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ON SOME PROPERTIES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $p(z)$ be analytic in $|z| < 1$, $p(0) = 1$, $p(z) \neq 0$ in $|z| < 1$ and suppose that $|p(z)|$ takes its maximum and minimum value at points $z = z_0$ and $z = z_1$ respectively on the closed disc $|z| = |z_0| = |z_1| \leq r$ $(0 < r < 1)$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = m \geq 0$$

and

$$\frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0.$$

1. INTRODUCTION.

In [2], Jack proved the following theorem.

Theorem A. *Let $w(z)$ be analytic in $\mathbb{E} = \{z : |z| < 1\}$ and suppose that $w(0) = 0$. If $|w(z)|$ takes its maximum value on the circle $|z| = r < 1$ at a point $z = z_0$, then we have*

$$\frac{z_0 w'(z_0)}{w(z_0)} = k$$

where k is real number and $k \geq 1$.

Fukui and Sakaguchi [1] generalized Theorem A and gave a simple and geometrical proof.

Furthermore, Miller and Mocanu [3] generalized Theorem A and obtained the following theorem.

Theorem B. *Let $w(z) = \sum_{k=n}^{\infty} a_k z^k$ be analytic in $\mathbb{E}$, $n \in \mathbb{N}$, $w(z) \not\equiv 0$. If $z_0 = r_0 e^{i\theta_0}$ $(0 < r_0 < 1)$ and*

$$|w(z_0)| = \max_{|z_0| \leq r_0} |w(z)|$$

then

$$\frac{z_0 w'(z_0)}{w(z_0)} = m$$

and

$$1 + \mathrm{Re}\frac{z_0 w''(z_0)}{w'(z_0)} \geq m$$

where $1 \leq n \leq m$.

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It is a purpose of the present paper to obtain some similar but a little different results from Theorem A and B.

2. Main result.

Theorem. *Let $p(z)$ be analytic in $\mathbb{E}$, $p(0) = 1$, $p(z) \neq 0$ in $\mathbb{E}$ and suppose that*

$$\max_{|z|\leq r} |p(z)| = |p(z_0)|$$

and

$$\min_{|z|\leq r} |p(z)| = |p(z_1)|$$

where $0 < r < 1$ and $|z_0| = |z_1| = r$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = m \geq 0,$$

$$\frac{z_1 p'(z_1)}{p(z_1)} = n \leq 0,$$

$$1 + \mathrm{Re}\frac{z_0 p''(z_0)}{p'(z_0)} \geq m$$

and

$$1 + \mathrm{Re}\frac{z_1 p''(z_1)}{p'(z_1)} \geq n$$

where m and n are real, $0 \leq m$ and $n \leq 0$.

Proof. Let us put

$$g(z) = \left(\frac{R+1}{R-1}\right)\left(\frac{R-p(z)}{R+p(z)}\right) = A\left(\frac{R-p(z)}{R+p(z)}\right), \quad g(0) = 1$$

where $R = |p(z_0)|$ and $A = (R+1)/(R-1)$. Then we have

$$p(z) = R\left(\frac{A-g(z)}{A+g(z)}\right).$$

When $|p(z)|$ takes its maximum value at a point $z = z_0$, then we have $\mathrm{Re}\, g(z) > 0$ for $|z| < |z_0|$ and $\mathrm{Re}\, g(z_0) = 0$. Putting $|z| = |z_0|$, $z = |z_0|e^{i\theta}$ and $0 \leq \theta \leq 2\pi$, we have

$$\begin{aligned} \frac{zp'(z)}{p(z)} &= \frac{d \arg p(z)}{d\theta} - i\frac{d \log |p(z)|}{d\theta} \\ &= -\frac{zg'(z)}{A-g(z)} - \frac{zg'(z)}{A+g(z)} \\ &= -\frac{2zg'(z)}{A^2 - g(z)^2}. \end{aligned} \tag{1}$$

