数理解析研究所講究録 1293 巻 2002 年 110-118

Asymptotics of Green functions and Martin boundaries for elliptic operators with periodic coefficients

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July 9, 2002

The main purpose of this talk is to give the asymptotics at infinity of a Green function for an elliptic equation with periodic coefficients on $\mathbf{R}^d$.

The second purpose is to completely determine the Martin compactification of $\mathbf{R}^d$ with respect to an elliptic equation with periodic coefficients by using the exact asymptotics at infinity of the Green function.

1. Asymptotics at infinity of Green functions

Let

$$L = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}(a_{jk}(x)\frac{\partial}{\partial x_j}) - \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j} + c(x)$$
$$= -\nabla \cdot a(x)\nabla - b(x)\cdot\nabla + c(x)$$

be a second order elliptic operator on $\mathbf{R}^d$ with smooth real-valued coefficients which are $\mathbf{Z}^d$-periodic. Here

$d \geq 2$,

$\nabla = (\partial/\partial x_1, \cdots, \partial/\partial x_d)$,

$a(x) = (a_{jk}(x))_{j,k=1}^d$, and $b(x) = (b_j(x))_{j=1}^d$.

For each $\zeta \in \mathbf{C}^d$, define an operator $L(\zeta)$ on the d-dimensional torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$ by

$$L(\zeta) = e^{-i\zeta\cdot x} L e^{i\zeta\cdot x}$$

$$= -(\nabla + i\zeta)\cdot a(x)(\nabla + i\zeta) - b(x)\cdot(\nabla + i\zeta) + c(x),$$

where $i = \sqrt{-1}$ is the imaginary unit.

$L(\zeta)$: a closed operator in $L^2(\mathbf{T}^d)$ with the domain $H^2(\mathbf{T}^d)$

$H^2(\mathbf{T}^d)$: the Sobolev space of order two

$L(\zeta)^*$: the formal adjointof $L(\zeta)$

For $\beta \in \mathbf{R}^d$,

$$E(\beta) : \text{the principal eigenvalue of } L(i\beta)$$

By the Krein-Rutman theorem, $E(\beta)$ is a real eigenvalue of multiplicity one such that the corresponding eigenspace is generated by a positive function.

$E(\beta)$ is also an eigenvalue of $L(i\beta)^*$.

$$C_L = \{u \in C^2(\mathbf{R}^d);\ Lu = 0 \text{ and } u > 0 \text{ in } \mathbf{R}^d\}.$$

L : subcritical when a positive Green function for L on $\mathbf{R}^d$ exists (In this case, $C_L \neq \emptyset$.)

L : critical when a positive Green function for L on $\mathbf{R}^d$ does not exist but $C_L \neq \emptyset$

For $\lambda \in \mathbf{R}$, put

$\Gamma_\lambda = \{\beta \in \mathbf{R}^d; \exists\psi \in C_{L-\lambda}$ of the form $\psi(x) = e^{-\beta\cdot x}u(x)$, where u is periodic$\}$

(Note: $L(i\beta)u = \lambda u$ on $\mathbf{T}^d$ and $E(\beta) = \lambda$)

$K_\lambda = \{\beta \in \mathbf{R}^d; \exists\psi \in C^2(\mathbf{R}^d)$ such that $(L-\lambda)\psi \geq 0$ and $\psi(x) = e^{-\beta\cdot x}u(x) > 0$, where u is periodic$\}$

Define K_λ^* and Γ_λ^* for $L^* - \lambda$ analogously to K_λ and Γ_λ for $L - \lambda$.

First suppose that $\sup_\beta E(\beta) > 0$. Then

L : subcritical

$\forall s \in \mathbf{S}^{d-1}$ (the unit sphere), $\exists_1 \beta_s \in \Gamma_0$ such that $\sup_{\beta\in\Gamma_0} \beta\cdot s = \beta_s\cdot s$.

