数理解析研究所講究録 1323 巻 2003 年 76-83

Derivation of variational problems from microscopic interface model

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1 Introduction

This note reviews recent results on the $\nabla\varphi$ interface model considered over the wall or under the weak effects of additional self-potentials. We are, in particular, interested in the scaling limit which passes from microscopic to macroscopic levels. The results are classified into two types:

1. Static results.
   - Large Deviation Principle
   - Derivation of Variational Problems (VP)
     - Wulff Shape, Winterbottom Shape
     - Alt-Caffarelli or Alt-Caffarelli-Friedman's VP
2. Dynamic results.
   - Motion by Mean Curvature (MMC) with anisotropy
   - Dynamic Large Deviation Principle
   - MMC with reflection
     - Evolutionary Variational Inequality
   - Fluctuation
     - Stochastic PDE with reflection

2 Static results

2.1. Let us introduce the **$\nabla\varphi$ interface model** briefly. We are concerned with a surface (interface) in $\mathbb{R}^{d+1}$, which separates two distinct pure phases, described by the height variables $\phi = \{\phi(x) \in \mathbb{R}, x \in \Gamma\}$ measured from a reference hyperplane Γ located in $\mathbb{R}^d$. To avoid complications, we assume that the interface has no overhangs nor bubbles. The variables ϕ are microscopic objects, and the space Γ is discretized and taken as $\Gamma = D_N (\equiv ND \cap \mathbb{Z}^d)$, lattice torus $(\mathbb{Z}/N\mathbb{Z})^d (\equiv \{1, 2, \dots, N\}^d)$ or $\mathbb{Z}^d$. Here D is a (macroscopic) bounded domain in $\mathbb{R}^d$ and N represents the size of the microscopic system.

An **energy** is associated with each height variable $\phi : \Gamma \to \mathbb{R}$ as the sum over all bonds $\langle x, y\rangle$ in Γ (or in $\Gamma \cup \partial\Gamma$)

$$H(\phi) \equiv H_\Gamma^\psi(\phi) = \sum_{\langle x,y\rangle \subset \Gamma (\text{or } \Gamma\cup\partial\Gamma)} V(\phi(x) - \phi(y)),$$

and the **equilibrium state (Gibbs measure)** is defined by

$$d\mu \equiv d\mu_\Gamma^\psi = Z^{-1} e^{-H(\phi)} \prod_{x\in\Gamma} d\phi(x),$$

where Z is a normalization constant. The **potential** V is symmetric, smooth and strictly convex $(0 <^\exists c_- \le V'' \le^\exists c_+ < \infty)$. Note that the boundary conditions $\psi = \{\psi(x), x \in \partial\Gamma\}$ are required to define $H(\phi)$ and μ when $\Gamma = D_N$. An infinite volume limit (thermodynamic limit) as $\Gamma \nearrow \mathbb{Z}^d$ exists when $d \ge 3$ and the limit measure μ has a long correlation. More information on the $\nabla\varphi$ interface model can be found in [7] and [13].

2.2. Our main interest is in the **scaling limit**, which passes from microscopic to macroscopic levels, defined by

$$h^N(\theta) = \frac{1}{N}\phi([N\theta]), \quad \theta \in D \ (\text{or} \in \mathbb{T}^d \equiv [0,1)^d,\ \mathbb{R}^d),$$

where $[N\theta]$ stands for the integral part of $N\theta (\in \mathbb{R}^d)$ taken componentwise. The function h^N is the macroscopic height variable associated with the microscopic $\phi : \Gamma \to \mathbb{R}$. The **surface tension** $\sigma = \sigma(u)$ is the macroscopic energy for a surface with tilt $u \in \mathbb{R}^d$ determined by

$$\mu\left(\text{tilt of } h^N \sim u\right) \underset{N\to\infty}{\sim} \exp\{-N^d \sigma(u)\}.$$

