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Interfaces in Activator-Inhibitor Systems
- Asymptotics and Degeneracy -

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1. Activator-Inhibitor System

A system of reaction-diffusion equations

$$\frac{\partial u}{\partial t} = d_1\Delta u + f(u,v), \qquad \frac{\partial v}{\partial t} = d_2\Delta v + g(u,v),$$

is called an activator-inhibitor system when the reaction terms (f, g) satisfy

(A-I) (i) $f_u > 0$, (ii) $f_v < 0$, (iii) $g_u > 0$, (iv) $g_v < 0$

on some region in (u, v)-plane. In such a case, u is called an activator and v an inhibitor. As long as the conditions in (A-I) are valid, u has self-activation and v-enhancing effects, while the increase in v tends to inhibit the production of both u and v itself. A typical example is:

(FH-N) $f(u,v) = u - u^3 - v, \qquad g(u,v) = u - \beta v \quad (\beta > 0)$

for which conditions (A-I)-(ii), (ii), (iv) are satisfied for all $(u, v) \in \mathbb{R}^2$, while the condition (A-I)-(i) is valid only when $-1/\sqrt{3} < u < 1/\sqrt{3}$. Another example is

(CAM) $f(u,v) = (1-u^2)(u - \tanh v), \qquad g(u,v) = u - \beta v \quad (\beta > 0).$

For f in (CAM), the condition (A-I)-(ii) is valid only for $-1 < u < 1$. This is a significant difference from f in (FH-N), which will turn out to be important later.

For f in (CAM), we define $h^{\pm}(v) \equiv \pm 1$ and $h^0(v) \equiv \tanh v$. Similarly for f in (FH-N), $h^{\pm}(v)$ and $h^0(v)$ are three roots of $u - u^3 = v$ (for $|v| < 2\sqrt{3}/9$) satisfying

$$h^-(v) < h^0(v) < h^+(v).$$

We will deal in this article a situation where the activator u diffuses slowly and reacts fast, compared with the inhibitor v. Namely, we consider the following system

$$(1.1) \qquad \begin{cases} \varepsilon u_t = \varepsilon^2 \Delta u + f(u,v), & \\ & x \in \Omega \subset \mathbb{R}^N \quad (N \geq 2) \quad t > 0 \\ v_t = D\Delta v + g(u,v), & \\ 0 = \partial u/\partial \mathbf{n} = \partial v/\partial \mathbf{n} \quad x \in \partial\Omega \quad t > 0, & \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\mathbf{n}$ the outward unit vector on $\partial\Omega$, and $\varepsilon \geq 0$ is a small parameter (called a *layer parameter*).

We first look at the equation for u in (1.1) on the entire one-dimensional space, with v frozen so that the functions $h^{\pm}(v)$ are defined. This problem has a special type of solution $u(t,x) = Q((x-ct)/\varepsilon) = Q(z)$, called a travelling wave solution which satisfies

$$\text{(TW)}\qquad \frac{d^2Q}{dz^2} + c\frac{dQ}{dz} + f(Q,v) = 0, \quad z \in \mathbb{R}, \quad \lim_{z\to\pm\infty} Q(z) = h^{\pm}(v), \quad Q(0) = 0.$$

This problem has a unique solution pair $(Q(z;v), c(v))$ for each v chosen appropriately.

2. Transition Layer and Interface

When the layer parameter $\varepsilon > 0$ is small, the solution $(u(t,x), v(t,x))$ of (1.1) with appropriate initial conditions will develop a transition layer in its u-component, i.e., $u(t,x)$ has the following behavior;

$$u(t,x) \approx h^{\pm}(v(t,x)), \quad x \in \Omega^{\pm}(t)\backslash\Gamma(t)^{-\varepsilon\log\varepsilon},$$

where

$$\Gamma(t) = \{x \in \Omega \mid u(t,x) = 0\}$$

is called an interface,

$$\Omega^{\pm}(t) = \{x \in \Omega \mid \pm u(t,x) > 0\}$$

bulk regions, and $\Gamma(t)^{\delta}$ ($\delta > 0$) stands for the δ-neighborhood of the interface. Since $u(t,x)$ makes a sharp transition from $u \approx h^-(v)$ to $u \approx h^+(v)$ across $\Gamma(t)$ within a narrow region $\Gamma(t)^{-\varepsilon\log\varepsilon}$, $u(t,x)$ is said to be a transition layer solution. This transition layer structure is known to persists during an extended period of time. To keep track of the transition layer it suffices to describe the normal speed of the interface $\Gamma(t)$. Let ν be the unit normal vector on $\Gamma(t)$ pointing into the '+'-bulk region $\Omega^+(t)$, and $\mathbf{v}(x;\Gamma(t))$ the normal speed in ν-direction. Since we have identified the interface as the 0-level set of $u(t,x)$, differentiating $u(\Gamma(t),t) \equiv 0$ with respect to t, we obtain

$$0 = u_t + (\nabla_\nu u)\mathbf{v} = \frac{1}{\varepsilon}\left\{\varepsilon u_t + (\nabla_{\tilde{\nu}} u)\mathbf{v}\right\},$$

where $\nu = \varepsilon\tilde{\nu}$. Using the equation for u and the expression of the Laplacian near $\Gamma(t)$;

$$\Delta \approx \frac{1}{\varepsilon^2}\nabla_{\tilde{\nu}}^2 + \frac{\kappa}{\varepsilon}\nabla_{\tilde{\nu}},$$

where $\kappa = \kappa(x;\Gamma(t))$ is the sum of principal curvatures of the interface at $x \in \Gamma$, we obtain

$$\begin{aligned} 0 =& \varepsilon\Delta u + (\nabla_{\tilde{\nu}} u)\mathbf{v} + f(u,v) \\ =& \nabla_{\tilde{\nu}}^2 u + (\mathbf{v} + \varepsilon\kappa)\nabla_{\tilde{\nu}} u + f(u,v). \end{aligned}$$