$\{e_{s,1}, \cdots, e_{s,d-1}, s\}$: orthonormal basis of $\mathbf{R}^d$ $(\forall s \in \mathbf{S}^{d-1})$

For $\beta \in \mathbf{R}^d$,

u_β : positive solution to $L(i\beta)u = E(\beta)u$

v_β : positive solution to $L(i\beta)^*v = E(\beta)v$

For functions u and v in $L^2(\mathbf{T}^d)$, put $(u, v) = \int_{\mathbf{T}^d} u(x)\bar{v}(x)dx$.

Theorem 1 *Suppose that* $\sup_\beta E(\beta) > 0$. *Then the minimal Green function* G *of* L *on* $\mathbf{R}^d$ *has the following asymptotics as* $|x-y| \to \infty$:

$$G(x,y) = \frac{e^{-(x-y)\cdot\beta_s}}{(2\pi|x-y|)^{(d-1)/2}}$$
$$\times \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{(\det(-e_{s,j}\cdot \operatorname{Hess} E(\beta_s)e_{s,k})_{jk})^{1/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})}$$
$\times(1+O(|x-y|^{-1}))$,
where $s = (x-y)/|x-y|$.

Here, let us recall some more facts.
λ_c : The generalized principal eigenvalue of L on $\mathbf{R}^d$, i.e.

$$\lambda_c = \sup\{\lambda \in \mathbf{R};\ L-\lambda \text{ is subcritical }\}$$

Then $-\infty < \lambda_c < \infty$, $L-\lambda$ is subcritical for $\lambda < \lambda_c$, and $L-\lambda_c$ is subcritical or critical.

The formal adjoint operator L^* of L is subcritical (or critical) if and only if L is subcritical (or critical).

The generalized principal eigenvalue of L and L^* coincide.

Theorem (Agmon & Pinsky) (i) If $\lambda < \lambda_c$, then K_λ is a d-dimensional strictry convex compact set with smooth boundary and
$\Gamma_\lambda = \partial K_\lambda$.
(ii) If $\lambda = \lambda_c$, then
$\Gamma_\lambda = K_\lambda = \{\beta_0\}$ for some $\beta_0 \in \mathbf{R}^d$.
(iii) If $\lambda > \lambda_c$, then $\Gamma_\lambda = K_\lambda = \emptyset$.
(iv) The function $E(\beta)$ is real analytic and strictly concave.
Its Hessian Hess $E(\beta)$ is negative definite for any $\beta \in \mathbf{R}^d$.
The equality $\lambda_c = \sup_\beta E(\beta)$ holds, and the supremum is attained uniquely at β_0 in (ii).
$\nabla_\beta E(\beta) = 0$ if and only if $\beta = \beta_0$.
(v) For any $\lambda \in \mathbf{R}$,
$\Gamma_\lambda = \{\beta \in \mathbf{R}^d; E(\beta) = \lambda\}$
$K_\lambda = \{\beta \in \mathbf{R}^d; E(\beta) \geq \lambda\}$.
(vi) $K^*_\lambda = -K_\lambda$, and $\beta_0 = 0$ if $L^* = L$.

Now, let us look at the asymptotics of the Green function again. Note that its main term is positive because of the assertion (iv).

Theorem 1. Suppose that $\lambda_c > 0$. Then the minimal Green function G of L on $\mathbf{R}^d$ has the following asymptotics as
$|x-y| \to \infty$:

$$G(x,y) = \frac{e^{-(x-y)\cdot\beta_s}}{(2\pi|x-y|)^{(d-1)/2}}$$
$$\times \frac{|\nabla E(\beta_s)|^{(d-3)/2}}{(\det(-e_{s,j}\cdot \operatorname{Hess} E(\beta_s)e_{s,k})_{jk})^{1/2}} \frac{u_{\beta_s}(x)v_{\beta_s}(y)}{(u_{\beta_s}, v_{\beta_s})}$$

$\times(1+O(|x-y|^{-1}))$,
where $s=(x-y)/|x-y|$.