Theorem 1. (**Large Deviation**, Deuschel-Giacomin-Ioffe [4]) Consider the Gibbs measure $\mu^0_{D_N}$ on $\Gamma = D_N$ with 0-boundary conditions $\psi(x) = 0, x \in \partial D_N$. Then, the probability that h^N is close to a given macroscopic surface $h \in H^1_0(D)$ behaves as

$$\mu^0_{D_N}\left(h^N \sim h\right) \underset{N\to\infty}{\sim} \exp\{-N^d \Sigma_D(h)\},$$

where $\Sigma_D(h)$ is the total surface tension of h defined by

$$\Sigma_D(h) = \int_D \sigma(\nabla h(\theta))\, d\theta. \qquad (1) \qquad \square$$

This result is an analogue of Dobrushin-Kotecký-Shlosman [5] for the Ising model.

Corollary 2. (Wall and constant volume conditions) For every $v \geq 0$, under the conditional probability $\mu^0_{D_N}\left(\cdot\,\middle|h^N \geq 0, \int_D h^N(\theta)d\theta \geq v\right)$, the law of large numbers $h^N \longrightarrow \bar{h}_v$ (as $N \to \infty$) holds, where $\bar{h}_v$ is the minimizer (called **Wulff shape**) of the variational problem

$$\min\left\{\Sigma_D(h); h \in H^1_0(D), h \geq 0, \int_D h(\theta)d\theta = v\right\}. \qquad \square$$

We add a **weak self-potential** term to the energy $H^\psi_{D_N}(\phi)$:

$$H^{\psi,Q,W}_{D_N}(\phi) = H^\psi_{D_N}(\phi) + \sum_{x\in D_N} Q\left(\frac{x}{N}\right) W(\phi(x)),$$

having the boundary conditions $\psi(x) = Nf\,(x/N)\,, x \in \partial D_N$ determined from macroscopic function $f : \partial D \to \mathbb{R}$, where $Q : D \to [0,\infty)$ and $W : \mathbb{R} \to \mathbb{R}$ satisfies $\alpha_\pm =^\exists \lim_{r\to\pm\infty} W(r)$ such that $\alpha_+ \wedge \alpha_- \leq W \leq \alpha_+ \vee \alpha_-$. The Gibbs measure is associated and defined by

$$d\mu^{\psi,Q,W}_{D_N} = Z^{-1}_{\psi,Q,W} e^{-H^{\psi,Q,W}_{D_N}(\phi)} \prod_{x\in D_N} d\phi(x).$$

Theorem 3. (**Large Deviation**, Funaki-Sakagawa [11]) Assume $A := \alpha_+ - \alpha_- \geq 0$. Then,

$$\mu^{\psi,Q,W}_{D_N}\left(h^N \sim h\right) \underset{N\to\infty}{\sim} \exp\{-N^d I^A_D(h)\},$$

where

$$I^A_D(h) = \Sigma^A_D(h) - \inf_{h'\in H^1_f(D)} \Sigma^A_D(h'),$$

$$\Sigma^A_D(h) = \Sigma_D(h) - A\int_D Q(\theta) 1_{\{h(\theta)\leq 0\}}\, d\theta,$$

and $H^1_f(D)$ is the space of all $h \in H^1(D)$ having boundary conditions f. $\square$

Corollary 4. The law of large numbers $h^N \longrightarrow \bar{h}_A$ (as $N \to \infty$) holds under $\mu_{D_N}^{\psi,Q,W}$, where $\bar{h}_A$ is the minimizer of the variational problem

$$\min\left\{\Sigma_D^A(h); h \in H_f^1(D)\right\}. \qquad \square$$

Remark 1. (1) The variational problem obtained in Corollary 4 was studied by Alt-Caffarelli [1], Alt-Caffarelli-Friedman [2], Weiss [18] and others.
(2) The large deviation for the Gibbs measure with δ-pinning instead of weak self-potentials is discussed by [11] in one dimension.
(3) Bolthausen-Ioffe [3] proved the law of large numbers for the Gibbs measure on the wall with δ-pinning and quadratic potential under the constant volume condition in two dimension. The limit called **Winterbottom shape** is uniquely (except translation) characterized by a certain variational problem. $\square$