Comparing the last equation with that in (TW), we arrive at an *interface equation*

$$\text{(2.1)}\qquad \mathbf{v}(x;\Gamma(t)) = c(v(t,x)) - \varepsilon\kappa(x;\Gamma(t)), \quad (x \in \Gamma(t),\ t > 0). \quad \Gamma(0) = \Gamma_0.$$

Although the derivation above is rather formal, it can be made a little more rigorous[1] thanks to matched asymptotic expansions. By using such expansions, we find that $v(t,x)$ is a solution of the following problem defined in the bulk regions $\Omega^{\pm}(t)$.

$$(2.2)\qquad \begin{cases} \text{(i)} \quad v_t = D\Delta v + g^*(v,x;\Gamma(t)), \quad x \in \Omega\backslash\Gamma(t),\ t>0, \\ \text{(ii)} \quad \partial v(t,x)/\partial \mathbf{n} = 0, \quad x \in \partial\Omega, \qquad v(0,x) = \psi(x), \quad x \in \Omega \\ \text{(iii)} \quad v(t,\cdot) \in C^1(\overline{\Omega}) \cap C^2(\Omega\backslash\Gamma(t)), \quad t>0, \end{cases}$$

where g^* is defined by

$$g^*(v,x;\Gamma(t)) = g(h^{\pm}(v),v), \qquad x \in \Omega^{\pm}(t).$$

We call (2.1)-(2.2) the interface equation $(\mathrm{IFE})_\varepsilon$ for (1.1). When the curvature term $-\varepsilon\kappa$ is neglected in (2.1), we represent the interface equation by $(\mathrm{IFE})_0$.

We now summarize known results on the existence and uniqueness of solutions for $(\mathrm{IFE})_\varepsilon$.

Theorem 2.1 (Classical Solution [2]). *Let $\Gamma_0 \subset \Omega$ be of class $C^{2+\alpha}$ and let ψ be of class $C^{1+\alpha}$ for some $\alpha \in (0,1)$. Then there exists a classical solution pair $(\Gamma(t), v(t,x))$ of $(\mathrm{IFE})_\varepsilon$ $(\varepsilon > 0)$ on a time interval $[0,T]$. To be more precise, let $\gamma(t,\cdot) : \Gamma_0 \to \Omega$ be a representation of $\Gamma(t)$. Then there exists a $\beta \in (0,\alpha)$ such that*

$$\gamma \in C^{1+\beta/2,2+\beta}([0,T]\times\Gamma_0), \quad v \in C^{1+\beta/2,2+\beta}\left([0,T]\times\Omega\backslash(\cup_{0\le t\le T}\{t\}\times\Gamma(t))\right).$$

Theorem 2.2 (Semi-Classical Solution [1]). *Let $\psi \in C^2(\overline{\Omega})$ and Γ_0 be of C^2 class. Then there exists a positive constant $T > 0$ so that $(\mathrm{IFE})_0$ has a unique solution on the time interval $[0,T]$ satisfying*

$$\gamma \in W^{2,2}_\infty([0,T]\times\Omega), \qquad v \in W^{1,2}_\infty([0,T]\times\Gamma_0).$$

Theorem 2.3 (Weak Solution [5]). *Let $\psi \in C^2(\overline{\Omega})$ and Γ_0 be of C^0 class. Then for each $T>0$, $(\mathrm{IFE})_\varepsilon$ $(\varepsilon \ge 0)$ has a solution on $[0,T]$ with*

$$\gamma \in C^0 \text{ (viscosity solution)}, \qquad v \in C([0,T]\times\overline{\Omega}), \quad \nabla_x v \in C([0,T]\times\overline{\Omega}).$$

It is not, in general, expected to have a global-in-time solution of $(\mathrm{IFE})_\varepsilon$ $(\varepsilon \ge 0)$. This is why the weak (viscosity) solutions as in Theorem 2.3 are important. Our next interest is how well the interface equation $(\mathrm{IFE})_\varepsilon$ approximates the reaction-diffusion system (1.1).