This theorem is derived from the following theorem, where we regard L as a closed operator in $L^2(\mathbf{R}^d)$ with the domain $H^2(\mathbf{R}^d)$.

Theorem 2 *Assume $E(0) > 0$. Then the resolvent L^{-1} exists, and the integral kernel G of L^{-1} has the same asymptotics as in Theorem 1.*

Actually, consider the operator

$$L_1 = e^{\beta_0\cdot x} L e^{-\beta_0\cdot x}.$$

Then L_1 satisfies the assumption of Theorem 2, and the minimal Green function G_1 of L_1 satisfies

$$G_1(x,y) = e^{\beta_0\cdot x} G(x,y) e^{-\beta_0\cdot y}.$$

Thus Theorem 1 follows from Theorem 2.

Later, I will give an outline of the proof of Theorem 2.

Next, suppose that $\sup_\beta E(\beta) = 0$. Then
L is critical if $d \leq 2$, and subcritical if $d \geq 3$

Our second main theorem is the following

Theorem 3 *Let $d \geq 3$. Suppose that*
$\lambda_c = E(\beta_0) = 0$. *Put* $H = -\mathrm{Hess}E(\beta_0)$.
Then the minimal Green function G of L on $\mathbf{R}^d$ has the following asymptotics as $|x-y| \to \infty$:

$$G(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}(\det H)^{1/2}} \frac{e^{-(x-y)\cdot\beta_0}}{|H^{-1/2}(x-y)|^{d-2}}$$
$$\times \frac{u_{\beta_0}(x)v_{\beta_0}(y)}{(u_{\beta_0}, v_{\beta_0})}(1 + O(|x-y|^{-1})).$$

2. Martin boundaries

Now, let us determine explicitly the Martin compactification of $\mathbf{R}^d$ with respect to L in the case $\lambda_c > 0$.

Fix a reference point x_0 in $\mathbf{R}^d$. Then the following proposition is a direct consequence of Theorem 1.

Proposition 1. Suppose that $\lambda_c > 0$. Then for any sequence $\{y_n\}$ in $\mathbf{R}^d$ such that
$|y_n| \to \infty$ and $y_n/|y_n| \to \nu$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{G(x,y_n)}{G(x_0,y_n)} = e^{-(x-x_0)\cdot\beta_{-\nu}} \frac{u_{\beta_{-\nu}}(x)}{u_{\beta_{-\nu}}(x_0)}, \quad x \in \mathbf{R}^d.$$

Denote this right hand side by $K(x,\nu)$. Then
$K(\cdot,\nu) \in C_L$, $K(x_0,\nu) = 1$,
$K(\cdot,\nu) \neq K(\cdot,\mu)$ if $\nu \neq \mu$
$\forall\nu \in \mathbf{S}^{d-1}$, $K(\cdot,\nu)$ is minimal in C_L, i.e.,
If $\psi \in C_L$ satisfies $\psi(x) \leq K(x,\nu)$ on $\mathbf{R}^d$, then $\psi(x) = \exists c K(x,\nu)$
Hence we can explicitly determine the Martin compactification of $\mathbf{R}^d$ for L as follows.

Theorem 4 *Suppose that $\lambda_c > 0$. Then the Martin boundary and the minimal Martin*
boundary of $\mathbf{R}^d$ for L are both equal to
the sphere $\mathbf{S}^{d-1}$ at infinity which is homeomorphic to Γ_0;
the Martin kernel at $\nu \in \mathbf{S}^{d-1}$ is equal to $K(\cdot,\nu)$;
the Martin compactification of $\mathbf{R}^d$ for L is equal to

$$\{x \in \mathbf{R}^d; |x| < 1\} \cup [1,\infty] \times \mathbf{S}^{d-1}$$

equipped with the standard topology.

In the case $\lambda_c = 0$ and $d \geq 3$, we obtain directly from Theorem 3 the following theorem.

This result, however, is also a simple consequence of the known result that C_L is one dimensional in this case.

