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Game Theoretic Analysis of a Stochastic Inventory Control Problem

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1. The Model

There are n retailers treating a kind of product. Each of them buys products from a supplier and sells them to customers. Each retailer faces an independent demand. We treat a case of a single period with no inventory at the beginning of the period. At the beginning of the period, each retailer tries to choose his best order quantity in order to meet a (uncertain) demand in that period. The inventory carrying cost occurs for the units that are not sold at the end of the period. The penalty cost occurs when a retailer does not order enough products to meet a demand. If some of retailers make a coalition, menbers in the coalition can transfer their demands among them. If one is short of supply its excess demand is fulfilled by those who have excess supplies in the coalition. This transfer takes place freely. So the coalition decides order quantities of its menbers. All retailers are assumed to have knowledge of all the parameter values of all participants. The purpose of this report is to model this situation as a coalitional game where the characteristic function is the expected profit, and we examine whether the core of this game is nonempty.

If all of the n retailers are identical, that is, they have the same parameters (sales price, buying cost, salvage value, and lost sales penalty), then the model reduces to the model in [3]. Game-theoretic analysis of economic-lot-size problems is analyzed in detail in [2]. In [5], the demand transfer is included in the model with 3 retailers, and fractions of excess demand are transfered to other excess supplies, and fractions are given constants.

We use the following notation. :

$N := \{1, \ldots, n\}$: The set of n retailers. We call a subset of N a coalition ;

q_i : Order quantity of Retaler i;

X_i : Random demand for Retailer i's product;

$f_i(x)$: Probability density function of X_i;

$F_i(x)$: Cumulative distribution function of X_i;

s_i : Sales price/unit for Retailer i' s product;

c_i : Buying cost/unit for Retailer i' s product;

r_i : Salvage value/unit for Retailer i' s product;

p_i : Lost sales penalty/unit for Retailer i' s product;

Π_i : Random profit for Retailer i;

For an n-tuple $\{z_i\}_{i\in N}$ and a coalition $S \subseteq N$, we let $z(S) := \sum_{i\in S} z_i$, $z_S := \{z_i\}_{i\in S}(\in R^S)$, and $\overline{z_S} := (z_S, 0_{N\setminus S})(\in R^N)$. We assume

$$r_i < c_i < s_i, \text{ for all } i \in N. \tag{1}$$

In this report, all of the random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. Also we assume $X_i \in \mathcal{L}^1(\Omega, \mathcal{F}, P)$ for all $i \in N$. The salvage value defined above can be considered as the resale value of Retailer i' s product minus its inventory carrying cost. For a realized value x_i of the random variable X_i and for a given order quantity q_i, the profit of Retailer i when he behaves by himself is

$$\Pi_i(q_i,x_i) = \begin{cases} (s_i - c_i)q_i - p_i(x_i - q_i), & \text{if } x_i \geq q_i; \\ s_i x_i - c_i q_i + r_i(q_i - x_i), & \text{if } x_i \leq q_i, \end{cases}$$

$$= (s_i - c_i)q_i - \begin{cases} p_i(x_i - q_i), & \text{if } x_i \geq q_i; \\ (s_i - r_i)(q_i - x_i), & \text{if } x_i \leq q_i, \end{cases} \tag{2}$$

$$= (s_i - c_i)q_i - max\{p_i(x_i - q_i), (s_i - r_i)(q_i - x_i)\}$$

2. Profit of a Coalition

A coalition $S \subseteq N$, for realized values $\{x_i\}_{i \in S}$, transfer demands so that the sum of the menbers' profits is maximized. For this purpose, they must solve a problem :

$$\begin{aligned} \text{Maximize} \quad & \sum_{i \in S} \Pi_i(q_i, \xi_i) \\ \text{s.t.} \quad & \xi(S) = x(S), \quad \xi_i \geq 0 \quad \forall i \in S. \end{aligned} \tag{3}$$

We let the maximum value of this problem be $\Pi_S(q_S, x_S)$ and then let $J_S(q_S) := E[\Pi_S(q_S, X_S)]$.