3 Dynamic results

3.1. One can introduce **microscopic dynamics** (stationary and reversible under the Gibbs measure μ) for the interfaces by the SDEs (Langevin equation)

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)\, dt + \sqrt{2} dw_t(x), \quad x \in \Gamma,$$

where $\{w_t(x), x \in \Gamma\}$ is a family of independent Brownian motions and

$$\frac{\partial H}{\partial \phi(x)}(\phi) = \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y)).$$

The goal is to discuss the **space-time diffusive scaling limit** for $\phi_t = \{\phi_t(x), x \in \Gamma\}$:

$$h^N(t,\theta) = \frac{1}{N}\phi_{N^2 t}([N\theta]).$$

Theorem 5. (**Hydrodynamic Limit**, Funaki-Spohn [12] on the torus $\mathbb{T}^d$, Nishikawa [14] on D with boundary conditions) As $N \to \infty$, $h^N(t,\theta) \longrightarrow h(t,\theta)$. The limit $h(t,\theta)$ is a unique weak solution of the nonlinear PDE (MMC with anisotropy):

$$\begin{aligned} \frac{\partial h}{\partial t}(t) &= \operatorname{div}\left\{\nabla\sigma(\nabla h(t))\right\} \\ &\equiv \sum_{i=1}^d \frac{\partial}{\partial \theta_i}\left\{\frac{\partial \sigma}{\partial u_i}(\nabla h(t))\right\}. \end{aligned} \tag{2}$$

$\square$

The surface tension has the following properties: $\sigma \in C^1(\mathbb{R}^d)$, $\nabla\sigma$ is Lipschitz continuous and σ is strictly convex. The PDE (2) can be regarded as the gradient flow for $\Sigma = \Sigma_{\mathbb{T}^d}$ or Σ_D, the total surface tension (1) on $\mathbb{T}^d$ or D:

$$\frac{\partial h}{\partial t}(t) = -\frac{\delta}{\delta h(\theta)}\Sigma(h(t)).$$

Theorem 6. (**Dynamic Large Deviation**, Funaki-Nishikawa [9] on $\mathbb{T}^d$)

$$P\left(h^N(t) \sim h(t),\ t \le T\right) \underset{N\to\infty}{\sim} \exp\{-N^d I_T(h)\},$$

where $h(t) = h(t,\theta)$ is a given motion of surface and

$$I_T(h) = \frac{1}{4}\int_0^T dt \int_{\mathbb{T}^d} \left[\frac{\partial h}{\partial t} - \text{div}\,\{\nabla\sigma(\nabla h(t))\}\right]^2 d\theta.$$

The relation to the static large deviation (Theorem 1) is given by

$$\lim_{T\to\infty} S_T(\bar{h}) = \Sigma_{\mathbb{T}^d}(\bar{h}), \quad \bar{h} = \bar{h}(\theta),$$

where

$$S_T(\bar{h}) = \inf\left\{I_T(h); h(T,\theta) = \bar{h}(\theta)\right\}. \qquad \square$$

3.2. **Dynamics on the wall** are introduced by SDEs of Skorohod type:

$$(3) \quad d\phi_t(x) = -\frac{\partial H}{\partial\phi(x)}(\phi_t)\,dt + \sqrt{2}dw_t(x) + \frac{1}{N}f\left(\frac{t}{N^2}, \frac{x}{N}, \frac{1}{N}\phi_t(x)\right)dt + d\ell_t(x),$$

subject to the conditions

$$\phi_t(x) \ge 0, \quad \ell_t(x) \nearrow \quad \text{and} \quad \int_0^\infty \phi_t(x)d\ell_t(x) = 0,$$

where $f = f(t,\theta,h)$ is a given macroscopic external field. Note that $\ell_t(x)$ increases only when $\phi_t(x) = 0$. The unique invariant (stationary) measure (when $f = 0, \Gamma = D_N$ with 0-boundary conditions) is given by $\mu^0_{D_N}(\,\cdot\,|\phi \ge 0)$, which is reversible under the dynamics.