Theorem 5 *Suppose that* $d \geq 3$ *and* $\lambda_c = E(\beta_0) = 0$. *Then*

for any sequence $\{y_n\}$ *in* $\mathbf{R}^d$ *with* $|y_n| \to \infty$ *as* $n \to \infty$,

$$\lim_{n\to\infty} \frac{G(x, y_n)}{G(x_0, y_n)} = e^{-(x-x_0)\cdot\beta_0} \frac{u_{\beta_0}(x)}{u_{\beta_0}(x_0)}, \quad x \in \mathbf{R}^d;$$

the Martin boundary and the minimal Martin boundary are both equal to one point ∞ *at infinity;*

the Martin kernel at ∞ *is equal to this right hand side;*

the Martin compactification of $\mathbf{R}^d$ *for* L *is equal to the one point compactification* $\mathbf{R}^d \cup \{\infty\}$ *of* $\mathbf{R}^d$.

3. Proof of Theorem 2

Finally, let us give an outline of the proof of Theorem 2.

Basic ingredients in establishing the asymptotics are an integral representation of the Green function and the saddle point method in complex integrations.

Let us give an integral expression of the resolvent of L.

$2\pi\mathbf{T}^d = \mathbf{R}^d/(2\pi\mathbf{Z})^d$

$$\mathcal{H} = L^2(2\pi\mathbf{T}^d, \frac{d\zeta}{(2\pi)^d}; L^2(\mathbf{T}^d)) = \int_{2\pi\mathbf{T}^d}^{\oplus} L^2(\mathbf{T}^d) \frac{d\zeta}{(2\pi)^d}.$$

$\mathcal{F} : L^2(\mathbf{R}^d) \to \mathcal{H}$

$$(\mathcal{F}f)(\zeta, x) = \sum_{l\in\mathbf{Z}^d} f(x-l)e^{-i(x-l)\cdot\zeta}.$$

Then $\mathcal{F}$ is a unitary operator, and an isomorphism from $H^2(\mathbf{R}^d)$ to $L^2(2\pi\mathbf{T}^d, (2\pi)^{-d}d\zeta;\ H^2(\mathbf{T}^d))$.

The adjoint $\mathcal{F}^*$ is given by, for $g \in \mathcal{H}$,

$$(\mathcal{F}^* g)(x-l) = \int_{2\pi \mathbf{T}^d} \frac{d\zeta}{(2\pi)^d} e^{i(x-l)\cdot\zeta} g(\zeta, x),$$

$$x \in \mathbf{T}^d,\ l \in \mathbf{Z}^d$$

$L = \mathcal{F}^* \tilde{L} \mathcal{F}$,
$\tilde{L} = (2\pi)^{-d} \int_{2\pi\mathbf{T}^d}^{\oplus} L(\zeta) d\zeta$
(since $(\nabla_x + i\zeta)\mathcal{F}f = \mathcal{F}(\nabla f)$)
Assume $E(0) > 0$. Then
$L(\zeta)^{-1}$ is a real analytic function from $2\pi\mathbf{T}^d$
to the Banach space of bounded operators on $L^2(\mathbf{T}^d)$.
$L^{-1} = \mathcal{F}^* M \mathcal{F}$
$M = (2\pi)^{-d} \int_{2\pi\mathbf{T}^d}^{\oplus} L(\zeta)^{-1} d\zeta$
That is, $\forall x \in \mathbf{T}^d$, $l \in \mathbf{Z}^d$, and $f \in L^2(\mathbf{R}^d)$,

$$L^{-1} f(x-l) = \int_{2\pi\mathbf{T}^d} F(\zeta) \frac{d\zeta}{(2\pi)^d},$$

where

$$F(\zeta) = e^{i(x-l)\cdot\zeta} L(\zeta)^{-1} \Big(\sum_{m \in \mathbf{Z}^d} f(\cdot - m) e^{-i(\cdot - m)\cdot\zeta}\Big)(x).$$

