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A positive solution of semilinear elliptic equation with G-invariant nonlinearity

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0. Introduction

In this note, we consider the following elliptic problem:

$$\begin{cases} -\Delta u+u=f(x,u) & \text{in } \mathbf{R}^N, \\ u>0 & \text{in } \mathbf{R}^N, \\ u\in H^1(\mathbf{R}^N), \end{cases} \tag{0.1}$$

where $f(x,u)$ is a superlinear and subcritical function in u. We assume that $f(x,u)$ is invariant under some finite group action G on x and we would like to show the existence of at least one positive solution of (0.1) via variational methods. More precisely we assume that $f(x,0)\equiv 0$ and $f(x,u)$ satisfies

(A0) $f(x,u)\in C(\mathbf{R}^N\times\mathbf{R},\mathbf{R})$,

(A1) there exist constants $\delta_0\in[0,1)$ and $m_0>0$ such that

$$0<f(x,u)\le\delta_0 u+m_0u^p \quad \text{for all } x\in\mathbf{R}^N \text{ and } u>0,$$

(A2) there exists a constant $\theta>2$ such that

$$0<\theta F(x,u)\le f(x,u)u \quad \text{for all } x\in\mathbf{R}^N \text{ and } u>0,$$

where $F(x,u)=\displaystyle\int_0^u f(x,\tau)\,d\tau$.

(0.1) or related problems were also studied by many authors such as **[BaYL]**, **[BaPLL]**, **[BWi1]**, **[BWi2]**, **[BWa]**, **[CR]**, **[DN]**, **[Li]**, **[PLL1]**, **[PLL2]**, **[R2]**, **[Y]** and the references therein. The main difficulty of these problems is a lack of compactness for corresponding functional and they overcome this difficulty by assuming some symmetric condition on $f(x,u)$. In particular, Bartsch-Willem [**BWi1**] assume radially symmetric condition on $f(x,u)$. If $f(x,u)$ is a radially symmetric function, then a functional corresponding to (0.1) satisfies Palais-Smale condition in a class of radially symmetric functions. Thus one can use many variational methods to show the existence of radially symmetric solutions. Bartsch-Wang [**BWa**] (c.f. Bartsch-Willem [**BWi2**]) consider the following G-invariant elliptic problem:

$$-\Delta u + b(x)u = f(x,u) \quad \text{in } \mathbf{R}^N,$$

where $b(x)$ and $f(x,u)$ are invariant under a group action G. That is, $b(gx) = b(x)$, $f(gx,u) = f(x,u)$ for all $g \in G$ and $x \in \mathbf{R}^N$. Here G is a subgroup of the orthogonal group $O(N) = \{A\,;\, N\times N \text{ matrix}, {}^tAA = I_N\}$, where I_N is an unit matrix. They assume that G is an infinite subgroup such that for all $x \in \mathbf{R}^N\setminus\{0\}$, $Gx = \{gx\,; g \in G\}$ has infinitely many elements. For such a group action G, they show that G-invariant subspace E_G of $H^1(\mathbf{R}^N)$ is compactly embedded into $L^{p+1}(\mathbf{R}^N)$, where $1 < p < \dfrac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 1,\ 2$. As to other type of group action, we refer to Coti Zelati-Rabinowitz [**CR**]. In [**CR**], they consider the case where $f(x,u)$ is periodic in each x_i and obtain infinitely many solutions modulo $\mathbf{Z}^N$ symmetries.

We are interested in a finite group action G, that is, $|G| < \infty$. We consider the existence of positive solutions of (0.1) with $f(x,u)$ symmetric with respect to a finite group action $G \subset O(N)$. For such a finite group action G, the embedding from E_G into $L^{p+1}(\mathbf{R}^N)$ is not compact any more. We assume that $f(x,u)$ has a limit $f^\infty(u) \in C^1(\mathbf{R}, \mathbf{R})$ as $|x| \to \infty$ and we regard (0.1) as a perturbation of the following autonomous problem:

$$\begin{cases} -\Delta u + u = f^\infty(u) & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases} \tag{0.2}$$

We request more precise conditions on the behavior of $f^\infty(u)$:

(H1) $f^\infty(u) > 0$ for all $u > 0$,
$\displaystyle\limsup_{u\to\infty} \frac{f^\infty(u)}{u^p} < \infty,$
for some $\eta > 0$ and $c_0 > 0$, $\dfrac{f^\infty(u)}{u^{1+\eta}} \to c_0$ as $u \downarrow 0$,

(H2) $\dfrac{f^\infty(u)}{u}$ is increasing in $u > 0$.

(H1) gives the behavior of $f^\infty(u)$ near ∞ and 0. (H2) is a kind of convexity condition of $F^\infty(u) = \displaystyle\int_0^u f^\infty(\tau)\,d\tau$, which gives a good characterization of the mountain pass critical point. See Section 1 below.

We first state a result with respect to $G = \{id, -id\}$, which is an example of $G \subset O(N)$, for simplicity. Later in Theorem 0.3, we state our existence result in the setting of more general group actions.

Theorem 0.1. (0.1) *has at least one even positive solution, if* $f(x,u)$ *satisfies* (A0)–(A2) *and*

(A3) $f(x,u) = f(-x,u)$ *for all* $x \in \mathbf{R}^N$, $u \geq 0$,

(A4) *there exists a limit function* $f^\infty(u) \in C^1(\mathbf{R}, \mathbf{R})$ *satisfying* (H1) *and* (H2) *such that*

$$f(x,u) \to f^\infty(u) \quad \text{as } |x| \to \infty$$

uniformly on any compact subset of $[0, \infty)$,

(A5) *there exists a constant* $\lambda > 2$ *such that for any* $\varepsilon > 0$ *we can find a constant* $C_\varepsilon > 0$ *which satisfies*

$$f(x,u) - f^\infty(u) \geq -e^{-\lambda|x|}(\varepsilon u + C_\varepsilon u^p) \quad \text{for all } x \in \mathbf{R}^N \text{ and } u \geq 0.$$

Remark 0.2. (i) (A3) means, in other words, $f(x,u)$ is invariant under the group action $G = \{id, -id\}$ on x.
(ii) If $f(x,u)$ satisfies (A2) and (A4), then the limit function $f^\infty(u)$ also satisfies (H2) with the same constant θ.
(iii) λ (in (A5)) corresponds to a convergent rate (from below) and $\lambda > 2$ plays an essential role in our existence result.