3. Convergence and Asymptotics

When we have a solution (Γ, v) of $(\mathrm{IFE})_\varepsilon$, a solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) is said to *converge* to (Γ, v) if the following are valid;

$$\lim_{\varepsilon\to 0} v^\varepsilon(t,x) = v(t,x) \text{ uniformly on } [0,T]\times\overline{\Omega},$$
$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = h^{\pm}(v(t,x)) \text{ uniformly on } \Omega_T^{\pm}\backslash\Gamma_T^{\delta} \text{ for each } \delta > 0,$$

[1]This does not mean that the matched asymptotic expansion method justifies the interface equation in a mathematically precise sense.

$$\Omega_T^{\pm} = \{(t,x) \mid t \in [0,T],\ \ x \in \Omega^{\pm}(t)\},$$
$$\Gamma_T = \{(t,x) \mid t \in [0,T],\ \ x \in \Gamma(t)\},$$
$$\Gamma_T^{\delta} = \{(t,x) \mid t \in [0,T],\ \ x \in \Gamma(t)^{\delta}\}.$$

A convergence result for (1.1) was first given by Chen [1] when *the nonlinearity (f,g) is of (FH-N) type.*

Theorem 3.1 ([1]). *Let (Γ, v) be a solution of* $(\mathrm{IFE})_0$ *on a time interval $[0,T]$, in the sense of Theorem 2.2. Then there exists a solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) that converges to (Γ, v). More precisely, there exists a constant $M > 0$, independent of $\varepsilon > 0$, such that*

$$\sup\{|v^\varepsilon(t,x) - v(t,x)|\ ;\ x \in \overline{\Omega}\} \le M\varepsilon \log\frac{1}{\varepsilon},$$
$$\sup\{|u^\varepsilon(t,x) - u(t,x)|\ ;\ x \in \overline{\Omega}\backslash\Gamma(t)^{M\varepsilon\log\frac{1}{\varepsilon}}\} \le M\varepsilon \log\frac{1}{\varepsilon}$$

unifromly on $t \in [0,T]$, where $u(t,x) = h^{\pm}(v(t,x))$ for $x \in \Omega^{\pm}(t)$.

Extending Chen's method of proof [1], Soravia and Souganidis [11][2] was able to prove a *global-in-time* convergence result for nonlinearities of (FH-N) type.

Theorem 3.2 (Global-in-time convergence to viscosity solutions [11]). *Let (Γ, v) be the weak solution of Theorem 2.3 defined on the infinite time interval $[0,\infty)$. Assume that $\{(t,x) \mid t \in [0,\infty),\ \ x \in \Gamma(t)\}$ is a null-set. Then there exists a solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) that converges to (Γ, v) uniformly on $t \in [0,T]$ for any $T > 0$.*

These convergence results are very nice. However, they apply to (1.1) only when the nonlinearity (f,g) has appropriate monotonicity properties;

$$f \text{ is monotone in } v \text{ and } g \text{ is monotone in } u.$$

These monotonicity properties are used in the proof to apply the maximum principle (comparison principle). Therefore the proofs in [1] and [11] do not apply when (f,g) is of (CAM)-type. For scalar reaction-diffusion equations, de Mottoni and Schatzman [4] developed a method of proof of convergence which does not depend on the maximum principle.

3.1. **Asymptotic methods in convergence proof.** We now present a convergence result for (1.1) in the spirit of [4].

Theorem 3.3 (Convergence by approximation [7]). *Assume that* $(\mathrm{IFE})_0$ *has a smooth solution (Γ, v) on a time interval $[0,T]$, enjoiying the regularity properties;*

$$\Gamma \in C^{1+\frac{\alpha}{2}, l+\alpha}([0,T] \times \Gamma_0), \quad v \in C^{1+\frac{\alpha}{2}, l+\alpha}([0,T] \times \overline{\Omega}\backslash\Gamma_T) \cap C^1([0,T] \times \overline{\Omega})$$

with $l \ge 2$ and $\alpha \in (0,1)$.

[2] I am indebted to Professor Y. Giga for bringing the reference [11] to my attention.

(i) *There exists a family of approximate solutions* $(u_A^\varepsilon, v_A^\varepsilon)$ *of* (1.1) *in the* $L^p(\Omega)$-*sense* $(p > N)$;

$$\begin{aligned}\left\|\partial_t u_A^\varepsilon - \varepsilon\Delta u_A^\varepsilon - \varepsilon^{-1} f(u_A^\varepsilon, v_A^\varepsilon)\right\|_{L^p} &= O(\varepsilon^l),\\ \|\partial_t v_A^\varepsilon - D\Delta v_A^\varepsilon - g(u_A^\varepsilon, v_A^\varepsilon)\|_{L^p} &= O(\varepsilon^l)\end{aligned}$$

satisfying

$$\lim_{\varepsilon\to 0} v_A^\varepsilon(t,x) = v(t,x) \ \textit{uniformly on}\ [0,T]\times\overline{\Omega},$$

$$\lim_{\varepsilon\to 0} u_A^\varepsilon(t,x) = h^\pm(v(t,x)) \ \textit{uniformly on}\ \overline{\Omega_T^\pm}\backslash\Gamma_T^\delta \ \textit{for each}\ \delta > 0.$$

(ii) *There exists a family of solutions* $(u^\varepsilon, v^\varepsilon)$ *of* (1.1) *satisfying*

$$\begin{aligned}\sup_{[0,T]\times\overline{\Omega}} |v^\varepsilon(t,x) - v_A^\varepsilon(t,x)| &\le M\varepsilon^{l-\frac{N}{2p}},\\ \sup_{[0,T]\times\overline{\Omega}} |u^\varepsilon(t,x) - u_A^\varepsilon(t,x)| &\le M\varepsilon^{l-\frac{N}{p}},\end{aligned}$$

where $M > 0$ *is a constant independent of* ε.

The outline of proof of Theorem 3.3 now follows.

