数理解析研究所講究録 1296 巻 2002 年 92-100

Non-spherical principal series Whittaker functions on $SL(3, \mathbf{R})$

Hiroyuki MANABE (眞鍋廣幸) and Takayuki ODA (織田孝幸)
Graduate School of Math.-Sci., the Univ. of Tokyo; 東京大学数理科学
(Talk presented by Oda at RIMS, Kyoto, June 5, 2002)

1 Introduction

This article is just to explain the main results of the master thesis by Hiroyuki MANABE ([1]).

The study of Whittaker models of algebraic groups over local fields has already some history. The Jacquet integral is named after the investigation of H.Jaquet. Multiplicity free theorem by J.Shalika for quasi-split groups, was later enhanced for the case of real fields by N.Wallach. For redutive groups over the real field, this theme was investigated by M.Hashizume, B.Kostant, D. Vogan, H.Matsumoto, and the joint work of R.Goodmann and N.Wallach.

More specifically $GL(n, \mathbf{R})$, explicit expressions for class 1 Whittaker functions are obtained, firstly for $n=3$ by D.Bump [2]. The main contributor for the case of general n seems to be E.Stade. Other related results will be find in the references of the papers of him ([7],[8]).

The purpose of the master thesis [1] is to discuss the Whittaker functions belonging to the non-spherical principal series representations of $SL(3, \mathbf{R})$. The minimal K-type of such representations is 3-dimensional. So we have to consider vector-valued functions. The main reults are, firstly, to obtain the holonomic system of the A-radial part of such Whittaker functions with minimal K-type explicitly, and secondly to have 6 formal solutions, which are consider as examples of conflunent hypergeometric series of two variables.

2 Whittaker model

Given an irreducible admissible representation (π, H) of $G=SL(3, \mathbf{R})$, we consider its model or realization in the space of Whittaker functions. This means, for a non-degenerate unitary character ψ of a maximal unipotent subgroup $N = \{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in G\}$ of G defined by

$$\psi(\begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix}) = \exp\{2\pi\sqrt{-1}(c_1x_{12} + c_2x_{23})\}$$

with $c_1, c_2 \in \mathbf{R}$ being non-zero, we consider a smooth induction $C^\infty\text{–}\mathrm{Ind}_N^G(\psi)$ to G, and the space of intertwining operators of smooth G-modules

$$\mathrm{Hom}_G(H^\infty, C^\infty\text{–}\mathrm{Ind}_N^G(\psi)).$$

Or more algebraically speaking, we might consider the corresponding space in the context of $(\mathfrak{g}, K)$-modules (with $\mathfrak{g} = \mathrm{Lie}(G), K = SO(3)$):

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H^\infty, C^\infty\text{–}\mathrm{Ind}_N^G(\psi)).$$

3 Principal series representations

Let P be a minimal parabolic subgroup of G given by the upper triangular matrices in G, and $P = MAN$ be a Langlands decomposition of P with $M = K \cap \{$ diagonals in $G\}$, $A = \exp\mathfrak{a}$, with

$$\mathfrak{a} = \{\mathrm{diag}(a_1, a_2, a_3) | a_i \in \mathbf{R}, a_1 + a_2 + a_3 = 0\}.$$

In order to define a principal series representation with respect to the minimal parabolic subgroup P of G, we firstly fix a character σ of the finite abelian group M of type $(2,2)$ and a linear form $\nu \in \mathfrak{a}^* \otimes_{\mathbf{R}} \mathbf{C} = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{a}, \mathbf{C})$. For such data, we can define a representation $\sigma \otimes e^\nu$ of MA, and extend this to P by the identification $P/N \cong MA$. Then we set

$$\pi_{\sigma,\nu} = L^2\text{-}\mathrm{Ind}_P^G(\sigma \otimes e^{\nu+\rho} \otimes 1_N).$$

Here ρ is the half-sum of positive roots of $(\mathfrak{g}, \mathfrak{a})$ for P.

