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Diophantine approximation of values of Gauß' hypergeometric function

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Abstract

In this article we present Diophantine Approximations concerning values of Gauß' hypergeometric function. Our estimate relies on a natural application of the method of Chudnovsky used in [2] for the quantitative theory of linear forms in elliptic logarithms. We regard Gauß' hypergeometric function as a "logarithmic function".

Keywords: Gauß' hypergeometric function, Transcendence measure, Diophantine Approximation, Linear forms in logarithms.

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1 Introduction

Let a, b, c be rational numbers where c is not zero neither a negative integer. Denote $(a)_k := a(a+1)\cdots(a+k-1)$. For $|z| < 1$, let us consider

$$F(z) := F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{2c(c+1)}z^2 + \cdots,$$

the classical hypergeometric function of Gauß.

This function satisfies the linear differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (c-(a+b+1)z)\frac{dF}{dz} - abF = 0.$$

Let us write also Gauß' continued fraction

$$G(z) := G(a,b,c;z) = \frac{F(a,b+1,c+1;z)}{F(a,b,c;z)} = \frac{1|}{|1} - \frac{g_1 z|}{|1} - \frac{g_2 z|}{|1} \cdots,$$

where $g_{2n-1} = \dfrac{(a+n-1)(c-b+n-1)}{(c+2n-2)(c+2n-1)}$, $g_{2n} = \dfrac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}$ for $n \in \mathbf{Z}, n \geq 1$.

Arithmetical property of the values of these functions is closely related to the monodromy group of the differential equation of $F(z)$. It is known that the singularities of the differential equation are $0, 1$ and ∞, so the fundamental group is that of the projective line $\mathbf{P}^1$ where these points are removed.

In 1985, J. Wolfart investigated when the function $F(z)$ had algebraic values. This result relied on Wüstholz' transcendence theorem concerning with abelian integrals [9], since we have the integral representation

$$F(a,b,c;z) \int_0^1 \omega(x,0) = \int_0^1 \omega(x,z)$$

where $\omega(x,z)$ denotes the rational differential form by $x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$.

F. Beukers and Wolfart gave in 1988 a condition for the algebraicity of the value of $F(z)$, and Wolfart also showed a condition for the algebraicity of the value of $G(z)$ (*see* [1]).

Let us recall Wolfart's criterion in 1985 :

Theorem (Wolfart)

Let $F(z)$ be Gauss' hypergeometric function defined as above. We have the following properties:

(i). *Suppose that $F(z)$ is algebraic over $\mathbf{C}(\mathbf{z})$, then $F(\xi) \in \overline{\mathbf{Q}}$ if $\xi \in \overline{\mathbf{Q}}$.*

(ii). *Suppose that the monodromy group of $F(z)$ is an arithmetic hyperbolic triangle group and that $c < 1, 0 < a < c, 0 < b < c, 1-c+|a-b|+|c-a-b| < 1$. Then there is a subset E of algebraic numbers which is dence in $\mathbf{C}$ such that $F(\xi) \in \overline{\mathbf{Q}}$ whenever $\xi \in E$.*

(iii). *In the case neither (i) nor (ii), there are only finitely many $\xi \in \overline{\mathbf{Q}}$ such that $F(\xi) \in \overline{\mathbf{Q}}$.*

If the monodromy group is finite, then $F(z)$ is an algebraic function from well-known Schwarz' list and there are 85 arithmetic triangle groups from Takeuchi's list. Then the triples (a,b,c) for the cases (i) (ii) of Wolfart's theorem above are determined. The statement (ii) says that the function $F(z)$, which is a typical example of G-function, allows algebraic values at algebraic points infinitely many often. Namely the analogy of Shidlovsky's theorem on E-function [6] [7] does not hold for G-function.

Now consider the function $G(z)$.

Theorem (Wolfart)

Let $G(z)$ be Gauss' continued fraction defined as above. If $G(z)$ is not algebraic over $\mathbf{C}(\mathbf{z})$, then $G(\xi)$ is transcendental at almost all $\xi \in \overline{\mathbf{Q}}$.

Indeed, we have the following property:

Suppose that none of $a, b, a-c, b-c$ is rational integer. Then almost all $\xi \in \overline{\mathbf{Q}} \Longleftrightarrow \xi \in \overline{\mathbf{Q}} - \{0, 1\}$.

Assume that none of $a, b, a-c, b-c$ is rational integer. We note that it is also explicitely known under which condition $G(z)$ is an algebraic function; both $F(a, b, c; z)$ and $F(a, b+1, c+1; z)$ are periods of the same abelian variety up to a normalization factor, and their monodromy groups are isomorphic. So

$F(a, b, c; z)$ is algebraic function of $z \Longleftrightarrow$ the monodromy group is finite $\Longleftrightarrow F(a, b+1, c+1; z)$ is algebraic function of z.

Our aim is to give Diophantine approximation of values of $F(z)$ and $G(z)$ whenever these values are transcendental.

We even remark that the usual assumptions $|z| < 1$ for $F(z)$ and z is not a real≥ 1 for $G(z)$ are not so important to state the algebraicity or the transcendence since the integral representations of $F(z)$ and $G(z)$ are valid for any analytic continuation ; the integration path has to be moved only. However, we need to restrict us to the case $|\xi| < 1$ to state our result.

2 Our statement

We assume that none of $a, b, c, a-c, b-c$ is rational integer.