We remark that if $f(x,u)$ satisfies $f(x,u) \geq f^\infty(u)$ for all $x \in \mathbf{R}^N$, $u \geq 0$, then it is well-known that the mountain pass minimax value for corresponding functional is attained. (c.f. Lions [**PLL1**], [**PLL2**].) However, without any order relation between $f(x,u)$ and $f^\infty(u)$, the mountain pass minimax value is not attained in general. For example, it is not attained under condition: $f(x,u) < f^\infty(u)$ for all $x \in \mathbf{R}^N$, $u > 0$. As far as we know, without any order relation, the existence of positive solutions of (0.1) is obtained by Bahri-Li [**BaYL**] (c.f. Bahri-Lions [**BaPLL**]) just for the case $f(x,u) = a(x)u^p$ with $a(x)$ satisfying

$$a(x) > 0 \quad \text{for all } x \in \mathbf{R}^N, \tag{0.3}$$

$$a(x) \to 1 \quad \text{as } |x| \to \infty, \tag{0.4}$$

$$a(x) - 1 \geq -Ce^{-\lambda|x|} \quad \text{for all } x \in \mathbf{R}^N. \tag{0.5}$$

Their proof essentially depends on the uniqueness of positive solutions for the limit problem: $-\Delta u + u = u^p$ in $\mathbf{R}^N$ which is obtained by Kwong [**K**]. See also Chen-Lin [**CL**] for uniqueness result. We remark that Bahri-Li's solution does not correspond to the mountain pass critical point.

Theorem 0.1 can be extended to the setting of more general group actions. We assume, instead of (A3),

(A3') let $G \subset$ be a subgroup of $O(N)$ which does not have a common fixed point on $S^{N-1} = \{x \in \mathbf{R}^N ; |x| = 1\}$, that is, for any $x \in S^{N-1}$, there exists $g \in G$ such that $gx \neq x$. We assume $f(x,u)$ is invariant under the group action $G \subset O(N)$ on x, that is,

$$f(gx,u) = f(x,u) \quad \text{for all } g \in G,\ x \in \mathbf{R}^N \text{ and } u \geq 0.$$

Let card$\{\ldots\}$ denote the cardinal number of $\{\ldots\}$. Moreover, we set

$$m = \min_{x \in S^{N-1}} \operatorname{card}\{gx\,; g \in G\} (\geq 2) \tag{0.6}$$

and choose $x_0 \in S^{N-1}$ such that card$\{gx_0 ; g \in G\} = m$. We denote $\{gx_0 ; g \in G\} = \{\tilde{e}_1, \ldots, \tilde{e}_m\}$ and set $\lambda_0 = \min_{i \neq j} |\tilde{e}_i - \tilde{e}_j| \in (0\,,2]$. We assume, instead of (A5),

(A5') there exists a constant $\lambda > \lambda_0$ such that for any $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ which satisfies

$$f(x,u) - f^\infty(u) \geq -e^{-\lambda|x|}(\varepsilon u + C_\varepsilon u^p) \quad \text{for all } x \in \mathbf{R}^N \text{ and } u \geq 0.$$

Our second existence result is the following

Theorem 0.3. *Suppose $f(x,u)$ satisfies* (A0)–(A2), (A3'), (A4) *and* (A5'). *Then* (0.1) *has at least one positive solution $u \in H^1(\mathbf{R}^N)$ which is invariant under the group action G on x, that is,*

$$u(gx) = u(x) \quad \textit{for all } g \in G,\ x \in \mathbf{R}^N. \tag{0.7}$$

In our setting, by virtue of G-invariant property, we do not need the uniqueness of positive solutions for the limit problem (0.2). Moreover, we have no order relation between $f(x,u)$ and $f^\infty(u)$. Since $H^1(\mathbf{R}^N)$ is not embedded compactly into $L^{p+1}(\mathbf{R}^N)$, the mountain pass minimax value for corresponding functional may not be attained without order relation. However if we assume that $f(x,u)$ is invariant under finite effective group action G on x, then we can show that the mountain pass minimax value for functional restricted to G-invariant subspace of $H^1(\mathbf{R}^N)$ is attained without order relation.

In the following sections, we prove Theorem 0.3 by variational arguments. Since Theorem 0.1 is a special case of Theorem 0.3, we show the existence of positive solution of (0.1) in the setting of Theorem 0.3. Our paper organized as follows. In Section 1, we give a functional framework and give some known results for the limit problem. We also give a concentration-compactness lemma in our setting. Using G-invariant property, we study where Palais-Smale condition breaks down. In Section 2, we establish some energy estimate which is a key of our existence result. In Section 3, we complete a proof of Theorem 0.3. Lastly, in Section 4, we give proofs of some remaining lemmas.

1. Preliminaries

In this section, we state some known results which are important to our existence result. First of all, we give a functional framework.