Theorem 7. (**Hydrodynamic Limit**, Funaki [8] on $\mathbb{T}^d$) As $N \to \infty$, $h^N(t,\theta) \longrightarrow h(t,\theta)$. The limit $h(t,\theta)$ is a unique solution of the **evolutionary variational in-**

equality (MMC with reflection (obstacle)):

(a) $\quad h \in L^2(0,T;V),\ \dfrac{\partial h}{\partial t} \in L^2(0,T;V'), \quad {}^{\forall}T > 0,$

(b) $$\left(\frac{\partial h}{\partial t}(t), h(t) - v\right) + (\nabla\sigma(\nabla h(t)), \nabla h(t) - \nabla v) \le (f(t,h(t)), h(t) - v),$$
$$\text{a.e. } t, \quad {}^{\forall}v \in V : v \ge 0,$$

(c) $\quad h(t,\theta) \ge 0, \qquad$ a.e.,

(d) $\quad h(0,\theta) = h_0(\theta),$

where $V = H^1(\mathbb{T}^d), H = L^2(\mathbb{T}^d), V' = H^{-1}(\mathbb{T}^d)$ and $(\cdot,\cdot)$ denotes the inner product of H (or H^d) or the duality between V' and V. □

Remark 2. Rezakhanlou [15], [16] derived a Hamilton-Jacobi equation under hyperbolic scaling from growing SOS dynamics ($\phi(x) \in \mathbb{Z}$) with constraints on the gradients (e.g. $\nabla\phi(x) \le v$). Related results were obtained by Evans-Rezakhanlou [6] and Seppäläinen [17]. □

Let us consider the **equilibrium dynamics** ϕ_t on the wall in one dimension, i.e., ϕ_t is a solution of the SDE (3) with $d = 1, \Gamma = \{1, 2, \dots, N-1\}, f = 0$ and with 0-boundary conditions $\phi_t(0) = \phi_t(N) = 0$ and an initial distribution $\mu_\Gamma^0(\,\cdot\,|\phi \ge 0)$. **Macroscopic fluctuation field** (around the hydrodynamic limit $h(t,\theta) = 0$) is defined by

$$\Phi^N(t,\theta) = \sqrt{N} h^N(t,\theta) \ (\ge 0), \quad \theta \in [0,1].$$

Theorem 8. (Equilibrium Fluctuation, Funaki-Olla [10]) As $N \to \infty$, $\Phi^N(t,\theta) \Longrightarrow \Phi(t,\theta)$. The limit $\Phi(t,\theta)$ is a unique weak stationary solution of the stochastic PDE with reflection of Nualart-Pardoux type:

$$\begin{aligned}
&\frac{\partial\Phi}{\partial t}(t,\theta) = q\frac{\partial^2\Phi}{\partial\theta^2}(t,\theta) + \sqrt{2}\dot{B}(t,\theta) + \xi(t,\theta), \quad \theta \in [0,1],\\
&\Phi(t,\theta) \ge 0, \quad \int_0^\infty \int_0^1 \Phi(t,\theta)\,\xi(dtd\theta) = 0,\\
&\Phi(t,0) = \Phi(t,1) = 0, \quad \xi\text{: random measure,}
\end{aligned}$$

where $\dot{B}(t,\theta)$ is a space-time white noise and

$$q = 1/\langle\eta^2\rangle_\nu, \quad \nu(d\eta) = e^{-V(\eta)}\,d\eta \Big/ \int_{\mathbb{R}} e^{-V(\eta')}\,d\eta'. \qquad \square$$

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