Part (i): Construction of approximate solutions.
Let us first agree to identify the interface $\Gamma_\varepsilon(t)$ as the 0-level set of $u^\varepsilon(t,x)$;

$$\Gamma_\varepsilon(t) = \{x \in \Omega \mid u^\varepsilon(t,x) = 0\} \approx \Gamma(t),$$

where $\Gamma(t)$ is obtained from a solution (Γ, v) of $(\mathrm{IFE})_0$. We now intend to express $\Gamma_\varepsilon(t)$ as a graph over $\Gamma(t)$, i.e.,

$$\Gamma_\varepsilon(t) = \{\gamma(t,y) + \varepsilon R^\varepsilon(t,y)\nu(t,y) \mid y \in \Gamma_0,\ \ t \in [0,T]\}.$$

Note that $R^\varepsilon(t,y)$ is a priori unknown (to be determined). Let us decompose the domain Ω by the interface;

$$\Omega = \Omega_\varepsilon^-(t) \cup \Gamma_\varepsilon(t) \cup \Omega_\varepsilon^+(t)$$

and consider the following approximate problem.

$$\text{(3.1)}\qquad \begin{cases} \partial_t u^{\pm,\varepsilon} = \varepsilon\Delta u^{\pm,\varepsilon} + \varepsilon^{-1} f(u^{\pm,\varepsilon}, v^{\pm,\varepsilon}), \\[2ex] \partial_t v^{\pm,\varepsilon} = D\Delta v^{\pm,\varepsilon} + g(u^{\pm,\varepsilon}, v^{\pm,\varepsilon}), \end{cases} \qquad x \in \Omega_\varepsilon^\pm(t),\ t > 0,$$

with the boundary conditions

$$\text{(3.2)}\qquad u^{\pm,\varepsilon}|_{\Gamma_\varepsilon(t)} = 0, \quad v^{\pm,\varepsilon}|_{\Gamma_\varepsilon(t)} = b^\varepsilon, \qquad \frac{u^{\pm,\varepsilon}}{\partial\mathbf{n}} = 0 = \frac{v^{\pm,\varepsilon}}{\partial\mathbf{n}},\ \ x \in \partial\Omega,\ t > 0.$$

Here, b^ε is to be determined.

We now substitute formal expressions

$$R^\varepsilon = R_1 + \varepsilon R_2 + \varepsilon^2 R_3 + \dots, \qquad b^\varepsilon = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \dots$$

into (3.1)-(3.2) to construct formal approximate solutions $(u^{\pm,\varepsilon}, v^{\pm,\varepsilon})$. This construction consists of two stages, outer and inner expansions.

Once the formal approximations are obtained, we impose on them C^1-matching conditions;

$$(3.3)\qquad \frac{u^{-,\varepsilon}}{\partial\nu}=\frac{u^{+,\varepsilon}}{\partial\nu},\quad \frac{v^{-,\varepsilon}}{\partial\nu}=\frac{v^{+,\varepsilon}}{\partial\nu},\qquad \text{on } \Gamma_\varepsilon(t),\ t>0.$$

These conditions give rise to a series of equations; the lowest order (0-th order) equation is nothing but $(\mathrm{IFE})_0$. The k-th ($k\geq 1$) order equation is a linear *inhomogeneous* parabolic system for (R_k, b_{k-1}) with the inhomogeneous terms depending only on known quantities and (R_j, b_{j-1}) with lower indices ($0\leq j<k$). The principal part of the equation is the same for all order $k\geq 1$, which is the linearization of $(\mathrm{IFE})_0$. So, these equations are solvable and we obtain the desired approximation as in Theorem 3.3 (i).

Part (ii): Spectral estimate.

We first linearize (1.1) around the approximate solution $U_A^\varepsilon=(u_A^\varepsilon, v_A^\varepsilon)$. For each $t\in[0,T]$ fixed, let us denote the linearized operator by $\mathcal{L}^\varepsilon(t)$;

$$\mathcal{L}^\varepsilon(t)=\begin{pmatrix} \varepsilon\Delta+\frac{1}{\varepsilon}f_u^A & \frac{1}{\varepsilon}f_v^A \\ g_u^A & D\Delta+g_v^A \end{pmatrix},$$

where $f_u^A=f_u(U_A^\varepsilon)$ and similarly for f_v^A, g_u^A and g_v^A. It is shown that $-\mathcal{L}^\varepsilon(t)$ is a sectorial operator for each $t\in[0,T]$. More precisely, we have the following

Lemma 3.1 (Resolvent estimate). *There exist $\lambda_*>0$, $\theta_0\in(0,\pi/2)$ and $M>0$, which depend only on the solution (Γ, v) of the interface equation $(\mathrm{IFE})_0$ such that*

$$(3.4)\qquad \|(\lambda-\mathcal{L}^\varepsilon(t))^{-1}\|\leq\frac{M}{|\lambda-\lambda_*|},\quad \lambda\in\{\lambda\in\mathbb{C}\mid \arg(\lambda-\lambda_*)\leq\frac{\pi}{2}+\theta_0\}.$$

We now rescale $\mathcal{L}^\varepsilon(t)$ and look for a solution $U^\varepsilon(t,x)$ of (1.1) as follows.

$$A^\varepsilon(t):=\varepsilon\mathcal{L}^\varepsilon(\varepsilon t),\qquad U^\varepsilon(\varepsilon t,x)=U_A^\varepsilon(\varepsilon t,x)+\varphi(t,x),\quad t\in[0,\frac{T}{\varepsilon}].$$