1.1. Functional framework

We use notation:

$$\|u\| = \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2)\,dx\right)^{\frac{1}{2}},$$

$$\langle u, v\rangle = \int_{\mathbf{R}^N} (\nabla u \cdot \nabla v + uv)\,dx$$

for u, $v \in H^1(\mathbf{R}^N)$. The functional corresponding to (0.1) is

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x,u)\,dx : H^1(\mathbf{R}^N) \to \mathbf{R}. \tag{1.1}$$

Since we look for only positive solutions, we may assume without loss of generality that

$$f(x,u) = 0 \quad \text{for all } x \in \mathbf{R}^N \text{ and } u \le 0.$$

Then it follows from standard functional analysis and the maximum principle that the functional $I(u)$ given in (1.1) belongs to $C^1(H^1(\mathbf{R}^N), \mathbf{R})$ and nontrivial critical points of $I(u)$ are positive solutions of (0.1). See [**AT1**], Coti Zelati-Rabinowitz [**CR**] and Rabinowitz [**R1**]. We remark that $I(u)$ possesses a *mountain pass structure*, that is, $I(u)$ satisfies the following three properties:

(i) $I(0) = 0$,

(ii) there exist constants α_0, $\rho_0 > 0$ such that

$$I(u) \ge \alpha_0 > 0 \quad \text{for all } u \in H^1(\mathbf{R}^N) \text{ with } \|u\| = \rho_0,$$

(iii) $Z_0 = \{u \in H^1(\mathbf{R}^N);\ \|u\| > \rho_0 \text{ and } I(u) < 0\} \ne \emptyset$.

The proof that $I(u)$ possesses a mountain pass structure has been established in Coti Zelati-Rabinowitz [**CR**], Rabinowitz [**R1**] and [**R2**].

Moreover, we set

$$E = E_G = \{u \in H^1(\mathbf{R}^N);\ u(gx) = u(x) \text{ for all } g \in G \text{ and } x \in \mathbf{R}^N\}.$$

By the well-known principle of symmetric criticality, we see that if the restriction $I|_E(u)$ has a critical point, then it is in fact a critical point of $I(u)$ and therefore it is a positive solution of (0.1) which satisfies (0.7). See Palais [**P**]. Thus it suffices to find a critical point of $I|_E(u)$. We find a critical point of $I|_E(u)$ by the Mountain Pass Theorem. The mountain pass minimax value for $I(u)$ is not attained, however, we show the restriction $I|_E(u)$ satisfies Palais-Smale condition in a range of the mountain pass minimax level.

1.2. Some properties of the limit equation

We use concentration-compactness lemma given by Lions [**PLL1**], [**PLL2**] to study where Palais-Smale condition for $I(u)$ or $I|_E(u)$ breaks down. To classify levels of breakdown of Palais-Smale condition, the limit equation (0.2) and corresponding functional

$$I^\infty(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F^\infty(u)\,dx : H^1(\mathbf{R}^N) \to \mathbf{R}$$

play important roles. We state here some known results for (0.2). Berestycki-Lions [**BeL**] showed that (0.2) has a positive radial solution $w(x) = w(|x|) > 0$, which we call a ground-state solution, as a minimizer of the following minimization problem on the Nehari manifold:

$$\inf\{I^\infty(u)\,;\, u \in H^1(\mathbf{R}^N),\ u \not\equiv 0,\ I^{\infty\prime}(u)u = 0\} > 0.$$

$w(x)$ satisfies

$$0 < I^\infty(w) \le I^\infty(u) \quad \text{for any nontrivial solution } u \text{ of (0.2).}$$

Moreover, Gidas-Ni-Nirenberg [**GNN**] showed the exponential decay property of $w(x)$: there exist constants $a_1, a_2 > 0$ such that

$$a_1(|x|+1)^{-\frac{N-1}{2}}e^{-|x|} \le w(x) \le a_2(|x|+1)^{-\frac{N-1}{2}}e^{-|x|} \quad \text{for all } x \in \mathbf{R}^N. \tag{1.2}$$

From (H2), we can easily see that $w(x)$ is also characterized as a mountain pass critical point of $I^\infty(u)$ and it also satisfies

$$\sup_{t\ge 0} I^\infty(tw) = I^\infty(w). \tag{1.3}$$

1.3. Breakdown of Palais-Smale condition

Definition 1.1. For $c \in \mathbf{R}$ we say that $(u_n)_{n=1}^\infty \subset H^1(\mathbf{R}^N)$ is a *$(PS)_c$-sequence* for $I(u)$, if and only if $(u_n)_{n=1}^\infty$ satisfies as $n \to \infty$,

$$\begin{aligned} &I(u_n) \to c, \\ &I'(u_n) \to 0 \quad \text{in } H^{-1}(\mathbf{R}^N). \end{aligned}$$

We also say $I(u)$ satisfies *$(PS)_c$-condition* if any $(\mathrm{PS})_c$-sequence possesses a strongly convergent subsequence in $H^1(\mathbf{R}^N)$.

The following lemma provides a precise description of a behavior of $(\mathrm{PS})_c$-sequence for $I(u)$. The proof of this lemma can be given in [**PLL1**] and [**PLL2**].

Lemma 1.2. *Let $(u_n) \subset H^1(\mathbf{R}^N)$ be a $(PS)_c$-sequence for $I(u)$. Then there exists a subsequence — still denoted by (u_n) — for which the following holds: there exist a solution*

$u_0(x)$ of (0.1), an integer $k \geq 0$, for $i = 1, \dots, k$, sequences of points $(x_n^i) \subset \mathbf{R}^N$ and nontrivial solutions of $v_i(x)$ of the limit equation (0.2) satisfying

$$u_n \rightharpoonup u_0 \quad \textit{weakly in } H^1(\mathbf{R}^N),$$

$$I(u_n) \to c = I(u_0) + \sum_{i=1}^{k} I^\infty(v_i),$$

$$u_n - \left(u_0 + \sum_{i=1}^{k} v_i(x - x_n^i)\right) \to 0 \quad \textit{strongly in } H^1(\mathbf{R}^N),$$

$$|x_n^i| \to \infty, \ |x_n^i - x_n^j| \to \infty \quad \textit{for } 1 \leq i \neq j \leq k,$$

where we agree that in the case $k = 0$, the above holds without v_i and x_n^i. ∎

The following corollary is obtained from Lemma 1.2.

Corollary 1.3. *$I|_E(u)$ satisfies $(PS)_c$-condition for the level*

$$c \in (-\infty\,,\, mI^\infty(w)),$$

where m is given in (0.6) and w is a ground state solution of (0.2).