Therefore we have

$$\begin{aligned}\frac{z_0p'(z_0)}{p(z_0)} &= -\frac{2z_0g'(z_0)}{A^2-g(z_0)^2}\\ &= \left(\frac{d\arg p(z)}{d\theta}\right)_{z=z_0} - i\left(\frac{1}{|p(z)|}\right)\left(\frac{d|p(z)|}{d\theta}\right)_{z=z_0}\\ &= \left(\frac{d\arg p(z)}{d\theta}\right)_{z=z_0} = m \geq 0\end{aligned}$$

where m is a real and $0 \leq m$. By logarithmic differentiation of (1), we have

$$1+\frac{zp''(z)}{p'(z)} = \frac{zp'(z)}{p(z)} + 1 + \frac{zg''(z)}{g'(z)} - \frac{2zg'(z)g(z)}{A^2-g(z)^2}.$$

Then we have

$$\begin{aligned}1+\mathrm{Re}\frac{z_0p''(z_0)}{p'(z_0)} &= \mathrm{Re}\frac{z_0p'(z_0)}{p(z_0)} + 1 + \mathrm{Re}\frac{z_0g''(z_0)}{g'(z_0)} - \mathrm{Re}\frac{2z_0g'(z_0)g(z_0)}{A^2-g(z_0)^2}\\ &= m+1+\mathrm{Re}\frac{z_0g''(z_0)}{g'(z_0)} - m\mathrm{Re}\, g(z_0)\\ &= m+1+\mathrm{Re}\frac{z_0g''(z_0)}{g'(z_0)}.\end{aligned}$$

From the hypothesis, we have

$$\mathrm{Re}\, g(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\mathrm{Re}\, g(z_0) = 0,$$

then from the geometrical property of $g(z)$, we have

$$1+\mathrm{Re}\frac{z_0g''(z_0)}{g'(z_0)} \geq 0.$$

This shows that

$$1+\mathrm{Re}\frac{z_0p''(z_0)}{p'(z_0)} \geq m.$$

On the other hand, let us put

$$h(z) = \left(\frac{p(z)-l}{p(z)+l}\right)\left(\frac{1+l}{1-l}\right), \quad h(0)=1$$

where $0 < l = \min_{|z|\leq|z_1|}|p(z)| < 1$, then we have

$$p(z) = l\left(\frac{B+h(z)}{B-h(z)}\right) \tag{2}$$

where $B = (1+l)/(1-l)$. From the hypothesis of the theorem, we have

$$\mathrm{Re}\, h(z) > 0 \quad \text{for } |z| < |z_1|$$

and

$$\mathrm{Re}\, h(z_1) = 0.$$

From (2), we have

$$\frac{zp'(z)}{p(z)} = \frac{2zh'(z)}{B^2 - h(z)^2}. \tag{3}$$

By the same reason as the above, we have on the circle $|z| = |z_1|e^{i\theta}$ and $0 \leq \theta \leq 2\pi$

$$\left(\frac{d|p(z)|}{d\theta}\right)_{z=z_1} = 0$$

and from the geometrical property, we have

$$\left(\frac{d\arg p(z)}{d\theta}\right)_{z=z_1} \leq 0.$$

This shows that

$$\frac{z_1p'(z_1)}{p(z_1)} = \text{Re}\frac{z_1p'(z_1)}{p(z_1)} = n \leq 0.$$

where n is a real number. From (3), we have

$$1 + \frac{zp''(z)}{p'(z)} = \frac{zp'(z)}{p(z)} + 1 + \frac{zh''(z)}{h'(z)} + \frac{2zh'(z)h(z)}{B^2 - h(z)^2}$$

and therefore we have

$$\begin{aligned} 1 + \text{Re}\frac{z_1p''(z_1)}{p'(z_1)} &= n + 1 + \text{Re}\frac{z_1h''(z_1)}{h'(z_1)} - \text{Re}\frac{2z_1h'(z_1)h(z_1)}{B^2 - h(z_1)^2} \\ &= n + 1 + \text{Re}\frac{z_1h''(z_1)}{h'(z_1)} - n\text{Re}\, h(z_1) \\ &= n + 1 + \text{Re}\frac{z_1h''(z_1)}{h'(z_1)}. \end{aligned}$$

Applying the same reason as the above, we have

$$1 + \text{Re}\frac{z_1h''(z_1)}{h'(z_1)} \geq 0$$

and this shows that

$$1 + \text{Re}\frac{z_1p''(z_1)}{p'(z_1)} \geq n$$

where n is a real number and $n \leq 0$. This completes the proof. □

References

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