Fact 1 (i) If σ is the trivial character of M, the representation $\pi_{\sigma,\nu}$ is spherical or class 1, i.e. it has a (unique) K-invariant vector in the representation space $H_{\sigma,\nu}$.
(ii) If σ is not trivial, then the minimal K-type of the restriction $\pi_{\sigma,\nu}|K$ to K is a 3-dimensional representation of $K = SO(3)$, which is isomorphic to the unique standard one (τ_2, V_2). The multiplicity of this minimal K-type is one, i.e.

$$\dim_{\mathbf{C}} \mathrm{Hom}_K(\tau_2, H_{\sigma,\nu}) = 1,$$

i.e. there is a unique non-zero K-homomorphism

$$\iota : (\tau_2, V_2) \to (\pi_{\sigma.\nu}|K, H_{\sigma,\nu})$$

up to constant multiple.

4 Standard basis of (τ_2, V_2)

In order to specify a basis $\{v_0, v_1, v_2\}$ in V_2, which we call standard, firstly we define generators of the Lie algbera $\mathfrak{k} = \mathrm{Lie}(K)$ by

$$K_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, K_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in \mathfrak{k}.$$

Fact 1 There is a basis $\{v_0, v_1, v_2\}$ of V_2 such that

$$K_1 v_j = (1-j)\sqrt{-1}v_j \quad (j = 0, 1, 2),$$
$$K_2 v_j = -\frac{1}{2} j v_{j-1} + (1 - \frac{1}{2}j) v_{j+1},$$
$$K_3 v_j = -\frac{1}{2} j \sqrt{-1} v_{j-1} - (1 - \frac{1}{2}j)\sqrt{-1} v_{j+1}.$$

Or in other words,

$$X_+ v_j = (2-j) v_{j+1}, X_- v_j = -j v_{j-1}.$$

For non-trivail σ, there exists a unique non-zero K-homomorphism

$$\iota : \tau_2 \to \pi_{\sigma,\nu}$$

up to constant multiple.

5 Standard Whittaker functions and the holonomic system for their A-radial parts

Given a Whittaker functional

$$W : (\pi_{\sigma,\nu}, H^{\infty}_{\sigma,\nu}) \to C^{\infty}\text{–}\mathrm{Ind}_N^G(\psi),$$

we can consider the images $\varphi_i(g) = W(\iota(v_i))$ of each v_i. For each i it satisfies $\varphi_i(xg) = \psi(x)\varphi(g)$ for any $x \in N$ and $g \in G$. Since both ι and W are K-homomorphisms, we have an intertwinig property

$$^t(\varphi_0(gk), \varphi_1(gk), \varphi_2(gk)) = \tau_2(k)^t(\varphi_0(g), \varphi_1(g), \varphi_2(g))$$

for $k \in K$ and $g \in G$.

Therefore to specifty the functions $\varphi_i(g)$ on G it sufficies to investigate their A-radial parts $F_i = \varphi_i|A$.

Now we can state our 1-st main result.

Theorem A Let $F(a) = {}^t(F_0(a), F_1(a), F_2(a))$ be the vector of the A-radial part of the standard Whittaker functions with minimal K-type of the principal series representation $\pi_{\sigma,\nu}$ with non-trivial σ. Then it satisfies the following partial differential equations:

(i)

$$\begin{pmatrix} -(\partial_1+\partial_2)+3 & 6\sqrt{-1}\eta_2(a) & 3(\partial_1-\partial_2)-6\sqrt{-1}\eta_1(a)-3 \\ -3\sqrt{-1}\eta_2(a) & 2(\partial_1+\partial_2)-6 & -3\sqrt{-1}\eta_2(a) \\ 3(\partial_1-\partial_2)+6\sqrt{-1}\eta_1(a)-3 & 6\sqrt{-1}\eta_2(a) & -(\partial_1+\partial_2)+3 \end{pmatrix} \begin{pmatrix} F_0(a) \\ F_1(a) \\ F_2(a) \end{pmatrix}$$