Proposition 1.

$$\begin{aligned} J_S(q_S) = & \sum_{i \in S}(s_i - c_i)q_i - \min_{i \in S}\{p_i\} \int_{x(S) > q(S)} [x(S) - q(S)] \Pi_{i \in S} f_i(x_i) dx_i \\ & - \min_{i \in S}\{s_i - r_i\} \int_{x(S) < q(S)} [q(S) - x(S)] \Pi_{i \in S} f_i(x_i) dx_i. \end{aligned}$$

Proof : It suffices to prove

$$\Pi_S(q_S, x_S) = \sum_{i \in S}(s_i - c_i)q_i - max\{\min_{i \in S}\{p_i\}(x(S) - q(S)), \min_{i \in S}\{s_i - r_i\}(q(S) - x(S))\}. \tag{4}$$

By (2),

$$\sum_{i \in S} \Pi_i(q_i, \xi_i) = \sum_{i \in S}(s_i - c_i)q_i - \sum_{i \in S} max\{p_i(\xi_i - q_i), (s_i - r_i)(q_i - \xi_i)\}.$$

So

$$\Pi_S(q_S, x_S) = \sum_{i \in S}(s_i - c_i)q_i - \min_{\xi} \sum_{i \in S} max\{p_i(\xi_i - q_i), (s_i - r_i)(q_i - \xi_i)\}.$$

Case (I): $x(S) \geq q(S)$.

(i) $\xi_i \geq q_i$ for all $i \in S$.

We consider $min_\xi \sum_{i \in S} p_i(\xi_i - q_i)$. This is linear in ξ, so it is minimized at an extreme point of the feasible region. For each $i \in S$, an extreme point is given by :$\xi_i = x(S) - q(S \setminus \{i\}), \xi_j = q_j$ for all $j \in S \setminus \{i\}$. Checking at extreme points, we have the minimum : $min_{i \in S}\{p_i\}(x(S) - q(S))$.

(ii) $\xi_i \le q_i$ for $i \in T(\subset S)$ and $\xi_i \ge q_i$ for $i \in S \setminus T$.
We consider

$$\sum_{i \in T}(s_i - r_i)(q_i - \xi_i) + \sum_{i \in S \setminus T} p_i(\xi_i - q_i). \tag{5}$$

Suppose the minimum is attained at ξ^* and assume $\xi_i^* < q_i$ for some $i \in T$. Since $x(S) = \xi(S) \ge q(S)$, there exists $j \in S \setminus T$ such that $\xi_j^* > q_j$. Define ξ^ϵ by $\xi_x^\epsilon = \xi_i^* + \epsilon$ for $x = i$, $= \xi_j^* - \epsilon$ for $x = j$ and $= \xi_x^*$ for $x \ne i, j$. Then the value of the function (5) decreases. This contradicts the optimality of ξ^*. Hence $\xi_i^* = q_i$ for all $i \in T$. Then the argument in (i) applies and the minimum of the function (5) is $min_{i \in S \setminus T}\{p_i\}(x(S) - q(S))$.

Consequently, the minimum is $min_{i \in S}\{p_i\}(x(S) - q(S))$.

Case (II): $x(S) \le q(S)$.
In the same way as in Case (I), the minimum is $min_{i \in S}\{s_i - r_i\}(q(S) - x(S))$.

So we get the equation (4). ∎

When his order quantuty is q_i, from Proposition 1 the expected profit of Retailer i is

$$J_i(q_i) = (s_i - c_i)q_i - (s_i - r_i)\int_0^{q_i}(q_i - x)f_i(x)dx - p_i\int_{q_i}^{\infty}(x - q_i)f_i(x)dx.$$

We let

$$v(\{i\}) = \max_{q_i \ge 0} J_i(q_i).$$

Then we have

$$\frac{dJ_i(q_i)}{dq_i} = (r_i - s_i - p_i)\int_0^{q_i} f_i(x)dx + s_i - c_i + p_i, \quad \frac{d^2J_i}{dq_i^2} = (r_i - s_i - p_i)f_i(q_i) < 0,$$

so $J_i(q_i)$ is maximized at q_i^* where

$$F_i(q_i^*) = \frac{s_i + p_i - c_i}{s_i + p_i - r_i} \quad (< 1, \text{ by (1) }).$$

So $v(\{i\}) \ge 0$ if and only if

$$\int_0^{q_i^*} xf_i(x)dx \ge \frac{p_i}{s_i + p_i - r_i}E(X_i).$$

3. Cooperative Games and the Core

In this section we derive a cooperative game and show that its core is nonempty. We define a function $v : 2^N \to R^1$ by

$$v(S) = \max_{q_S \ge 0} J_S(q_S). \tag{6}$$

For $S \subseteq N$, suppose demands $\{x_i\}_{i \in S}$ are realozed. Members in the coalition S will transfer demnds (i.e., Problem (3)) and determine order quantities (i.e., Problem (6)) cooperatively so that the sum of the members' profits is maximized. We regard the pair (N, v) as a (cooperative) game with sidepayments. The core of the game (N, v) is a solution-concept in cooperative games and it is defined by the set of n-tuples $y = (y_1, \dots, y_n)$ satisfying