Theorem 2.1 *Let $B \geq e$, $D \in \mathbf{Z}, D \geq 1$. Let $\beta \in \overline{\mathbf{Q}}$ with $h(\beta) \leq \log B, [\mathbf{Q}(\beta) : \mathbf{Q}] \leq D$. Then for any $\xi \in \overline{\mathbf{Q}}, \xi \neq 0, |\xi| < 1$, we have*

(i). *either $F(a, b, c : \xi) = \beta$ or*

(ii). *there exists $C_1 > 0$ an effectively calculable constant depending only on $F(\xi)$ and D, independent of B, satisfying*

$$|F(a, b, c : \xi) - \beta| > B^{-C_1}.$$

Corollary 2.1 *When the assumption of Theorem is verified, the value* $F(a,b,c$ $\xi)$ *is not a Liouville number.*

Concerning $G(z)$, we have also the following. Now assume that none of $a, b, a-c, b-c$ is rational integer.

Theorem 2.2 *Let* $B \geq e$, $D \in \mathbf{Z}, D \geq 1$. *Let* $\beta \in \overline{\mathbf{Q}}$ *with* $h(\beta) \leq \log B, [\mathbf{Q}(\beta) : \mathbf{Q}] \leq D$. *Then for any* $\xi \in \overline{\mathbf{Q}}, \xi \neq 0, |\xi| < 1$, *we have*

(i). *either* $G(a,b,c : \xi) = \beta$ *or*

(ii). *there exists* $C_2 > 0$ *an effectively calculable constant depending only on* $G(\xi)$ *and* D , *independent of* B, *satisfying*

$$|G(a,b,c : \xi) - \beta| > B^{-C_2}.$$

Corollary 2.2 *When the assumption of Theorem is verified, the value* $G(a,b,c$ $\xi)$ *is not a Liouville number.*

Indeed, these results correspond to a refinement of our previous estimate [5].

Idea of the proof

The proof uses then integral representation :

$$F(a,b,c:z) = \frac{1}{B(b,c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx \quad (z \neq 0, |z| < 1).$$

Consider the curve defined by

$$y^N = x^A(1-x)^B(1-zx)^C$$

where $N =$ l.c.m. of (denominator of a, denominator of b, denominator of c), $A = (1-b)N, B = (b+1-c)N, C = aN$. Thus as before, we see

$$\frac{dx}{y} = x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

and $F(a,b,c : \xi)$ is regarded as the quatient of two periods; the one is a quasi-period of an abelian variety defined over $\overline{\mathbf{Q}}$ of dimension $\phi(N)$, the other is a quasi-period of an abelian variety defined over $\overline{\mathbf{Q}}$ of dimension $\frac{\phi(N)}{2}$. Similarly, $G(a,b,c : \xi)$ is regarded as the quatient of two quasi-periods of abelian varieties defined over $\overline{\mathbf{Q}}$ of dimension $\phi(N)$ for both. The idea viewing our values as quotients of periods is already used in the articles of

Wolfart. Our contribution is regarding the hypergeometric function as a logarithmic function of such abelian varieties, namely a local inverse map of the exponential maps of algebraic groups. The usefulness of the logarithmic function is showed in our previous work [2] [3]; logarithmic function has the Taylor expansion where the height of the n-th term denominator is not too increasing, namely of order n. However the exponential function has the Taylor expansion where the height of the n-th term denominator is of order $n!$. The hypergeometric function $F(z)$ has Taylor expansion whose n-th coefficient f_n verifies the following: there exists a constant $C > 1$ such that l.c.m. of (denominators of f_i) for $i = 1, \cdots, n$ is bounded by C^n, which is nothing but the fact that it is G-function.

Remark. Recently, E. Gaudron obtained a generalization of [2] which is valid for quasi-period of abelian varieties. From this we may also get the same type of approximation.

References

[1] F. Beukers & G. Wolfart, Algebraic Values of hypergeometric functions, in New advances in Transcendence Theory (ed. A. Baker), Cambridge Univ. Press, 68–81, 1988.

[2] S. David & N. Hirata-Kohno, Recent progress on linear forms in elliptic logarithms, in A Panorama in Number Theory (ed. G. Wüstholz) Cambridge University Press, 26–37, 2002.

[3] S. David and N. Hirata-Kohno Linear forms in elliptic logarithms, preprint.

[4] N. Hirata-Kohno, Formes linéaires de logarithmes de points algébriques sur les groupes algébriques, Invent. Math. 104, 401–433, 1991.

[5] N. Hirata-Kohno, Nouvelles mesures de transcendance liees aux groupes algebriques commutatifs, Approximations Diophantiennes et Nombres Transcendants, ed. P. Philippon, Walter de Gruyter, 165-172, 1992.

[6] A. N. Parshin & I. R. Schfarevich (eds.), N. I. Fel'dman & Yu. V. Nesterenko (authors), Number Theory IV, Encyclopaedia of Mathematical Sciences Vol. 44, 1998.

[7] A. B. Shidlovski, Transcendental Numbers, Walter de Gruyter, 1989.

[8] M. Waldschmidt, Diophantine Approximation on Linear Algebraic Groups, Grundlehren der Math. Wissenschften 326, Springer, 2000.

[9] G. Wüstholz, Algebraische Punkte auf analytischen Untergruppen algebraischer Gruppen, Ann. Math. 129, 501–517, 1989.

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