$$= \lambda_i \begin{pmatrix} F_0(a) \\ F_1(a) \\ F_2(a) \end{pmatrix}$$

$$\text{(ii)} \qquad \{\partial_1^2+\partial_2^2-\partial_1\partial_2-3\partial_1+3(\eta_1(a)^2+\eta_2(a)^2)\} \begin{pmatrix} F_0(a) \\ F_1(a) \\ F_2(a) \end{pmatrix}$$

$$+3\sqrt{-1}\eta_1(a) \begin{pmatrix} F_0(a) \\ 0 \\ -F_2(a) \end{pmatrix} - 3\sqrt{-1}\eta_2(a) \begin{pmatrix} F_1(a) \\ \frac{1}{2}(F_0(a)+F_2(a)) \\ F_1(a) \end{pmatrix} = \mu \begin{pmatrix} F_0(a) \\ F_1(a) \\ F_2(a) \end{pmatrix}$$

Here we set $\partial_j = a_j \dfrac{\partial}{\partial a_j} (j=1,2)$ and

$$\eta_1(a) = 2\pi\sqrt{-1}c_1 a_1 a_2^{-1}, \eta_2(a) = 2\pi\sqrt{-1}c_2 a_1 a_2^2.$$

Moreover the eigenvalues λ_i and μ depending on the representation $\pi_{\sigma,\nu}$ are given by

$$\begin{cases} \lambda_1 = -2(2\nu_1-\nu_2) & (\sigma=\sigma_1) \\ \lambda_2 = 2(\nu_1-2\nu_2) & (\sigma=\sigma_2) \\ \lambda_3 = 2(\nu_1+\nu_2) & (\sigma=\sigma_3) \end{cases} \quad \text{and } \mu = \nu_1^2+\nu_2^2-\nu_1\nu_2.$$

Here the characters σ_j of M is identified as follows.

The group M consisting of 4 elements is a finite abelian group of $(2,2)$ type, and its elements except for the unity is given by the matrices

$$m_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, m_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since M is commutative, all the irreducible unitary representations of it is 1-dimensional. For any $\sigma \in \widehat{M}$, we have $\sigma^2 = 1$. Therefore the set $\widehat{M}$ consisting of 4 characters $\{\sigma_j : j = 0,1,2,3\}$, where each σ_j, except for the tirivial character σ_0, is specified by the following table of values at the elements m_i.

	m_1	m_2	m_3
σ_1	1	-1	-1
σ_2	-1	1	-1
σ_3	-1	-1	1

Remark We can write the differential equations (i) and (ii) of Theorem A as

$$\text{(i)}: \quad \mathcal{D}_1 F = \lambda_i F \qquad \text{(ii)}: \quad \mathcal{D}_2 F = \mu F,$$

with $\mathcal{D}_i$ $(i = 1,2)$ 3 by 3 matrix-valued differential operators. Then we have

$$\mathcal{D}_1 \cdot \mathcal{D}_2 - \mathcal{D}_2 \cdot \mathcal{D}_1 = 0.$$

This is natural, because our system of equations is holonomic, hence involutive.We can use this computation for error check of calculation.

6 Solutions

6.1 Variable change

The holonomic system obtained in Theorem A has regular singularities at the point $(a_1, a_2) = (0,0)$. The rank of this system is 6, i.e. the order of the Weyl group of $SL(3,\mathbf{R})$, for generic values of parameter ν. We want to determine the characteristic indices and the convergent formal power series solutions at $(a_1, a_2) = 0$.

Before to do so, it is more convenient to rewrite the system in new variables (y_1, y_2) correponding to positive roots, given by

$$y_1 = a_1 a_2^2, \quad y_2 = a_1 a_2^{-1}.$$

The reason to name the variales as y_* is to make it easier to compare with the results and formulae, say, of Bump [2].

Also after some computation, by inspection, we find that it is convenient to introduce scalar functions $\Phi_i(y_1, y_2)$ $(i = 0,1,2)$ by

$$F(y) = \Phi_0(y) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \Phi_1(y) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \Phi_2(y) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Here to abbridge the notation, we write the set of variales (y_1, y_2) as y collectively.