Proof. Let $(u_n) \subset E$ be a $(\mathrm{PS})_c$-sequence for $I(u)$. Then it follows from the usual concentration-compactness argument that (u_n) is bounded and if (u_n) does not have a convergent subsequence, then there exists a sequence $(x_n) \subset \mathbf{R}^N$ and $a > 0$ such that $|x_n| \to \infty$ as $n \to \infty$ and

$$\liminf_{n\to\infty} \int_{B_1(x_n)} |u_n|^2\, dx > a,$$

where $B_1(x_n) = \{x \in \mathbf{R}^N\,;\, |x - x_n| < 1\}$. Since $(u_n) \subset E$, we see that

$$\liminf_{n\to\infty} \int_{B_1(gx_n)} |u_n|^2\, dx > a \quad \text{for all } g \in G.$$

By (0.6), we can find m sequences $\{(y_n^i)\}_{i=1}^m \subset \mathbf{R}^N$ such that

$$B_1(y_n^i) \subset \bigcup_{g\in G} B_1(gx_n) \quad \text{for all } i = 1, \dots, m,$$

$$\operatorname{dist}(B_1(y_n^i)\,,\, B_1(y_n^j)) \to \infty \quad \text{as } n \to \infty \text{ for } 1 \leq i \neq j \leq m.$$

Thus it follows from Lemma 1.2 that

$$\liminf_{n\to\infty} I(u_n) \geq mI^\infty(w).$$

By the principle of symmetric criticality, we see that $(\mathrm{PS})_c$-sequences for $I|_E(u)$ are in fact $(\mathrm{PS})_c$-sequences for $I(u)$. Therefore the first level of breakdown of $(\mathrm{PS})_c$-condition for $I|_E(u)$ is $mI^\infty(w)$. ∎

2. Energy estimates

To obtain a positive solution of (0.1) through the Mountain Pass Theorem, by Corollary 1.3, we need only to show the mountain pass minimax value for $I|_E(u)$ is strictly less than $mI^\infty(w)$. That is, we find a test path which lies below $mI^\infty(w)$. The following proposition plays an important role to find a desired test path.

Proposition 2.1. *For any integer $\ell \geq 2$ and any $e_1, \dots, e_\ell \in S^{N-1}$, we suppose that there exists a constant $\lambda > \lambda_0$ such that for any $\varepsilon > 0$ we can find a constant $C_\varepsilon > 0$ which satisfies*

$$f(x,u) - f^\infty(u) \geq -e^{-\lambda|x|}(\varepsilon u + C_\varepsilon u^p) \quad \text{for all } x \in \mathbf{R}^N \text{ and } u \geq 0,$$

where $\lambda_0 = \min_{i \neq j} |e_i - e_j| \in (0, 2]$. Then there exists a constant $S_0 \geq 1$ such that

$$I(t\sum_{i=1}^{\ell} w(x - se_i)) < \ell I^\infty(w) \quad \text{for all } t \geq 0 \text{ and } s \geq S_0. \tag{2.1}$$

Remark 2.2. This type of estimate was used successfully in Bahri-Li [**BaYL**], Bahri-Lions [**BaPLL**] to obtain the existence of positive solutions of (0.1) with $f(x,u) = a(x)u^p$. They used an interaction phenomenon among $w(x - se_i)$ in a sense of Taubes [**T**]. See also [**AT1**], [**AT2**] for nonhomogeneous perturbed problem.

We remark that we may assume $\lambda \in (\lambda_0, p+1)$ without loss of generality. To give a proof of Proposition 2.1, we need some lemmas.

Lemma 2.3. *For any integer $\ell \geq 2$, $\alpha \in (\frac{1}{2}, 1)$ and $M \geq 0$, there exists a constant $\beta = \beta(\ell, \alpha, M) \geq 0$ such that*

$$F^\infty(\sum_{i=1}^{\ell} u_i) - \sum_{i=1}^{\ell} F^\infty(u_i) - \alpha \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} f^\infty(u_i)u_j + \beta \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} u_i^{\frac{2+\eta}{2}} u_j^{\frac{2+\eta}{2}} \geq 0 \tag{2.2}$$

for all $0 \leq u_1, \dots, u_\ell \leq M$, where $\eta > 0$ is given in (H1).

Lemma 2.4. *There exist constants C_1, C_2, $C_3 > 0$ such that*

$$\int_{\mathbf{R}^N} e^{-\lambda|x|} w(x - se_i)^2 \, dx \leq \begin{cases} C_1 e^{-\lambda s} & \text{if } \lambda \leq 2, \\ C_2 s^{-(N-1)} e^{-2s} & \text{if } \lambda > 2, \end{cases} \tag{2.3}$$

$$\int_{\mathbf{R}^N} e^{-\lambda|x|} w(x - se_i)^{p+1} \, dx \leq C_3 e^{-\lambda s} \tag{2.4}$$

for all $e_i \in S^{N-1}$ *and* $s \geq 1$. *Moreover, for all* $\mu \in (1, \frac{2+\eta}{2})$, *there exists a constant* $C_4 > 0$ *such that*

$$\int_{\mathbf{R}^N} w(x - se_i)^{\frac{2+\eta}{2}} w(x - se_j)^{\frac{2+\eta}{2}} \, dx \leq C_4 e^{-\mu s |e_i - e_j|} \tag{2.5}$$

for all e_i, $e_j \in S^{N-1}$ *and* $s \geq 1$.

Lemmas 2.3 and 2.4 are important to use an interaction phenomenon, but those proofs are essentially elementary. We leave proofs of Lemmas 2.3 and 2.4 for a while and we proceed the proof of Proposition 2.1. We give proofs of Lemmas 2.3 and 2.4 in last section.