Then (1.1) is expressed as

$$(3.5)\qquad \varphi_t=A^\varepsilon(t)\varphi+\mathbf{N}^\varepsilon(t,\varphi)+\mathcal{R}^\varepsilon(t),$$

where $\mathbf{N}^\varepsilon(t,\varphi)=O(|\varphi|^2)$ and

$$\|\mathcal{R}^\varepsilon(t)\|_{L^p}=O(\varepsilon^{l+1}),\quad t\in[0,\frac{T}{\varepsilon}].$$

Now our task is to give a uniform estimate on φ in the time interval $[0,\frac{T}{\varepsilon}]$. To do this, let us set up appropriate function spaces. We define the basic space X_0^ε and the domain X_1^ε of $A^\varepsilon(t)$ by

$$(3.6)\qquad X_0^\varepsilon:=L^p(\Omega)\times L^p(\Omega),\quad X_1^\varepsilon:=W^{2,p}_{\varepsilon,N}(\Omega)\times W^{2,p}_{\sqrt{\varepsilon},N}(\Omega),$$

where, as sets,

$$W^{2,p}_{\varepsilon,N}(\Omega)=W^{2,p}_N(\Omega):=\{u\in W^{2,p}(\Omega)\mid \frac{\partial u}{\partial\mathbf{n}}|_{\partial\Omega}=0\}$$

with a weighted norm

$$\|u\|_{W^{2,p}_{\varepsilon,N}} = \|u\|_{L^p} + \varepsilon\|\nabla u\|_{L^p} + \varepsilon^2\|\nabla^2 u\|_{L^p}.$$

We denote by X^ε_α, $\alpha \in (0,1)$, the interpolation spaces between X^ε_0 and X^ε_1, i.e.,

$$X^\varepsilon_\alpha = W^{2\alpha,p}_{\varepsilon,N}(\Omega) \times W^{2\alpha,p}_{\sqrt{\varepsilon},N}(\Omega).$$

We also introduce weighted Hölder spaces $C^\beta_{\varepsilon,p}$. It is the same as the usual Hölder space $C^\beta(\overline{\Omega})$ as sets, with the weighted norm:

$$\|u\|_{C^\beta_{\varepsilon,p}} := \varepsilon^{\frac{N}{p}}|u|_\infty + \varepsilon^{\beta+\frac{N}{p}}[u]_\beta.$$

These Hölder spaces are introduced to deal with the quadratic term $\mathbf{N}^\varepsilon$ in (3.5). The weighted Sobolev spaces have usual embedding properties; if $\alpha, \beta \in (0,1)$ satisfy the relation $2\alpha - \frac{N}{p} > \beta$ then $W^{2\alpha,p}_{\varepsilon,N}$ is continuously embedded in $C^\beta_{\varepsilon,p}$;

$$2\alpha - \frac{N}{p} > \beta \quad \Longrightarrow \quad W^{2\alpha,p}_{\varepsilon,N} \hookrightarrow C^\beta_{\varepsilon,p} \tag{3.7}$$

with embedding constants being *independent* of $\varepsilon > 0$.

When we consider a bounded linear operator $B : X^\varepsilon_\alpha \to X^\varepsilon_\beta$, its norm is denoted by $\|B\|_{\alpha,\beta}$. Now let us recast Lemma 3.1 in terms of A^ε.

Lemma 3.2. *$-A^\varepsilon(t)$ is sectorial for each $t \in [0, \frac{T}{\varepsilon}]$ and the following estimate is valid;*

$$\|(\lambda - A^\varepsilon(t))^{-1}\|_{0,0} \le \frac{M}{|\lambda - \varepsilon\lambda_*|}, \quad \lambda \in \{\lambda \in \mathbb{C} \mid \arg(\lambda - \varepsilon\lambda_*) \le \frac{\pi}{2} + \theta_0\}. \tag{3.8}$$

Note that the operator $A^\varepsilon(t) - A^\varepsilon(s)$ for $0 \le s,t \le \frac{T}{\varepsilon}$ is a multiplication operator. This difference does not involve any differential operator. Therefore, we can easily show that there exists a constant $M_1 > 0$ such that for $0 \le \beta \le \alpha \le 1$

$$\|A^\varepsilon(t) - A^\varepsilon(s)\|_{\alpha,\beta} \le M_1\varepsilon(t-s), \quad 0 \le s \le t \le \frac{T}{\varepsilon} \tag{3.9}$$

Moreover, the estimate (3.8) implies

$$\|e^{(t-s)A^\varepsilon(s)}\|_{0,1} \le \frac{M_1}{t-s}, \quad 0 \le s \le t \le \frac{T}{\varepsilon}. \tag{3.10}$$

Therefore there exists a constant $K > 0$ such that the *evolution operator* $\Phi(t,s)$ associated with the family $\{A^\varepsilon(t)\}_{0 \le t \le \frac{T}{\varepsilon}}$ satisfies for $0 \le \alpha, \beta \le 1$

$$\|\Phi(t,s)\|_{\alpha,\beta} \le M_1(t-s)^{\alpha-\beta}e^{\varepsilon(\lambda_*+K)(t-s)}, \quad 0 \le s \le t \le \frac{T}{\varepsilon}. \tag{3.11}$$

