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Lindelöf type theorems for monotone Sobolev functions on half spaces

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Abstract

This paper deals with Lindelöf type theorems for monotone functions in weighted Sobolev spaces.

1 Introduction

Let $\mathbf{R}^n$ $(n \geq 2)$ denote the n-dimensional Euclidean space. We use the notation $\mathbf{D}$ to denote the upper half space of $\mathbf{R}^n$, that is,

$$\mathbf{D} = \left\{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\right\}.$$

We denote by $\rho_{\mathbf{D}}(x)$ the distance of x from the boundary $\partial\mathbf{D}$, that is, $\rho_{\mathbf{D}}(x) = |x_n|$ for $x = (x', x_n)$. Denote by $B(x, r)$ the open ball centered at x with radius r, and set $\sigma B(x, r) = B(x, \sigma r)$ for $\sigma > 0$ and $S(x, r) = \partial B(x, r)$.

A continuous function u on $\mathbf{D}$ is called monotone in the sense of Lebesgue (see [6]) if the equalities

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u$$

hold whenever G is a domain with compact closure $\overline{G} \subset \mathbf{D}$. If u is a monotone Sobolev function in $\mathbf{D}$ and $p > n - 1$, then

$$|u(x) - u(y)| \leq Mr \left(\frac{1}{r^n} \int_{2B} |\nabla u(z)|^p dz\right)^{1/p} \tag{1}$$

for all $x, y \in B$, where B is an arbitrary ball of radius r with $2B \subset \mathbf{D}$ (see [7, Theorem 1] and [5, Theorem 2.8]). For further results of monotone functions, we refer to [3], [14] and [16].

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Our aim in the present note is to extend the second author's result [13, Theorem 2] to weighted case.

Let μ be a Borel measure on $\mathbf{R}^n$ satisfying the doubling condition :

$$\mu(2B) \le c_1\ \mu(B)$$

for every ball $B \subset \mathbf{R}^n$. We further assume that

$$\frac{\mu(B')}{\mu(B)} \ge c_2 \left(\frac{r'}{r}\right)^s \tag{2}$$

for all $B' = B(\xi', r')$ and $B = B(\xi, r)$ with $\xi', \xi \in \partial\mathbf{D}$ and $B' \subset B$, where $s > 1$.

THEOREM 1. *Let u be a Sobolev function on $\mathbf{D}$ satisfying*

$$|u(x) - u(y)| \le M\rho_{\mathbf{D}}(z) \left(⨍_{\sigma B} |\nabla u(z)|^p d\mu\right)^{1/p} \tag{3}$$

for every $x, y \in B = B(z, \rho_{\mathbf{D}}(z)/(2\sigma))$ with $z \in \mathbf{D}$ and

$$\int_{\mathbf{D}} |\nabla u(z)|^p d\mu(z) < \infty.$$

Define

$$E_1 = \left\{\xi \in \partial\mathbf{D} : \int_{B(\xi,1)\cap\mathbf{D}} |\xi - y|^{1-n} |\nabla u(y)| dy = \infty\right\}$$

and

$$E_2 = \left\{\xi \in \partial\mathbf{D} : \limsup_{r\to 0} \left(r^{-p}\mu(B(\xi, r))\right)^{-1} \int_{B(\xi,r)\cap\mathbf{D}} |\nabla u(y)|^p d\mu(y) > 0\right\}.$$

Then u has a nontangential limit at every $\xi \in \partial\mathbf{D} \setminus (E_1 \cup E_2)$.

Remark 1. Note here that $E_1 \cup E_2$ is of $C_{1,p,\mu}$-capacity zero. In Manfredi-Villamor [9], the exceptional sets are characterized by Hausdorff dimension, so that their result follows from this nontangential limit result.

THEOREM 2. *Let u be a function on $\mathbf{D}$ for which there exist a nonnegative function $g \in L^p_{loc}(\mathbf{D}; \mu)$, $M > 0$ and $\sigma \ge 1$ such that*

$$|u(x) - u(y)| \le M\rho_{\mathbf{D}}(z) \left(⨍_{\sigma B} g^p d\mu\right)^{1/p} \tag{4}$$

for every $x, y \in B = B(z, \rho_{\mathbf{D}}(z)/(2\sigma))$ with $z \in \mathbf{D}$ and

$$\int_{\mathbf{D}} g(z)^p d\mu(z) < \infty. \tag{5}$$

Suppose $p > s-1$ and set

$$E = \left\{ \xi \in \partial \mathbf{D} : \limsup_{r \to 0} \left(r^{-p} \mu(B(\xi, r)) \right)^{-1} \int_{B(\xi,r) \cap \mathbf{D}} g(z)^p d\mu(z) > 0 \right\}.$$

If $\xi \in \partial \mathbf{D} \setminus E$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit β, then u has a nontangential limit β at ξ.