6.2 Characteristic indices

Now we can determine the 6 pairs (ρ_1, ρ_2) of characteristic indices, and the corresponding initial values conditions for F or Φ_i.

Lemma B.1 When $\sigma = \sigma_i$ for $i = 1$, 2 or 3, we have the following:
(i) If $(\rho_1, \rho_2) = (\frac{1}{6}\lambda_2 + 1, -\frac{1}{6}\lambda_k + 1) \quad (k \neq i)$,

$$F(0,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ i.e. } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_0(0,0) = 1, \text{ and } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_j(0) = 0 \text{ for other } j.$$

(ii) If $(rho_1, \rho_2) = (\frac{1}{6}\lambda_k + 1, -\frac{1}{6}\lambda_l + 1) \quad (k \neq i, l \neq i, k \neq l)$,

$$F(0,0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ i.e. } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_1(0,0) = 1, \text{ and } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_j(0) = 0 \text{ for other } j.$$

(iii) If $(\rho_1, \rho_2) = (\frac{1}{6}\lambda_k + 1, -\frac{1}{6}\lambda_2 + 1) \quad (k \neq i)$,

$$F(0,0) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ i.e. } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_2(0,0) = 1, \text{ and } y_1^{-\rho_1} y_2^{-\rho_2} \Phi_j(0) = 0 \text{ for other } j.$$

6.3 The holonomic system for $\Phi_i(y)$

Proposition A.2 The holonomic system in Theorem A is equivalent to the following system for $\Phi_k(y_1, y_2)(k = 0, 1, 2,)$.

(i)

(i-i) $\left\{(y_1 \frac{\partial}{\partial y_1} - 1) - \frac{1}{6}\lambda_i\right\} \Phi_0(y_1, y_2) + 2\pi y_1 \Phi_1(y_1, y_2) = 0$

(i-ii) $\left\{(y_1 \frac{\partial}{\partial y_1} - 1) - (y_2 \frac{\partial}{\partial y_2} - 1) + \frac{1}{6}\lambda_i\right\} \Phi_1(y_1, y_2)$
$-2\pi y_1 \Phi_0(y_1, y_2) - 2\pi y_2 \Phi_2(y_1, y_2) = 0$

(i-iii) $\left\{(y_2 \frac{\partial}{\partial y_2} - 1) + \frac{1}{6}\lambda_i\right\} \Phi_2(y_1, y_2) - 2\pi y_2 \Phi_1(y_1, y_2) = 0$

(ii)

(ii-i) $(\Delta - \frac{1}{3}\mu)\Phi_0(y_1, y_2) + 2\pi y_1 \Phi_1(y_1, y_2) = 0$

(ii-ii) $(\Delta - \frac{1}{3}\mu)\Phi_1(y_1, y_2) + 2\pi y_1 \Phi_0(y_1, y_2) - 2\pi y_2 \Phi_2(y_1, y_2) = 0$

(ii-iii) $(\Delta - \frac{1}{3}\mu)\Phi_2(y_1,y_2) - 2\pi y_2\Phi_1(y_1,y_2) = 0$

Here we set

$$\Delta = y_1^2\frac{\partial^2}{\partial y_1^2} + y_2^2\frac{\partial^2}{\partial y_2^2} - y_1y_2\frac{\partial^2}{\partial y_1\partial y_2} - 4\pi^2(y_1^2+y_2^2).$$

7 Power series solutions

Let

$$\Phi_k(y_1,y_2) = y_1^{e_1+1}y_2^{-e_2+1}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} c_{k;n_1,n_2}(\pi y_1)^{n_1}(\pi y_2)^{n_2}$$

be the power series solution at $(y_1,y_2) = (0,0)$ for each k. Then Propostion (A.2) is equivalent to some recurrence relations among the coefficients.

Before to describe the solutions of these recurrence relations, we have to introduce certain multinomial coefficients.

Definition Given 3 complex numbers α,β,γ with $\alpha+\beta+\gamma=0$, and let (m_1,m_2) be a pair of non-negative integers.