Applying the variation of constants formula to (3.5), we obtain

$$\varphi(t) = \Phi(t,0)\varphi(0) + \int_0^t \Phi(t,s)\mathbf{N}^\varepsilon(s,\varphi(s))ds + \int_0^t \Phi(t,s)\mathcal{R}^\varepsilon(s)ds. \tag{3.12}$$

Since the existence of solutions to this equation is well established, we only need to have an estimate on $\|\varphi(t)\|_\alpha$, where $\|\cdot\|_\alpha$ is the norm of X_α^ε. Let $C>0$ be a constant (independent of $\varepsilon>0$) such that

$$\|\mathcal{R}^\varepsilon(s)\|_{L^p}\le C\varepsilon^{l+1},\quad |\mathbf{N}^\varepsilon(s,\varphi)|\le C|\varphi|^2,\qquad 0\le s\le \frac{T}{\varepsilon}.$$

Then we have for $2\beta-\frac{N}{p}>0$

$$\|\mathbf{N}^\varepsilon(s,\varphi(s))\|_{L^p}\le C|\varphi(s)|_\infty\|\varphi(s)\|_{L^p}\le C\|\varphi(s)\|_\beta^2.$$

Now using these estimates and (3.11) in (3.12), we have

$$\begin{aligned}
r(t)\le & M_1r(0)+CM_1\varepsilon^{l+1}\int_0^t(t-s)^{-\beta}ds\\
&+CM_1\int_0^t(t-s)^{-\beta}e^{\varepsilon(\lambda_*+K)s}r(s)^2ds\\
\le & M_1r(0)+\frac{CM_1T^{1-\beta}}{1-\beta}\varepsilon^{l+\beta}\\
&+CM_1e^{(\lambda_*+K)T}\int_0^t(t-s)^{-\beta}r(s)^2ds,\quad 0\le t\le\frac{T}{\varepsilon},
\end{aligned}\tag{3.13}$$

where $r(t):=\|\varphi(t)\|_\beta e^{-\varepsilon(\lambda_*+K)t}$ is a continuous function of $t\in[0,\frac{T}{\varepsilon}]$. Now we *choose* the initial function $\varphi(0)$ so that

$$r(0)=\|\varphi(0)\|_\beta\le\varepsilon^{l+1}.$$

Then, from the continuity of $r(t)$, we have

$$r(t)\le\varepsilon^l\tag{3.14}$$

for t near 0. Let $T_1>0$ be defined by

$$\sup\{t\in[0,\frac{T}{\varepsilon}]\mid r(s)\le\varepsilon^l,\ 0\le s\le t\}.$$

We have either $T_1=\frac{T}{\varepsilon}$ or $r(T_1)=\varepsilon^l$. We will show that the latter possibility does not occur by choosing $\varepsilon>0$ small enough. From (3.13), we have

$$\begin{aligned}
r(T_1)\le & M_1\varepsilon^{l+1}+\frac{CM_1T^{1-\beta}}{1-\beta}\varepsilon^{l+\beta}+\frac{CM_1e^{(\lambda_*+K)T}T^{1-\beta}}{1-\beta}\varepsilon^{2l}\\
= & \varepsilon^l\left\{M_1\varepsilon+\frac{CM_1T^{1-\beta}}{1-\beta}\varepsilon^\beta+\frac{CM_1e^{(\lambda_*+K)T}T^{1-\beta}}{1-\beta}\varepsilon^l\right\}\le\frac{1}{2}\varepsilon^l,
\end{aligned}$$

arriving at a contradiction. Therefore, (3.14) is valid for $0\le t\le\frac{T}{\varepsilon}$. Now by using (3.7), we obtain

$$\varepsilon^{\frac{N}{p}}|\varphi^u(t)|_\infty+\varepsilon^{\frac{N}{2p}}|\varphi^v(t)|_\infty\le M\varepsilon^l,\quad 0\le t\le\frac{T}{\varepsilon},$$

for some $M>0$ independent of $\varepsilon>0$, where $\varphi(t)=(\varphi^u(t),\varphi^v(t))$. This completes the outline of proof of Theorem 3.3.

4. Degeneracy

In the previous section, we have discussed a relationship between the reaction-diffusion system (1.1) and its interface equation $(\mathrm{IFE})_0$ on *finite time intervals.* Does $(\mathrm{IFE})_0$ capture asymptotic (as $t \to \infty$) behaviors of solutions to (1.1)? We will show by an example that the answer is no! We will also show that $(\mathrm{IFE})_\varepsilon$ is more appropriate to describe the asymptotic behavior of (1.1).

Let us consider (1.1) on the N-dimensional unit disk; $\Omega = \{x \in \mathbb{R}^N \mid |x| < 1\}$, and look for its equilibrium solutions with spherical transition layers.