For $\alpha > -1$, we consider

$$d\nu(x) = |x_n|^\alpha dx$$

as a measure, which satisfies

$$\nu(B(\xi, r)) = \nu(B(0,1)) r^{n+\alpha} \quad \text{for all } \xi \in \partial \mathbf{D} \text{ and } r > 0.$$

Then we obtain the following result.

COROLLARY 1. *Let u be a monotone Sobolev function on $\mathbf{D}$ satisfying*

$$\int_{\mathbf{D}} |\nabla u(z)|^p z_n^\alpha dz < \infty$$

for $p > n-1$ and $-1 < \alpha < p-n+1$. Consider the set

$$E_{n+\alpha-p} = \left\{ \xi \in \partial \mathbf{D} : \limsup_{r \to 0} r^{p-\alpha-n} \int_{B(\xi,r) \cap \mathbf{D}} |\nabla u(z)|^p z_n^\alpha dz > 0 \right\}.$$

If $\xi \in \partial \mathbf{D} \setminus E_{n+\alpha-p}$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit β, then u has a nontangential limit β at ξ.

REMARK 2. We know that $\mathcal{H}^{n+\alpha-p}(E_{n+\alpha-p}) = 0$, where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure, and hence it is of $C_{1-\alpha/p,p}$-capacity zero; for these results, see Meyers [10, 11] and the second author's book [14].

2 Proof of Theorem 2

A sequence $\{x_j\}$ is called regular at $\xi \in \partial \mathbf{D}$ if $x_j \to \xi$ and

$$|x_{j+1} - \xi| < |x_j - \xi| < c|x_{j+1} - \xi|$$

for some constant $c > 1$.

First we give the following result, which can be proved by (4).

LEMMA 1. *Let u and g be as in Theorem 2. If $\xi \in \partial \mathbf{D} \setminus E$ and there exists a regular sequence $\{x_j\} \subset \mathbf{D}$ with $x_j = \xi + (0, ..., 0, r_j)$ such that $u(x_j)$ has a finite limit β, then u has a nontangential limit β at ξ.*

PROOF OF THEOREM 2 : For $r > 0$ sufficiently small, take $C(r) \in \gamma \cap S(\xi, r)$. Letting $C_1(r) = \xi + (0, ..., 0, r)$, take an end point $C_2(r) \in \partial \mathbf{D}$ of a quarter of circle containing $C_1(r)$ and $C(r)$.

We take a finite chain of balls B_1, $B_2, \ldots$, B_N (N may depend on r) with the following properties:

(i) $B_j = B(z_j, \rho_{\mathbf{D}}(z_j)/(2\sigma))$ with $z_j \in \widehat{C(r)C_1}(r)$, $z_1 = C(r)$ and $z_N = C_1(r)$;

(ii) $\rho_{\mathbf{D}}(z_j) \le \rho_{\mathbf{D}}(z_{j+1})$ and $z_{j+1} \notin B_j$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each j;

(iv) $|C_2(r) - z| \le 3\rho_{\mathbf{D}}(z)$ for $z \in A(\xi, r) = \bigcup_{j=1}^N \sigma B_j \subset B(\xi, 2r) \cap \mathbf{D}$;

(v) $\sum_j \chi_{\sigma B_j} \le c_3$, where χ_A denotes the characteristic function of A and c_3 is a constant depending only on c_1 and σ;

see Heinonen [2] and Hajłasz-Koskela [1].

Pick $x_j \in B_{j+1} \cap B_j$ for $1 \le j \le N-1$. By (4), we see that

$$|u(x_j) - u(x_{j-1})| \le M\rho_{\mathbf{D}}(z_j) \left(\fint_{\sigma B_j} g(z)^p d\mu(z) \right)^{1/p}$$

for $1 \le j \le N$, where $x_0 = C(r)$ and $x_N = C_1(r)$.