$$y(S) \ge v(S), \text{ for all } S \subseteq N, \text{ and } y(N) \le v(N).$$

A balanced set $\{S_1, \ldots, S_k\}$ is a collection of subsets of N with the property that there exist positive numbers $\lambda_1, \ldots, \lambda_k$, called balancing coefficients, such that for each $i \in N$ we have

$$\sum_{j: i \in S_j} \lambda_j = 1.$$

Theorem 2. ([1],[4]). A game has a nonempty core if and only if for every balanced set $\{S_1, \ldots, S_k\}$ with balancing coefficient $\lambda_1, \ldots, \lambda_k$,

$$\sum_{j=1}^{k} \lambda_j v(S_j) \leq v(N).$$

Theorem 3. The core of the game (N, v) is not empty.
Proof : For $S \subseteq N$, we see

$$\Pi_S(aq_S, ax_S) = a\Pi_S(q_S, x_S), \forall a \geq 0, \text{ and } \Pi_S(q_S + q'_S, x_S + x'_S) \geq \Pi_S(q_S, x_S) + \Pi_S(q'_S, x'_S).$$

Let $\{S_1, \ldots, S_k\}$ be a balanced set on N and let $\lambda_1, \ldots, \lambda_k$ be balancing coefficients. By Theorem 2, it suffices to show that the inequality $v(N) \geq \sum_{\ell=1}^{k} \lambda_\ell v(S_\ell)$ holds.

$$\begin{aligned}
\sum_{\ell=1}^{k} \lambda_\ell v(S_\ell) &= \sum_{\ell=1}^{k} \lambda_\ell \max_{q_{S_\ell}} E[\Pi_{S_\ell}(q_{S_\ell}, X_{S_\ell})] \\
&= \sum_{\ell=1}^{k} \lambda_\ell E[\Pi_{S_\ell}(q^*_{S_\ell}, X_{S_\ell})] \\
&= \sum_{\ell=1}^{k} E[\Pi_{S_\ell}(\lambda_\ell q^*_{S_\ell}, \lambda_\ell X_{S_\ell})] \\
&= E[\sum_{\ell=1}^{k} \Pi_{S_\ell}(\lambda_\ell q^*_{S_\ell}, \lambda_\ell X_{S_\ell})] \\
&\leq E[\Pi_N(\sum_{\ell=1}^{k} \lambda_\ell \overline{q^*_{S_\ell}}, \sum_{\ell=1}^{k} \lambda_\ell X_{S_\ell})] \\
&\leq \max_{q_N} E[\Pi_N(q_N, X_N)] = v(N).
\end{aligned}$$

Here $\overline{q^*_{S_\ell}}$ is a vector with n components and it is the extension of the vector $q^*_{S_\ell}$ and obtained by adding zeros. ∎

4. Discussions on Models with Two Periods

In this section we propose two models (Models I and II) with two periods and discuss on their properties. The difference between two models is whether or not excess demands in the first period could be transfered to the second period. Excess supplies could be transfered in both models.

Model I : Excess demands or excess supplies in the first period could be transfered to the second period. Suppose a retailer i behaves by himself. Suppose $x_i^t, (t = 1, 2)$ is a realized value of the demand $X_i^t, (t = 1, 2)$ and $q_i^t, (t = 1, 2)$ is an order quantity at the t-th period. We assume

X_i^1 and X_i^2 are independent for all $i \in N$. For these values, we denote by $\Pi_i^t(q_i^t, x_i^t), (t = 1, 2)$ the profit at the t-th period.

$$\Pi_i^1(q_i^1, x_i^1) = \begin{cases} (s_i - c_i)q_i^1, & \text{if } x_i^1 > q_i^1; \\ s_i x_i^1 - c_i q_i^1, & \text{if } x_i^1 < q_i^1. \end{cases}$$

When $x_i^1 < q_i^1$, the initial inventory level at the 2nd period is $q_i^1 - x_i^1$. So we have

$$\Pi_i^2(q_i^2, x_i^2) = \begin{cases} -c_i q_i^2 + s_i(q_i^1 + q_i^2 - x_i^1) - p_i(x_i^1 + x_i^2 - q_i^1 - q_i^2), & \text{if } x_i^2 > q_i^1 - x_i^1 + q_i^2; \\ s_i x_i^2 - c_i q_i^2 + r_i(q_i^1 + q_i^2 - x_i^1 - x_i^2), & \text{if } x_i^2 < q_i^1 - x_i^1 + q_i^2. \end{cases}$$