Meromorphic extension of $L(\zeta)^{-1}$
For each $s \in \mathbf{S}^{d-1}$, take $\beta_s \in \Gamma_0$ such that $\sup_{\beta\in\Gamma_0} \beta \cdot s = \beta_s \cdot s$
$\eta_s = \beta_s / |\beta_s|$
$\{e_{s,1}, \ldots, e_{s,d-1}, s\}$: orthonormal basis of $\mathbf{R}^d$
$e_s = (e_{s,1}, \ldots, e_{s,d-1})$
Introduce new coordinates (w, z) near $i\beta_s$ such that

$$\zeta = w\eta_s + z \cdot e_s = w\eta_s + \sum_{j=1}^{d-1} z_j e_{s,j},$$

$$w \in \mathbf{C}, \quad z = (z_1, \ldots, z_{d-1}) \in \mathbf{R}^{d-1}.$$

Proposition 2. $\exists r > 0$ such that
$\forall s \in \mathbf{S}^{d-1}$ and $z \in \mathbf{R}^{d-1}$ with $|z| < r$,
the inverse $L(w\eta_s + z \cdot e_s)^{-1}$ has a simple pole $w_s(z)$ as a function of w and has the following asymptotics at the pole

$$L(w\eta_s + z \cdot e_s)^{-1} = \frac{A_{s,z}}{w - w_s(z)} + O(1).$$

Here $A_{s,z}$ is a rank one operator-valued function with

$$A_{s,0} = \frac{i}{\eta_s \cdot \nabla E(\beta_s)} \frac{(\cdot, v_{\beta_s})u_{\beta_s}}{(u_{\beta_s}, v_{\beta_s})},$$

and $w_s(z)$ is a smooth function having the following properties:
$w_s(0) = i|\beta_s|$
$\frac{\partial w_s}{\partial z_j}(0) = 0 \quad (1 \leq j \leq d-1)$

$$\frac{\partial^2 w_s}{\partial z_j \partial z_k}(0) = i\frac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j \partial z_k}(0) = i\frac{e_{s,j} \cdot \mathrm{Hess}\, E(\beta_s) e_{s,k}}{\eta_s \cdot \nabla E(\beta_s)}$$

$$\mathrm{Hess}\, \mathrm{Im}\, w_s(0) = \left(\frac{\partial^2 \mathrm{Im}\, w_s}{\partial z_j \partial z_k}(0)\right)_{1 \leq j,k \leq d-1} > 0$$

Now, regard the integral expression of the resolvent of L via $L(w\eta_s + z \cdot e_s)^{-1}$ as a complex integration for w.

Deform the contour of the integral in w.

Finally, apply the **residue theorem** and the following **saddle point method** to get the asymptotics at infinity of the Green function.

Proposition 3. Let $n = d-1$. Let U be an open neighborhood of the origin in $\mathbf{R}^n$.

Let $\varphi(x)$ and $a(x)$ be C^∞-functions on a neighborhood of $\overline{U}$ satisfying $||\varphi||_{C^9(U)} \leq b_1$ and $||a||_{C^6(U)} \leq b_2$ for some constants b_1 and b_2.

Assume that $\mathrm{Hess}\, \varphi(0) = \mathrm{Hess}\, \mathrm{Re}\, \varphi(0)$ and it is positive definite.

Further suppose that $\exists p > 0$ such that
$p|x|^2 \leq x \cdot \mathrm{Hess}\, \varphi(0) x$ for $x \in \mathbf{R}^n$ and $\mathrm{Re}\,(\varphi(x) - \varphi(0)) \geq p|x|^2/4$ for $x \in U$.

Then the asymptotics

$$\int_U e^{-\lambda\varphi(x)} a(x) dx = \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{e^{-\lambda\varphi(0)}}{(\det \mathrm{Hess}\, \varphi(0))^{1/2}}$$

$$\times (a(0) + O(\lambda^{-1})) \quad \text{as } \lambda \to \infty$$

holds.

4. References

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