(i) If neither of $\frac{1}{2}(\alpha-\beta+1), \frac{1}{2}(\gamma-\beta+2), \frac{1}{2}(\alpha-\gamma+1)$ is non-postive integer, we put

$$A(\alpha,\beta,\gamma;m_1,m_2) = \frac{\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_1+m_2)}}{m_1!m_2!\{\frac{1}{2}(\alpha-\gamma+1)\}^{(m_1)}\{\frac{1}{2}(\gamma-\beta+2)\}^{(m_2)}\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_1)}\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_2)}}.$$

(ii) If neither of $\frac{1}{2}(\alpha-\beta+2), \frac{1}{2}(\gamma-\beta+1), \frac{1}{2}(\alpha-\gamma+1)$ is non-positive integer, we put

$$B(\alpha,\beta,\gamma;m_1,m_2) = \frac{\{\frac{1}{2}(\alpha-\beta+2)\}^{(m_1+m_2)}}{m_1!m_2!\{\frac{1}{2}(\alpha-\gamma+1)\}^{(m_1)}\{\frac{1}{2}(\gamma-\beta+1)\}^{(m_2)}\{\frac{1}{2}(\alpha-\beta+2)\}^{(m_1)}\{\frac{1}{2}(\alpha-\beta+2)\}^{(m_2)}}.$$

(iii) If neither of $\frac{1}{2}(\alpha-\beta+1), \frac{1}{2}(\gamma-\beta+1), \frac{1}{2}(\alpha-\gamma+2)$ is non-positive integer, we put

$$C(\alpha,\beta,\gamma;m_1,m_2) = \frac{\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_1+m_2)}}{m_1!m_2!\{\frac{1}{2}(\alpha-\gamma+2)\}^{(m_1)}\{\frac{1}{2}(\gamma-\beta+1)\}^{(m_2)}\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_1)}\{\frac{1}{2}(\alpha-\beta+1)\}^{(m_2)}}.$$

We might drop the last parameter in the symbol A, B, C to save space.

There are obvious relations:

$$A(-\beta,-\alpha,-\gamma;m_2,m_1) = C(\alpha,\beta,\gamma;m_1,m_2), \quad B(-\beta,-\alpha,-\gamma;m_2,m_1) = B(\alpha,\beta,\gamma;m_1,m_2).$$

Theorem B.2 Under the same notations as Lemma (B.1), we have the

(i) If $(\rho_1,\rho_2)=(\frac{1}{6}\lambda_2+1,-\frac{1}{6}\lambda_k+1)\quad(k\neq i)$, then the solution is given by a set of power series:

$$\Phi_0(y)=y_1^{+\frac{1}{6}\lambda_2+1}y_2^{-\frac{1}{6}\lambda_k+1}\sum_{n_1,n_2\geq 0}A(\frac{1}{6}\lambda_2,-\frac{1}{6}\lambda_k;\frac{n_1}{2},\frac{n_2}{2})(\pi y_1)^{n_1}(\pi y_2)^{n_2},$$

$$\Phi_1(y)=-y_1^{+\frac{1}{6}\lambda_2+1}y_2^{-\frac{1}{6}\lambda_k+1}\sum_{n_1,n_2\geq 0}A(\frac{1}{6}\lambda_2,-\frac{1}{6}\lambda_k;\frac{n_1}{2},\frac{n_2}{2})\cdot\frac{n_1+1}{2}(\pi y_1)^{n_1}(\pi y_2)^{n_2},$$

$$\Phi_2(y)=-y_1^{+\frac{1}{6}\lambda_2+1}y_2^{-\frac{1}{6}\lambda_k+1}\sum_{n_1,n_2\geq 0}A(\frac{1}{6}\lambda_2,-\frac{1}{6}\lambda_k;\frac{n_1}{2},\frac{n_2}{2})\cdot\frac{n_1+1}{n_2+\frac{1}{6}\lambda_2+\frac{1}{6}\lambda_k}(\pi y_1)^{n_1}(\pi y_2)^{n_2}.$$