When $x_i^1 > q_i^1$, the excess demand at the 2nd period is $x_i^1 - q_i^1$. So

$$\Pi_i^2(q_i^2, x_i^2) = \begin{cases} -c_i q_i^2 + s_i q_i^2 - p_i(x_i^1 + x_i^2 - q_i^1 - q_i^2), & \text{if } x_i^2 > q_i^1 - x_i^1 + q_i^2; \\ -c_i q_i^2 + s_i(x_i^1 + x_i^2 - q_i^1) + r_i(q_i^1 + q_i^2 - x_i^1 - x_i^2), & \text{if } x_i^2 < q_i^1 - x_i^1 + q_i^2. \end{cases}$$

From these,

$$\sum_{t=1}^{2} \Pi_i^t(q_i^t, x_i^t) = (s_i - c_i)(q_i^1 + q_i^2) - max\{p_i(x_i^1 + x_i^2 - q_i^1 - q_i^2), (s_i - r_i)(q_i^1 + q_i^2 - x_i^1 - x_i^2)\}. \tag{7}$$

Suppose a coalition S transfers demands at the both periods so that $\xi^1(S) = x^1(S)$ and $\xi^2(S) = x^2(S)$. By letting $q_i := q_i^1 + q_i^2$ and $\xi_i := \xi_i^1 + \xi_i^2$ for all $i \in S$, we see, from (7) and (2),

$$\sum_{t=1}^{2} \Pi_i^t(q_i^t, \xi_i^t) = \Pi_i(q_i, \xi_i).$$

Here $\xi(S) = x^1(S) + x^2(S)$. So, the analysis of this model reduces to that in the previous sections.

Model II: Excess demands could not be transfered but excess supplies in the first period could be transfered to the second period. Let y_i^t be the initial inventory level at the t-th period for the retailer i. The profit in the first period is:

$$\Pi_i^1(q_i^1, x_i^1, y^1(=0)) = \begin{cases} (s_i - c_i)q_i^1 - p_i(x_i^1 - q_i^1), & \text{if } x_i^1 > q_i^1; \\ s_i x_i^1 - c_i q_i^1, & \text{if } x_i^1 < q_i^1. \end{cases}$$

Suppose the initial inventory level is $y^2 \geq 0$ at the 2nd period. The profit in this period is :

$$\Pi_i^2(q_i^2, x_i^2, y^2) = \begin{cases} -c_i q_i^2 + s_i(y^2 + q_i^2) - p_i(x_i^2 - y^2 - q_i^2), & \text{if } x_i^2 > y^2 + q_i^2; \\ s_i x_i^2 - c_i q_i^2 + r_i(y^2 + q_i^2 - x_i^2), & \text{if } x_i^2 < y^2 + q_i^2. \end{cases}$$

When $x_i^1 < q_i^1$, we let $y^1 = 0, y^2 = q_i^1 - x_i^1$ and we have

$$\sum^{2} \Pi_i^t(q_i^t, x_i^t, y^t) = (s_i - c_i)(q_i^1 + q_i^2) - max\{p_i(x_i^1 + x_i^2 - q_i^1 - q_i^2), (s_i - r_i)(q_i^1 + q_i^2 - x_i^1 - x_i^2)\}.$$

When $x_i^1 > q_i^1$, we let $y^1 = 0, y^2 = 0$ and

$$\sum_{t=1}^{2} \Pi_i^t(q_i^t, x_i^t, y^t) = (s_i - c_i)(q_i^1 + q_i^2) - p_i(x_i^1 - q_i^1) - max\{p_i(x_i^2 - q_i^2), (s_i - r_i)(q_i^2 - x_i^2)\}.$$

The purpose of the retailer i is to determine $q_i^t, t = 1, 2$ so that $E[\sum_{t=1}^2 \Pi_i^t(q_i^t, X_i^t, y^t)]$ is maximized. From the above,

$$E[\Pi_i^2(q, X, y)] = \int_0^{q+y} \{s_i x - c_i q + r_i(q + y - x)\} f(x)dx + \int_{q+y}^{\infty} \{s_i(q + y) - c_i q - p_i(x - q - y)\} f(x)dx,$$

and so $\frac{\partial^2 E[\Pi_i^2(q,X,y)]}{\partial q^2} < 0$. Letting $\frac{\partial E[\Pi_i^2(q,X,y)]}{\partial q} = 0$, $E[\Pi_i^2(q, X, y)]$ is maximized when

$$\int_0^{q+y} f(x)dx = \frac{s_i + p_i - c_i}{s_i + p_i - r_i}.$$

In each period, demands are transfered among members in any coalition after demands are realized. This model would be analyzed in detail in the future.

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