq 1-\frac{(1-\alpha)+\sec\lambda}{2(1-\alpha)+\sec\lambda}r-\frac{1-\alpha}{2(1-\alpha)+\sec\lambda}r \quad \text{(by (2.1))}\\
&> 0.
\end{aligned}
$$

Thus (2.8) holds ture in U. Thus prove the first assertion. That $Ref(z) > -\frac{2(1-\alpha)+\sec\lambda}{(1-\alpha)+\sec\lambda}$ for $f(z) \in G(\lambda, \alpha)$ follows by taking $g(z) = \frac{z}{1-z}$ in (2.4).

To prove the sharpness of the constant $\{(1-\alpha)+\sec\lambda\}/2(2(1-\alpha)+\sec\lambda)$, we consider the function

$$
f_0(z) = z - \frac{(1-\alpha)}{(1-\alpha)+\sec\lambda}z^2 \quad \text{for } \left(|\lambda| < \frac{\pi}{2}\right), \tag{2.10}
$$

which is a member of the class $G(\lambda, \alpha)$. Thus from the relation (2.4), we obtain

$$
\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}f_0(z) \prec \frac{z}{1-z}. \tag{2.11}
$$

If can be verified that

$$
\min_{|z|\leq 1} Re\left[\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}f_0(z)\right] = -\frac{1}{2}. \tag{2.12}
$$

This shows that the constant $\frac{(1-\alpha)+\sec\lambda}{2(2(1-\alpha)+\sec\lambda)}$ is best possible.

Taking $\lambda = 0$, we obtain the following corollary.

Corollary 2.5. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is regular in U and satisfies the condition*

$$
\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}|a_n| \leq 1 \tag{2.13}
$$

then for every function g in K, we have

$$\frac{2-\alpha}{2(3-2\alpha)}(f*g)(z) \prec g(z). \tag{2.14}$$

In particular, $Ref(z) > -\frac{3-2\alpha}{2-\alpha}$, $z \in U$. *The constant* $\frac{2-\alpha}{2(3-2\alpha)}$ *is best possible.*

Remark 2. Putting $\alpha = 0$ in Theorem 2.4, we get the result in S. Singh [3].

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