Theorem 4.1 (Existence and stability of transition layers [8]). *Let Ω be the N-dimensional unit disk; $\Omega = \{x \,;\, |x| < 1\}$.*

(i) *There exists $R_* \in (0,1)$ such that for*

$$\Gamma_* = \{|x| = R_*\}, \quad \Omega^- = \{|x| < R_*\}, \quad \Omega^+ = \{R_* < |x| < 1\},$$

the problem

$$0 = D\Delta v + g^*(v, x; \Gamma_*), \quad x \in \Omega^\pm, \qquad \frac{\partial v}{\partial \mathbf{n}} = 0, \quad x \in \partial\Omega$$

has a unique spherically symmetric solution $v = v^*(x) = v^*(|x|)$ with regularity properties;

$$v^* \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega}\backslash\Gamma_*).$$

(ii) *There exists a family of spherically symmetric equilibrium solutions $(u^\varepsilon(x), v^\varepsilon(x))$ of (1.1) for small $\varepsilon > 0$. This solution has the following behavior;*

$$\lim_{\varepsilon \to 0} v^\varepsilon(x) = v^*(x), \quad \text{uniformly on } \overline{\Omega},$$

$$\lim_{\varepsilon \to 0} u^\varepsilon(x) = h^\pm(v^*(x)), \quad \text{uniformly on } \overline{\Omega}\backslash\Gamma_*^\delta \text{ for each } \delta > 0.$$

(iii) *The solution in (ii) is unstable; The linearization around it has spherically symmetric eigenfucntions. Let λ_j^ε be the eigenvalue associated with spherical harmonics of degree $j \geq 0$ which has the largest real part. Then they are all real and satisfy*

$$\begin{aligned}
&\lambda_0^\varepsilon < 0; \quad \text{breathing mode,}\\
&\lambda_1^\varepsilon < 0; \quad \text{translation mode,}\\
&\lambda_k^\varepsilon > 0 \ \ (2 \leq k \leq j_z^\varepsilon - 1); \quad \text{wiggly modes,}\\
&\lambda_k^\varepsilon \leq 0 \ \ (k \geq j_z^\varepsilon); \quad \text{wiggly modes,}
\end{aligned}$$

where $j_z^\varepsilon = O((\varepsilon D)^{-1/2})$. Moreover, λ_j^ε attains a maximum at $j = j_u^\varepsilon = O((\varepsilon D)^{-1/3})$.

(iv) *Let the space dimension be 2; $N = 2$. Then there exists a infinitely many critical values $\{\varepsilon_j\}_{j=j_0}^\infty$ with $j_0 \gg 1$ such that non-radial equilibrium solutions bifurcates at each $\varepsilon = \varepsilon_j$ from the equilibrium solution in (ii) and ε_j has the following characterization:*

$$\varepsilon_j = \frac{c'(0) v_r^*(R_*) R_*^2}{j^2} + O\!\left(\frac{1}{j^4}\right) \quad (\text{as } j \to \infty).$$

This theorem says that the spherically symmetric transition layer solution is highly unstable with $O(\varepsilon^{-1/2})$ many of unstable eigenvalues. It may be obscure how the interface equation $(\mathrm{IFE})_\varepsilon$ with $\varepsilon > 0$ is related to the results in Theorem 4.1. In oder to clarify this relationship, let us outline its proof.

Outline of Proof: Part (i) reduces to a boundary value problem for an ordinary differential equation.

For part (ii), we construct a pair of equilibrium solutions $(u^{\pm,\varepsilon}, v^{\pm,\varepsilon})$ of (1.1), respectively, on $\Omega^\pm$. Then the C^1-matching conditions

$$\frac{du^{-,\varepsilon}}{dr}(R_*) = \frac{du^{+,\varepsilon}}{dr}(R_*), \qquad \frac{dv^{-,\varepsilon}}{dr}(R_*) = \frac{dv^{+,\varepsilon}}{dr}(R_*)$$

give rise to an equation on Γ_*, i.e.,

$$\mathcal{A}^0 p := c'(0)v_r^*(R_*)p - c'(0)\Pi^{-1}p = q, \tag{4.1}$$

where q is known and (4.1) has to be uniquely solvable in p. In (4.1), Π is a Dirichlet-to-Neumann map, defied by

$$\Pi b := \frac{\partial v^-}{\partial \nu}|_{\Gamma_*} - \frac{\partial v^+}{\partial \nu}|_{\Gamma_*},$$

where $v^\pm$ are solutions of the boundary value problem;

$$D\Delta v^\pm + g_v^*(v, x; \Gamma_*)v^\pm = 0, \quad x \in \Omega^\pm, \qquad v^\pm|_{\Gamma_*} = b, \quad \frac{\partial v^+}{\partial \mathbf{n}}|_{\partial\Omega} = 0.$$

We emphasize that the C^1-matching condition is as simple as (4.1) only because we are dealing with spherically symmetric functions. For general functions, it is more involved and its solvability is not clear [6].

Part (iii). It turns out that the eigenvalues λ_j^ε in Theorem 4.1 (iii) has the following characterization;

$$\lambda_j^\varepsilon = \varepsilon\hat{\lambda}_j^\varepsilon + o(\varepsilon) \quad (\text{as } \varepsilon \to 0),$$

where $\hat{\lambda}_j^\varepsilon$ are eigenvalues of $\mathcal{A}^\varepsilon$ defined by

$$\mathcal{A}^\varepsilon := \varepsilon\left(\Delta^{\Gamma_*} + \frac{N-1}{R_*^2}\right) + \mathcal{A}^0 \tag{4.2}$$

with Δ^{Γ_*} being the Laplace-Beltrami operator on Γ_*. The ε-multiplied term in (4.2) exactly corresponds to $-\varepsilon\kappa$-term in $(\mathrm{IFE})_\varepsilon$. This is why $(\mathrm{IFE})_0$ cannot capture asymptotic behavior of solutions to (1.1).

In the proof of part (iv), we use an equivariant bifurcation theory developed in [3] and [12].