(ii) If $(\rho_1,\rho_2)=(\frac{1}{6}\lambda_k+1,-\frac{1}{6}\lambda_l+1)\quad(k\neq i,l\neq i,k\neq l)$,

$$\Phi_1(y)=y_1^{\frac{1}{6}\lambda_k+1}y_2^{-\frac{1}{6}\lambda_l+1}\sum_{n_1,n_2\geq 0}B(\frac{1}{6}\lambda_k,-\frac{1}{6}\lambda_l;\frac{n_1}{2},\frac{n_2}{2})(\pi y_1)^{n_1}(\pi y_2)^{n_2},$$

And there are similar formulae for Φ_0,Φ_2.
(iii) If $(\rho_1,\rho_2)=(\frac{1}{6}\lambda_k+1,-\frac{1}{6}\lambda_2+1)\quad(k\neq i)$,

$$\Phi_2(y)=y_1^{\frac{1}{6}\lambda_k+1}y_2^{-\frac{1}{6}\lambda_2+1}\sum_{n_1,n_2\geq 0}C(\frac{1}{6}\lambda_k,-\frac{1}{6}\lambda_2;\frac{n_1}{2},\frac{n_2}{2})(\pi y_1)^{n_1}(\pi y_2)^{n_2},$$

And similar formulae for Φ_0,Φ_1.

Note here if $\frac{n_1}{2}$ in the coefficient $A(*,*;\frac{n_1}{2},*)$ is not an integer, the coefficient is regarded as zero. We consider similarly for other entires $\frac{n_2}{2},\frac{n_1-1}{2}$, and other coefficients B,C.

Remark 1 The similarity of the form of our formal power series solution to that of the class 1 case is clear (*cf* Bump [2]). Therefore by a similar method as that of Stade [7], one can find integral expressions for the above power series solutions, using I-Bessel functions.

8 Further problems and comments

The solution of the following problem, which is also fundamental, seem to be in our reach (see [2], [3], [7]).

Problem 1 Write the Jacquet integral as a linear combination of the above 6 solutions.

A bit more delicate problem, but an important one, for it might have application to Number Theory, is the following.

Problem 2 Invetigate what happens when the characteristic indices become

9 Notation and symbols

Unfortunately some of notational definitions in the original paper [1] is not adequate. Firstly the author discusses the representation L^2−Ind($\sigma \otimes e^{\nu} \otimes 1_N$), not L^2−Ind($\sigma \otimes e^{\nu+\rho} \otimes 1_N$) as ours. This shift the parameters ν by ± 1.

The second point is the numbering of the representations of M is not adequate. His σ_1 is our σ_2, and his σ_2 is our σ_1. Accordingly λ_i is also changed in the same way. The efffect of the shift of ν does not appear in the symbol λ_i itself. The readers should be careful for these differences of notation.

参考文献

[1] Hiroyuki Manabe (眞鍋廣幸), On Whittaker functions for the principal series representations of $SL(3,\mathbf{R})$ (in Japanese), Master thesis, 2002 March, the University of Tokyo

[2] D. Bump, Automorphic Forms on $GL(3,\mathbb{R})$, Lecture Notes in Mathematics **1083**, Springer-Verlag Berlin Heidelberg New York Tokyo 1984

[3] D. Bump and J. Huntley, Unramified Whittaker functions for $GL(3,\mathbb{R})$, J. d'Analyse Mathématique, **65**, (1995), 19–44.

[4] A. W. Knapp, Representation theory of semisimple groups, -an overview based on examples-, Princeton Mathematical Series, vol. 36, Princeton Univercity Press, New Jersey, 1986

[5] B. Kostant, On Whittaker Vectors and Representation Theory, Inventions Math. 48(1978), 101-184

[6] W. Schmid, On the realization of the discrete series of a semisimple Lie group, Rice Univ. Stud. 56(1970), No.1, 99-108

[7] E.Stade, Poincaré series for $GL(3,\mathbb{R})$-Whittaker functions, Duke J. Math., **58** (1989), 131–165.

[8] E.Stade, On explicit formulas for $GL(n,\mathbb{R})$-Whittaker functions, Duke J. Math. **60** (1990), 313–362.