Proof of Proposition 2.1. By the continuity of $I(u)$ at 0 and the fact that $I(t \sum_{i=1}^{\ell} w(x - se_i)) \to -\infty$ as $t \to \infty$ uniformly in $s \geq 1$, we can find constants $\underline{t}$, $\overline{t} > 0$ such that

$$I(t \sum_{i=1}^{\ell} w(x - se_i)) < \ell I^\infty(w) \quad \text{for all } t \in [0, \underline{t}] \cup [\overline{t}, \infty) \text{ and } s \geq 1.$$

Thus we need to find a large $S_0 \geq 1$ such that (2.1) holds for $t \in [\underline{t}, \overline{t}]$. Simple calculation yields

$$\begin{aligned}
I(t \sum_{i=1}^{\ell} w(x - se_i)) &= \frac{1}{2} \| t \sum_{i=1}^{\ell} w(x - se_i) \|^2 - \int_{\mathbf{R}^N} F(x, t \sum_{i=1}^{\ell} w(x - se_i)) \, dx \\
&= \frac{1}{2} \sum_{i=1}^{\ell} \| t w(x - se_i) \|^2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} t^2 \langle w(x - se_i), w(x - se_j) \rangle \\
&\quad - \int_{\mathbf{R}^N} F^\infty(t \sum_{i=1}^{\ell} w(x - se_i)) \, dx \\
&\quad + \int_{\mathbf{R}^N} (F^\infty(t \sum_{i=1}^{\ell} w(x - se_i)) - F(x, t \sum_{i=1}^{\ell} w(x - se_i))) \, dx \\
&= \frac{1}{2} \sum_{i=1}^{\ell} \| t w(x - se_i) \|^2 - \sum_{i=1}^{\ell} \int_{\mathbf{R}^N} F^\infty(t w(x - se_i)) \, dx \\
&\quad - \int_{\mathbf{R}^N} F^\infty(t \sum_{i=1}^{\ell} w(x - se_i)) \, dx + \sum_{i=1}^{\ell} \int_{\mathbf{R}^N} F^\infty(t w(x - se_i)) \, dx \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} t^2 \langle w(x - se_i), w(x - se_j) \rangle \\
&\quad + \int_{\mathbf{R}^N} (F^\infty(t \sum_{i=1}^{\ell} w(x - se_i)) - F(x, t \sum_{i=1}^{\ell} w(x - se_i))) \, dx.
\end{aligned}$$

Fix $\alpha \in (\frac{1}{2}, 1)$ and we put $M = \bar{t} \max_{x \in \mathbf{R}^N} w(x)$. Applying Lemma 2.3, we have

$$
\begin{aligned}
I(t\sum_{i=1}^{\ell} w(x-se_i)) &\le \ell I^\infty(tw) \\
&- \alpha \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \int_{\mathbf{R}^N} f^\infty(tw(x-se_i))tw(x-se_j)\,dx \\
&+ \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{\ell} t^2\langle w(x-se_i), w(x-se_j)\rangle \\
&+ \int_{\mathbf{R}^N} (F^\infty(t\sum_{i=1}^{\ell} w(x-se_i)) - F(x, t\sum_{i=1}^{\ell} w(x-se_i)))\,dx \\
&+ \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \int_{\mathbf{R}^N} \beta(tw(x-se_i))^{\frac{2+\eta}{2}}(tw(x-se_j))^{\frac{2+\eta}{2}}\,dx. \\
&= \ell I^\infty(tw) - (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).
\end{aligned} \tag{2.6}
$$

We estimate each term of the right hand side of (2.6) respectively to show (2.1). First of all, we estimate (III) and (IV). We have from (A5) and Lemma 2.4,

$$
\begin{aligned}
(\mathrm{III}) &= \int_{\mathbf{R}^N} (F^\infty(t\sum_{i=1}^{\ell} w(x-se_i)) - F(x, t\sum_{i=1}^{\ell} w(x-se_i)))\,dx \\
&= \int_{\mathbf{R}^N}\int_0^{t\sum_{i=1}^{\ell} w(x-se_i)} (f^\infty(\tau) - f(x,\tau))\,d\tau dx \\
&\le \int_{\mathbf{R}^N}\int_0^{t\sum_{i=1}^{\ell} w(x-se_i)} e^{-\lambda|x|}(\varepsilon\tau + C_\varepsilon \tau^p)\,dx \\
&= \frac{\varepsilon}{2}\int_{\mathbf{R}^N} e^{-\lambda|x|}\left(t\sum_{i=1}^{\ell} w(x-se_i)\right)^2 dx \\
&\qquad + \frac{C_\varepsilon}{p+1}\int_{\mathbf{R}^N} e^{-\lambda|x|}\left(t\sum_{i=1}^{\ell} w(x-se_i)\right)^{p+1} dx \\
&\le \frac{\varepsilon}{2}C\int_{\mathbf{R}^N} e^{-\lambda|x|}\sum_{i=1}^{\ell}(tw(x-se_i))^2\,dx \\
&\qquad + \frac{C_\varepsilon}{p+1}C'\int_{\mathbf{R}^N} e^{-\lambda|x|}\sum_{i=1}^{\ell}(tw(x-se_i))^{p+1}\,dx \\
&\le \varepsilon A_1 \max\{e^{-\lambda s}, s^{-(N-1)}e^{-2s}\} + C_\varepsilon A_2 e^{-\lambda s},
\end{aligned} \tag{2.7}
$$

where A_1, $A_2 > 0$ are constants independent of $\varepsilon > 0$ and $s \geq 1$. Fix $\mu \in (1, \frac{2+\eta}{2})$. We also have from (2.5)

$$(\mathrm{IV}) = \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \int_{\mathbf{R}^N} \beta(tw(x - se_i))^{\frac{2+\eta}{2}} (tw(x - se_j))^{\frac{2+\eta}{2}} \, dx \leq A_3 e^{-\mu \lambda_0 s}, \tag{2.8}$$

where $A_3 > 0$ is a constant independent of $s \geq 1$. We remark that (2.7) and (2.8) hold for all $t \in [\underline{t}, \overline{t}]$.