Since $p > s-1$ by our assumption, there is $\delta > 0$ such that $s - p < \delta < 1$. We have by Hölder's inequality

$$\begin{aligned}
& |u(C_1(r)) - u(C(r))| \\
\le\ & |u(x_1) - u(x_0)| + |u(x_2) - u(x_1)| + \cdots + |u(x_N) - u(x_{N-1})| \\
\le\ & M \sum_{j=1}^N \rho_{\mathbf{D}}(z_j)^{1+\delta/p} \mu(\sigma B_j)^{-1/p} \left(\int_{\sigma B_j} g(z)^p \rho_{\mathbf{D}}(z)^{-\delta} d\mu(z) \right)^{1/p} \\
\le\ & M \left(\sum_{j=1}^N \rho_{\mathbf{D}}(z_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \left(\int_{A(\xi,r)} g(z)^p \rho_{\mathbf{D}}(z)^{-\delta} d\mu(z) \right)^{1/p} \\
\le\ & M \left(\sum_{j=1}^N \rho_{\mathbf{D}}(z_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \left(\int_{B(\xi,2r)\cap\mathbf{D}} g(z)^p |C_2(r) - z|^{-\delta} d\mu(z) \right)^{1/p}
\end{aligned}$$

where $1/p + 1/p' = 1$. Here note that $\mu(B(C_2(r), \rho_{\mathbf{D}}(z_j))) \le c_4 \mu(\sigma B_j)$, where c_4 is a positive constant depending only on the doubling constant c_1. Since $\delta > s - p$, we see from (2) that

$$\sum_{j=1}^N \rho_{\mathbf{D}}(z_j)^{p'(p+\delta)/p} \mu(\sigma B_j)^{-p'/p} \le M \sum_{j=1}^N \rho_{\mathbf{D}}(z_j)^{p'(p+\delta)/p} \mu(B(C_2(r), \rho_{\mathbf{D}}(z_j)))^{-p'/p}$$

$$
\begin{aligned}
&\leq M\sum_{j=1}^{N}\rho_{\mathbf{D}}(z_j)^{p'(p+\delta)/p}\left(\frac{\rho_{\mathbf{D}}(z_j)}{2r}\right)^{-sp'/p}\mu(B(\xi,2r))^{-p'/}\\
&\leq Mr^{sp/p'}\mu(B(\xi,r))^{-p'/p}\sum_{j=1}^{N}\rho_{\mathbf{D}}(z_j)^{p'(p+\delta-s)/p}\\
&\leq Mr^{sp'/p}\mu(B(\xi,r))^{-p'/p}\int_0^r t^{p'(p+\delta-s)/p}dt/t\\
&\leq Mr^{\delta p'/p}\left(r^{-p}\mu(B(\xi,r))\right)^{-p'/p}.
\end{aligned}
$$

Moreover, since $0<\delta<1$, we note that

$$
\int_{2^{-j}}^{2^{-j+1}}|C_2(r)-z|^{-\delta}dr \leq \int_{2^{-j}}^{2^{-j+1}}|r-|z||^{-\delta}dr \leq M2^{-j(1-\delta)}. \tag{6}
$$

Hence it follows from (6) that

$$
\begin{aligned}
&\int_{2^{-j}}^{2^{-j+1}}|u(C_1(r))-u(C(r))|^p\frac{dr}{r}\\
\leq\ & M\int_{2^{-j}}^{2^{-j+1}}r^{\delta}(r^{-p}\mu(B(\xi,r)))^{-1}\left(\int_{B(\xi,2r)\cap\mathbf{D}}g(z)^p|C_2(r)-z|^{-\delta}d\mu(z)\right)\frac{dr}{r}\\
\leq\ & M2^{-j(p+\delta-1)}\mu(B(\xi,2^{-j}))^{-1}\int_{B(\xi,2^{-j+2})\cap\mathbf{D}}g(z)^p\left(\int_{2^{-j}}^{2^{-j+1}}|C_2(r)-z|^{-\delta}dr\right)d\mu(z)\\
\leq\ & M\left(2^{jp}\mu(B(\xi,2^{-j}))\right)^{-1}\int_{B(\xi,2^{-j+2})\cap\mathbf{D}}g(z)^p d\mu(z).
\end{aligned}
$$

Since $\xi\in\partial\mathbf{D}\setminus E$, we can find a sequence $\{r_j\}$ such that $2^{-j}<r_j<2^{-j+1}$ and

$$
\lim_{j\to\infty}|u(C_1(r_j))-u(C(r_j))|=0.
$$

By our assumption we see that $u(C_1(r_j))$ has a finite limit β as $j\to\infty$. If we note that $\{C_1(r_j)\}$ is regular at ξ, then Lemma 1 proves the required conclusion of the theorem.