5. Rescaling

Theorem 4.1 says that as $t \to \infty$ $\Gamma(t)$ tends to develop fine scales. Theorem 4.1 (iii) says that (1.1) produces equilibrium transition layers in which the interface Γ has a typical length of scale $O((\varepsilon D)^{1/2}) = 1/j_z^\varepsilon$ and that the length scale of the most unstable mode is $O((\varepsilon D)^{1/3}) = 1/j_u^\varepsilon$. In this section, we will rescale $(\mathrm{IFE})_\varepsilon$ to obtain another interface equation which describes *meso-scale* (i.e., $\varepsilon^{1/3}$-scale) interfaces.

Let us simply write $(\text{IFE})_\varepsilon$ as

$$(\text{IFE}) \qquad \begin{cases} \mathbf{v} = & c(v) - \varepsilon\kappa, \\ v_t = & D\Delta v + g^*(v). \end{cases}$$

We now rescale the spatial variable x via;

$$\Omega \ni x \longmapsto \tilde{x} \in \tilde{\Omega}, \quad x = \varepsilon^\alpha \tilde{x}$$

where $0 < \alpha \leq 1$ is to be adequately determined. Under this rescaling, (IFE) becomes

$$(5.1) \qquad \begin{cases} \varepsilon^\alpha \tilde{\mathbf{v}} = & c(\tilde{v}) - \varepsilon^{1-\alpha}\tilde{\kappa}, \\ \varepsilon^{2\alpha}\tilde{v}_t = & D\Delta\tilde{v} + \varepsilon^{2\alpha} g^*(\tilde{v}). \end{cases}$$

The second equation in (5.1) implies $\tilde{v} = \varepsilon^{2\alpha}\overline{v}$ which upon substitution in the first of (5.1) gives

$$(5.2) \qquad \varepsilon^\alpha \tilde{\mathbf{v}} = \varepsilon^{2\alpha} c'(0)\overline{v} - \varepsilon^{1-\alpha}\tilde{\kappa}.$$

In order for the two terms on the right of (5.2) to have contributions of the same magnitude, it must be that $\varepsilon^{2\alpha} = \varepsilon^{1-\alpha}$. Hence, we obtain $\alpha = 1/3$. In this way, we naturally arrive at the meso-spatial scale $O(\varepsilon^{1/3})$ predicted in Theorem 4.1 (iii). The equation (5.2) also suggests us to rescale the time variable by $t = \varepsilon^{-1/3}\tilde{t}$. In terms of $(\tilde{t}, \tilde{x})$, (1.1) is written as

$$(5.3) \qquad \begin{cases} \tilde{\varepsilon}^4 u_{\tilde{t}} = \tilde{\varepsilon}^4 \tilde{\Delta} u + f(u, v) \\ \\ \tilde{\varepsilon}^3 v_{\tilde{t}} = D\tilde{\Delta} v + \tilde{\varepsilon}^2 g(u, v), \end{cases}$$

where $\tilde{\varepsilon} = \varepsilon^{1/3}$. An interface equation associated with (5.3) is

(5.4)

$$\begin{cases} \mathbf{v}(x; \Gamma(t)) = c'(0)\{v(t,x) - \overline{v}(t)\} - \{\kappa(x;\Gamma(t)) - \overline{\kappa}(t)\}, & x \in \Gamma(t), \quad t > 0, \\ \\ 0 = D\Delta v + g^*(x; \Gamma(t)), \quad x \in \overline{\Omega}\backslash\Gamma(t), \quad t > 0, & v(t,\cdot) \in C^1(\overline{\Omega}), \end{cases}$$

where $g^*(x;\Gamma(t)) = g(h^\pm(0), 0)$ for $x \in \Omega^\pm(t)$, $\overline{v}(t) = \int_{\Gamma(t)} v(t,x)dS_x$, and $\overline{\kappa}(t) = \int_{\Gamma(t)} \kappa(x;\Gamma(t))dS_x$. We can establish a relationship between (5.3) and (5.4) similar to Theorem 3.3.

Theorem 5.1 (Existence of classical solution [9]). *Let $\Gamma(0) = \Gamma_0$ be of $C^{2+\alpha}$-class for some $0 < \alpha < 1$. Then there exists a $T > 0$ so that (5.4) has a unique solution $(\Gamma(t), v(t,x))$ with regularity properties;*

$$\gamma(t,y) \in C^{1+\alpha/2, 2+\alpha}([0,T] \times \Gamma_0), \quad v(t,\cdot), v_t(t,\cdot) \in C^{2+\alpha}(\overline{\Omega}\backslash\Gamma(t)) \cap C^{1+1}(\overline{\Omega}).$$

We also have an analogue of Theorem 3.3.

Theorem 5.2 ([7]). *There exists a family of solutions $(u^\varepsilon, v^\varepsilon)$ of (5.3) such that*

$$\lim_{\varepsilon \to 0} v^\varepsilon(t,x) = v(t,x) \quad \textit{uniformly on } [0,T] \times \overline{\Omega},$$

$$\lim_{\varepsilon \to 0} u^\varepsilon = h^\pm(v(t,x)) \quad \textit{uniformly on } [0,T] \times \overline{\Omega}\backslash\Gamma_T^\delta \textit{ for each } \delta > 0.$$

The proof of this theorem is carried out in the same spirit as that of Theorem 3.3.

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