We treat (I) and (II) more carefully. Since $w(x)$ is a solution of (0.2), we have

$$\begin{aligned} t^2 \langle w(x - se_i), w(x - se_j) \rangle &= \int_{\mathbf{R}^N} t f^\infty(w(x - se_i)) tw(x - se_j) \, dx \\ &= \int_{\mathbf{R}^N} t f^\infty(w(x - se_j)) tw(x - se_i) \, dx. \end{aligned}$$

Thus we have

$$\begin{aligned} &- (\mathrm{I}) + (\mathrm{II}) \\ &= - \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{1}{2} \int_{\mathbf{R}^N} (2\alpha f^\infty(tw(x - se_i)) - t f^\infty(w(x - se_i))) tw(x - se_j) \, dx. \end{aligned} \tag{2.9}$$

From (H1), (H2) and $\alpha > \frac{1}{2}$, we can choose $t_1 \in (0, 1)$ and $\delta \in (0, 2\alpha - 1)$ such that

$$2\alpha f^\infty(tw(x - se_i)) - t f^\infty(w(x - se_i)) \geq \delta f^\infty(tw(x - se_i)) \tag{2.10}$$

for all $t \geq t_1$, $x \in \mathbf{R}^N$, $s \geq 1$ and $i = 1, \ldots, \ell$. Then we choose $t_1 \in (0, 1)$ and $\delta \in (0, 2\alpha - 1)$ satisfying (2.10) and fix them. We consider the following two cases: $t \in [t_1, \overline{t}]$ and $t \in [\underline{t}, t_1]$.

For $t \in [t_1, \overline{t}]$, we have from (1.2) and (2.10)

$$\begin{aligned} &\sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{1}{2} \int_{\mathbf{R}^N} (2\alpha f^\infty(tw(x - se_i)) - t f^\infty(w(x - se_i))) tw(x - se_j) \, dx \\ &\geq \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{\delta t_1}{2} \int_{\mathbf{R}^N} f^\infty(tw(x - se_i)) w(x - se_j) \, dx \\ &= \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{\delta t_1}{2} \int_{\mathbf{R}^N} f^\infty(tw(x)) w(x - s(e_j - e_i)) \, dx \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \frac{\delta t_1}{2}\int_{|x|\leq 1} f^\infty(tw(x))w(x-s(e_j-e_i))\,dx\\
&\geq \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \frac{\delta t_1 a_1}{2}\int_{|x|\leq 1} f^\infty(tw(x))(|x-s(e_j-e_i)|+1)^{-\frac{N-1}{2}}e^{-|x-s(e_j-e_i)|}\,dx\\
&\geq \sum_{\substack{i,j=1\\ i\neq j}}^{\ell} \frac{\delta t_1 a_1}{2}(s|e_j-e_i|+2)^{-\frac{N-1}{2}}e^{-s|e_j-e_i|-1}\int_{|x|\leq 1} f^\infty(tw(x))\,dx\\
&\geq A_0 s^{-\frac{N-1}{2}}e^{-\lambda_0 s}, \qquad (2.11)
\end{aligned}
$$

where $A_0 > 0$ is a constant independent of $s \geq 1$. Then taking ε small if necessary, we see that there exists a constant $S_1 \geq 1$ such that

$$
\begin{aligned}
&-A_0 s^{-\frac{N-1}{2}}e^{-\lambda_0 s} + \varepsilon A_1 \max\{e^{-\lambda s}, s^{-(N-1)}e^{-2s}\} + C_\varepsilon A_2 e^{-\lambda s} + A_3 e^{-\mu\lambda_0 s}\\
&< 0 \quad \text{for all } s \geq S_1. \qquad (2.12)
\end{aligned}
$$

Thus we have from (1.3), (2.6)–(2.12)

$$
I(t\sum_{i=1}^{\ell} w(x-se_i)) < \ell I^\infty(w) \quad \text{for all } s \geq S_1 \text{ and } t \in [t_1\,,\,\bar{t}].
$$

For $t \in [\underline{t}\,,\,t_1]$, it follows from (1.3) that

$$
I^\infty(tw) < I^\infty(w) \quad \text{for all } t \in [\underline{t}\,,\,t_1]. \qquad (2.13)
$$

On the other hand, (I) ≥ 0 is obvious. Moreover we have

$$
\begin{aligned}
\langle w(x-se_i), w(x-se_j)\rangle &= \langle w(x-s(e_i-e_j)), w(x)\rangle\\
&\to 0 \quad \text{as } s\to\infty \qquad (2.14)
\end{aligned}
$$

for all $i \neq j$. From (2.7), (2.8) and (2.14), we see that (II) + (III) + (IV) tends to 0 as $s \to \infty$ uniformly in t. Thus by (2.6) and (2.13), we find a constant $S_2 \geq 1$ such that

$$
I(t\sum_{i=1}^{\ell} w(x-se_i)) < \ell I^\infty(w) \quad \text{for all } s \geq S_2 \text{ and } t \in [\underline{t}\,,\,t_1].
$$

Finally, setting $S_0 = \max\{S_1\,,\,S_2\}$, we obtain (2.1) for this $S_0 \geq 1$. ∎

3. Proof of Theorem 0.3

Recall that $I(u)$ possesses a mountain pass structure (i)–(iii). Then we consider the following minimax value

$$b = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I|_E(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E)\,;\, \gamma(0) = 0,\ \gamma(1) \in Z_0\},$$
$$Z_0 = \{u \in E\,;\, \|u\| > \rho_0 \text{ and } I|_E(u) < 0\}.$$

Applying Proposition 2.1 with $\ell = m$ and $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$, we see that there exists a constant $S_0 \geq 1$ such that

$$I(t \sum_{i=1}^{m} w(x - s\tilde{e}_i)) < mI^\infty(w) \quad \text{for all } t \geq 0 \text{ and } s \geq S_0. \tag{3.1}$$

Since $I(t \sum_{i=1}^{m} w(x - s\tilde{e}_i)) \to -\infty$ as $t \to \infty$ uniformly in $s \geq S_0$, we choose $t_0 > 0$ such that $\|t_0 \sum_{i=1}^{m} w(x - s\tilde{e}_i)\| > \rho_0$ and $I(t_0 \sum_{i=1}^{m} w(x - s\tilde{e}_i)) < 0$. We define $\gamma_0(t)$ by

$$\gamma_0(t) = tt_0 \sum_{i=1}^{m} w(x - s\tilde{e}_i).$$

Since $|gx| = |x|$ for all $g \in G$, $x \in \mathbf{R}^N$ and w is a radially symmetric function, we see that $\gamma_0(t) \in E$ for all $t \in [0,1]$. Thus $\gamma_0(t) \in \Gamma$. Then it follows from Corollary 1.3 and (3.1) that we obtain a positive solution satisfying (0.7), which corresponds to the mountain pass minimax value b. ∎