3 A_q weights

Let w be a Muckenhoupt A_q weight, and define

$$
d\nu(y)=w(y)dy.
$$

Let u be a monotone Sobolev function on $\mathbf{D}$ such that

$$
\int_{\mathbf{D}}|\nabla u(x)|^p d\nu(x)<\infty.
$$

Suppose that $1 < q < p/(n-1)$. Since $p_1 = p/q > n-1$, applying inequality (1) we obtain

$$|u(x) - u(y)| \le Mr \left(\frac{1}{r^n} \int_{2B} |\nabla u(z)|^{p_1} dz \right)^{1/p_1}$$

for all $x, y \in B$, where B is an arbitrary ball of radius r with $2B \subset \mathbf{D}$. As in the proof of Theorem 2, we insist that

$$\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} \le M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \int_{B(\xi, 2^{-j+2}) \cap \mathbf{D}} |\nabla u(z)|^{p_1} dz.$$

Using Hölder inequality and A_q-condition of w, we have

$$\begin{aligned} &\int_{2^{-j}}^{2^{-j+1}} |u(C_1(r)) - u(C(r))|^{p_1} \frac{dr}{r} \\ \le\ & M 2^{-jp_1} |B(\xi, 2^{-j})|^{-1} \left(\int_{B(\xi, 2^{-j+2}) \cap \mathbf{D}} |\nabla u(z)|^{p_1 q} w(z) dz \right)^{1/q} \left(\int_{B(\xi, 2^{-j+2})} w(z)^{-q'/q} dz \right)^{1/q'} \\ \le\ & M \left(\left(2^{jp} \nu(B(\xi, 2^{-j})) \right)^{-1} \int_{B(\xi, 2^{-j+2}) \cap \mathbf{D}} |\nabla u(z)|^p d\nu(z) \right)^{1/q}, \end{aligned}$$

where $1/q + 1/q' = 1$. Thus we obtain the following result (cf. Manfredi-Villamor [9]), as in the proof of Theorem 2.

COROLLARY 2. *Let $1 \le q < p/(n-1)$. Let $w \in A_q$ and set $d\nu(y) = w(y)dy$. Suppose that u is a monotone Sobolev function on $\mathbf{D}$ satisfying*

$$\int_{\mathbf{D}} |\nabla u(z)|^p d\nu(z) < \infty. \tag{7}$$

Set

$$E = \left\{ \xi \in \partial \mathbf{D} : \limsup_{r \to 0} \left(r^{-p} \nu(B(\xi, r)) \right)^{-1} \int_{B(\xi, r) \cap \mathbf{D}} |\nabla u(z)|^p d\nu(z) > 0 \right\}.$$

If $\xi \in \partial \mathbf{D} \setminus E$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit β, then u has a nontangential limit β at ξ.

REMARK 3. Let $1 \le q < p/(n-1)$. Let w be a Muckenhoupt A_q weight, and define

$$d\nu(y) = w(y)dy.$$

Suppose that u is a monotone Sobolev function on $\mathbf{D}$ satisfying (7). Applying Hölder's inequality to (1) with p replaced by p/q, we see that

$$|u(x) - u(y)| \le Mr \left(\fint_{2B} |\nabla u(z)|^p d\nu(z) \right)^{1/p}$$

for all $x, y \in B$, where B is an arbitrary ball of radius r with $2B \subset \mathbf{D}$ (see also Manfredi-Villamor [9]).

REMARK 4. Consider $w(y) = |y_n|^\alpha$. Then $w \in A_q$ if and only if $-1 < \alpha < q-1$. In this case, Corollary 2 does not imply Corollary 1 when $n \ge 3$.