4. Proofs of Lemmas 2.3 and 2.4

Proof of Lemma 2.3. First we prove (2.2) with $\ell = 2$, that is, we show that for any $\alpha \in (\frac{1}{2}, 1)$ and $M \geq 0$, there exists a constant $\beta \geq 0$ such that

$$F^\infty(u+h) - F^\infty(u) - F^\infty(h) - \alpha f^\infty(u)h - \alpha f^\infty(h)u + \beta u^{\frac{2+\eta}{2}} h^{\frac{2+\eta}{2}} \geq 0 \tag{4.1}$$

for all $0 \leq h,\ u \leq M$. If $h = 0$ or $u = 0$, obviously (4.1) holds. Otherwise we assume, without loss of generality, that $0 < h \leq u \leq M$. It is easy to see that for $\alpha \in (\frac{1}{2}, 1)$,

$$\begin{aligned}
&F^\infty(u+h) - F^\infty(u) - F^\infty(h) - \alpha f^\infty(u)h - \alpha f^\infty(h)u + \beta u^{\frac{2+\eta}{2}} h^{\frac{2+\eta}{2}} \\
&= F^\infty(u+h) - F^\infty(u) - F^\infty(h) - f^\infty(u)h \\
&\quad + (1-\alpha) f^\infty(u)h - \alpha f^\infty(h)u + \beta u^{\frac{2+\eta}{2}} h^{\frac{2+\eta}{2}} \\
&= F^\infty(u+h) - F^\infty(u) - F^\infty(h) - f^\infty(u)h \\
&\quad + \left((1-\alpha)\frac{f^\infty(u)}{u} - \alpha \frac{f^\infty(h)}{h}\right) hu + \beta u^{\frac{2+\eta}{2}} h^{\frac{2+\eta}{2}}.
\end{aligned}$$

From (H2), we see that

$$\begin{aligned}
&F^\infty(u+h) - F^\infty(u) - F^\infty(h) - f^\infty(u)h \\
&\quad = \int_0^h (f^\infty(u+\tau) - f^\infty(\tau) - f^\infty(u))\, d\tau \\
&\quad = \int_0^h \left(\frac{f^\infty(u+\tau)}{u+\tau}(u+\tau) - \frac{f^\infty(\tau)}{\tau}\tau - \frac{f^\infty(u)}{u}u \right) d\tau \\
&\quad = \int_0^h \left(\frac{f^\infty(u+\tau)}{u+\tau} - \frac{f^\infty(\tau)}{\tau} \right) \tau\, d\tau + \int_0^h \left(\frac{f^\infty(u+\tau)}{u+\tau} - \frac{f^\infty(u)}{u} \right) u\, d\tau \\
&\quad \geq 0
\end{aligned}$$

for all $0 < h \leq u$. Thus if

$$(1-\alpha)\frac{f^\infty(u)}{u} \geq \alpha \frac{f^\infty(h)}{h},$$

(4.1) hold for any $\beta \geq 0$. The remaining case is

$$(1-\alpha)\frac{f^\infty(u)}{u} \leq \alpha \frac{f^\infty(h)}{h}.$$

It follows from (H1) that there exist constants $0 < c_1 \leq c_2$ such that

$$c_1 u^{1+\eta} \leq f^\infty(u) \leq c_2 u^{1+\eta} \quad \text{for } 0 < u \leq M.$$

Thus in this case we have $c_1(1-\alpha)u^\eta \leq c_2 \alpha h^\eta$, that is,

$$\left(\frac{c_1(1-\alpha)}{c_2\alpha} \right)^{\frac{1}{\eta}} \leq \frac{h}{u}.$$

Then

$$\begin{aligned}
-\alpha f^\infty(h)u + \beta u^{\frac{2+\eta}{2}} h^{\frac{2+\eta}{2}} &= u^{2+\eta} \left(-\alpha \frac{f^\infty(h)}{u^{2+\eta}} u + \beta \left(\frac{h}{u} \right)^{\frac{2+\eta}{2}} \right) \\
&\geq u^{2+\eta} \left(-\alpha \frac{f^\infty(h)}{h^{1+\eta}} + \beta \left(\frac{c_1(1-\alpha)}{c_2\alpha} \right)^{\frac{2+\eta}{2\eta}} \right) \\
&\geq 0
\end{aligned}$$

for $\beta \geq 0$ large enough.

Next we use induction argument to prove Lemma 2.3. We put $U_{\ell-1} = u_1 + \cdots + u_{\ell-1}$. By (4.1), we have for any $\alpha \in (\frac{1}{2}, 1)$, there exists a constant $\beta \geq 0$ such that

$$\begin{aligned}
&F^\infty(U_{\ell-1} + u_\ell) - F^\infty(U_{\ell-1}) - F^\infty(u_\ell) \\
&\quad - \alpha f^\infty(U_{\ell-1})u_\ell - \alpha f^\infty(u_\ell)U_{\ell-1} + \beta U_{\ell-1}^{\frac{2+\eta}{2}} u_\ell^{\frac{2+\eta}{2}} \geq 0.
\end{aligned} \tag{4.2}$$

It follows from the hypothesis of induction that for any $\alpha \in (\frac{1}{2}, 1)$, there exists a constant $\beta' \geq 0$ such that

$$F^\infty(U_{\ell-1}) - \sum_{i=1}^{\ell-1} F^\infty(u_i) - \alpha \sum_{\substack{i,j=1\\ i\neq j}}^{\ell-1} f^\infty(u_i)u_j + \beta' \sum_{\substack{i,j=1\\ i\neq j}}^{\ell-1} u_i^{\frac{2+\eta}{2}} u_j^{\frac{2+\eta}{2}} \geq 0. \tag{4.3}$$

By (H2), we have

$$f^\infty(U_{\ell-1}) - \sum_{i=1}^{\ell-1} f^\infty(u_i) = \sum_{i=1}^{\ell-1} \left(\frac{f^\infty(U_{\ell-1})}{U_{\ell-1}} - \frac{f^\infty(u_i)}{u_i} \right) u_i \geq 0. \tag{4.4}$$

We also see that there exists a constant $C \geq 1$ such that

$$U_{\ell-1}^{\frac{2+\eta}{2}} \leq C(u_1^{\frac{2+\eta}{2}} + \cdots + u_{\ell-1}^{\frac{2+\eta}{2}}). \tag{4.5}$$

From (4.2)–(4.5), putting $\beta'' = \max\{\beta', C\beta\}$, we have Lemma 2.3 for this β''. ∎

Remark 4.1. If $f(x,u) = a(x)u^p$ with $a(x)$ satisfying (0.3)–(0.5), then $f^\infty(u) = u^p$ and there exists a constant $\beta \geq 0$ such that Lemma 2.3 (with $\eta = p-1$) holds for $\alpha = 1$ and any h, $u \geq 0$. See Bahri-Li [**BaYL**], Bahri-Lions [**BaPLL**].

Proof of Lemma 2.4. In what follows, we denote various positive constants independent of e_i, $e_j \in S^{N-1}$ and $s \geq 1$ by C. We first show (2.5). From (1.2), we see that

$$w(x)^{\frac{2+\eta}{2}} \leq Ce^{-\mu|x|} \quad \text{for all } x \in \mathbf{R}^N,$$
$$\int_{\mathbf{R}^N} e^{\mu|x|} w(x)^{\frac{2+\eta}{2}}\, dx < \infty.$$

Thus we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} w(x-se_i)^{\frac{2+\eta}{2}} w(x-se_j)^{\frac{2+\eta}{2}}\, dx \\
&\leq C \int_{\mathbf{R}^N} e^{-\mu|x-se_i|} e^{-\mu|x-se_j|} e^{\mu|x-se_j|} w(x-se_j)^{\frac{2+\eta}{2}}\, dx \\
&= C \int_{\mathbf{R}^N} e^{-\mu|x-s(e_i-e_j)|} e^{-\mu|x|} e^{\mu|x|} w(x)^{\frac{2+\eta}{2}}\, dx \\
&\leq C \max_{x\in\mathbf{R}^N} e^{-\mu(|x-s(e_i-e_j)|+|x|)} \int_{\mathbf{R}^N} e^{\mu|x|} w(x)^{\frac{2+\eta}{2}}\, dx \\
&\leq Ce^{-\mu s|e_i-e_j|}
\end{aligned}$$

and we obtain (2.5). Next we show (2.4). It follows from (1.2) again that

$$w(x)^{p+1} \leq Ce^{-\lambda|x|} \quad \text{for all } x \in \mathbf{R}^N,$$
$$\int_{\mathbf{R}^N} e^{\lambda|x|} w(x)^{p+1}\, dx < \infty.$$

Thus in the same way as (2.5), we obtain (2.4). If $\lambda \leq 2$, then (2.3) is also obtained similarly. If $\lambda > 2$, we obtain (2.3) by the Lebesgue dominated convergent theorem. From (1.2), we have

$$\begin{aligned}
&\int_{\mathbf{R}^N} e^{-\lambda|x|} w(x - se_i)^2\, dx \\
&\quad \leq C \int_{\mathbf{R}^N} e^{-\lambda|x|} (|x - se_i| + 1)^{-(N-1)} e^{-2|x-se_i|}\, dx \\
&\quad = C \int_{\mathbf{R}^N} e^{-(\lambda-2)|x|} (|x - se_i| + 1)^{-(N-1)} e^{-2(|x-se_i|+|x|)}\, dx \\
&\quad \leq C s^{-(N-1)} e^{-2s} \int_{\mathbf{R}^N} e^{-(\lambda-2)|x|} \left(\frac{s}{|x - se_i| + 1}\right)^{N-1} dx.
\end{aligned}$$

We observe that

$$e^{-(\lambda-2)|x|} \left(\frac{s}{|x - se_i| + 1}\right)^{N-1} \to e^{-(\lambda-2)|x|} \text{ as } s \to \infty \text{ for all } x \in \mathbf{R}^N.$$

For $|x| \leq \frac{s}{2}$,

$$\begin{aligned}
e^{-(\lambda-2)|x|} \left(\frac{s}{|x - se_i| + 1}\right)^{N-1} &\leq e^{-(\lambda-2)|x|} \left(\frac{s}{\frac{s}{2} + 1}\right)^{N-1} \\
&\leq 2^{N-1} e^{-(\lambda-2)|x|}.
\end{aligned}$$

For $|x| \geq \frac{s}{2}$,

$$\begin{aligned}
e^{-(\lambda-2)|x|} \left(\frac{s}{|x - se_i| + 1}\right)^{N-1} &\leq e^{-(\lambda-2)|x|} s^{N-1} \\
&\leq 2^{N-1} e^{-(\lambda-2)|x|} |x|^{N-1}.
\end{aligned}$$

Thus

$$e^{-(\lambda-2)|x|} \left(\frac{s}{|x - se_i| + 1}\right)^{N-1} \leq 2^{N-1} e^{-(\lambda-2)|x|} \max\{1, |x|^{N-1}\} \in L^1(\mathbf{R}^N).$$

Therefore we can apply the Lebesgue dominated convergence theorem and we obtain

$$\int_{\mathbf{R}^N} e^{-\lambda|x|} w(x - se_i)^2\, dx \leq C s^{-(N-1)} e^{-2s} \left(\int_{\mathbf{R}^N} e^{-(\lambda-2)|x|}\, dx + o(1) \right)$$

as $s \to \infty$. Thus we obtain (2.3). ∎

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