4 Generalizations of Lindelöf theorems

For an integer d, $1 \leq d < n$, let $P_d : \mathbf{R}^n \longrightarrow \mathbf{R}^d$ be the projection, that is,

$$P_d(x) = (x_1, \dots, x_d, 0, \dots, 0) \quad \text{for } x = (x_1, x_2, \dots, x_n).$$

We say that $\Gamma \subset \mathbf{D}$ is a $(\lambda_1, \lambda_2, d)$-approach set at ξ, where $\lambda_1 \geq 1$ and $\lambda_2 > 0$, if there exists a sequence of positive numbers $\{r_j\}$ tending to zero such that $r_{j+1} < r_j < \lambda_1 r_{j+1}$ and

$$\mathcal{H}^d(P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1})))) \geq \lambda_2 r_j^d.$$

Theorem 3. *Let u be a function on $\mathbf{D}$ with g satisfying* (4) *and*

$$\int_{\mathbf{D}} g(z)^p d\mu(z) < \infty.$$

Suppose $p > s - d$, and define

$$E = \left\{ \xi \in \partial\mathbf{D} : \limsup_{r \to 0} \left(r^{-p}\mu(B(\xi, r))\right)^{-1} \int_{B(\xi,r)\cap\mathbf{D}} g(z)^p d\mu(z) > 0 \right\}.$$

If $\xi \in \partial\mathbf{D} \setminus E$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset \mathbf{D}$ at ξ along which u has a finite limit β at ξ, then u has a nontangential limit β at ξ.

Proof. By our assumption, we can take $\delta > 0$ such that $s - p < \delta < d$. Set

$$G_j = P_d(\Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))).$$

For $X \in G_j$, take $C(X) \in \Gamma \cap (B(\xi, r_j) \setminus B(\xi, r_{j+1}))$, and set $r(X) = r = |\xi - C(X)|$. Let $C_1(X) = \xi + (0, ..., 0, r)$ and $D(X) = P_{n-1}(C(X))$.

We take a finite chain of balls B_1, $B_2, \dots$, B_N with the following properties:

(i) $B_j = B(z_j, \rho_{\mathbf{D}}(z_j)/(2\sigma))$ with $z_j \in \widehat{C(X)C_1}(X)$, $z_1 = C(X)$ and $z_N = C_1(X)$;

(ii) $\rho_{\mathbf{D}}(z_j) \leq \rho_{\mathbf{D}}(z_{j+1})$ and $z_{j+1} \notin B_j$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each j;

(iv) $|D(X) - z| \leq 3\rho_{\mathbf{D}}(z)$ for $z \in A(\xi, r) = \bigcup_{j=1}^N \sigma B_j \subset B(\xi, 2r) \cap \mathbf{D}$;

(v) $\sum_j \chi_{\sigma B_j} \leq c_3$.

Since $\delta > s - p$, we have as in the proof of Theorem 2

$$|u(C_1(X)) - u(C(X))|^p \leq M r^\delta \left(r^{-p}\mu(B(\xi, r))\right)^{-1} \int_{B(\xi,2r)\cap\mathbf{D}} g(z)^p |D(X) - z|^{-\delta}\, d\mu(z).$$

Further, since P_d is 1-Lipschitz and $0 < \delta < d$, we see that

$$\begin{aligned}\int_{G_j} |D(X) - z|^{-\delta} d\mathcal{H}^d(X) &\leq \int_{G_j} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\ &\leq \int_{P_d(B(\xi, r_j))} |X - P_d(z)|^{-\delta} d\mathcal{H}^d(X) \\ &\leq M r_j^{d-\delta}.\end{aligned}$$

Hence we have

$$⨍_{G_j} |u(C_1(X)) - u(C(X))|^p d\mathcal{H}^d(X) \leq M \left(r_j^{-p} \mu(B(\xi, r_j))\right)^{-1} \int_{B(\xi, 2r_j) \cap \mathbf{D}} g(z)^p d\mu(z).$$

Thus we can find a sequence $\{X_j\}$ such that $X_j \in G_j$ and

$$\lim_{j \to \infty} |u(C_1(X_j)) - u(C(X_j))| = 0.$$

Thus we see that $u(C_1(X_j))$ has a finite limit β as $j \to \infty$. Since $\{C_1(X_j)\}$ is regular at ξ, we can show that u has a nontangential limit β at ξ by Lemma 1.

Corollary 3. *Let u be a harmonic function on $\mathbf{D}$ satisfying*

$$\int_{\mathbf{D} \cap B(0,N)} |\nabla u(z)|^p z_n^\alpha dz < \infty$$

for every $N > 0$, and $-1 < \alpha < p - n + d$. If $\xi \in \partial\mathbf{D} \setminus E_{n+\alpha-p}$ and there exists a $(\lambda_1, \lambda_2, d)$-approach set $\Gamma \subset \mathbf{D}$ at ξ along which u has a finite limit β at ξ, then u has a nontangential limit β at ξ.

REMARK 5. The conclusion of Corollary 3 is still valid for $\mathcal{A}$-harmonic functions and polyharmonic functions.

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