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GREEN CURRENTS FOR MODULAR CYCLES IN ARITHMETIC QUOTIENTS OF COMPLEX HYPERBALLS

MASAO TSUZUKI)
(都築正男
上智大学理工学部)

0. Introduction and basic notations

0.1. Introduction. Let X be a complex manifold and Y its analytic subvariety of codimension r. The Green current for Y is defined to be a current $\mathcal{G}$ of $(r-1,r-1)$-type on X such that $dd^c\mathcal{G}+\delta_Y$ is represented by a C^∞-form of (r,r)-type on X. In the arithmetic intersection theory developed by Gillet and Soulé, the role played by the algebraic cycles in the conventional intersection theory is replaced with the arithmetic cycles. In a heuristic sense, the Green currents is regarded as the 'archimedean' ingredient of such arithmetic cycles ([2]).

Let us consider the case when X is the quotient of a Hermitian symmetric domain G/K by an arithmetic lattice Γ in the semisimple Lie group G, and Y is a modular cycle stemming from a modular imbeding $H/H\cap K \hookrightarrow G/K$, where H is a reductive subgroup of G such that $H\cap K$ is maximally compact in H. Then inspired by the classical works on the resolvent kernel functions of the Laplacian on Riemannian surfaces and also by a series of works of Miatello and Wallach ([5], [6]), T. Oda posed a plan to construct a Green current for Y making use of a 'secondary spherical function' on $H\backslash G$, giving an evidence for divisorial case with some conjectures. Among many possible choices of the Green currents for a modular cycle Y, this construction may provide a way to fix a natural one. If $r=1$, namely Y is a modular divisor, we already obtained a satisfactory result by introducing the secondary spherical functions properly ([7]). Here we focus on the case when G/K is an n-dimensional complex hyperball and $H/H\cap K$ is a complex sub-hyperball of codimension $r>1$, and show that the same method also works well.

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0.2. Notations. The Lie algebra of a Lie group G is denoted by $\mathrm{Lie}(G)$. For a complex matrix $X=(x_{ij})_{ij}$, put $X^*:=(\bar{x}_{ji})_{ij}$.

1. Invariant tensors

Let n and r be integers such that $2\leqslant r<n/2$.

Let us consider the two involutions σ and θ in the Lie group $G=\mathsf{U}(n,1):=\{g\in \mathrm{GL}_{n+1}(\mathbb{C})|\, g^*\mathrm{I}_{n,1}g=\mathrm{I}_{n,1}\}$ defined by $\theta(g)=\mathrm{I}_{n,1}\,g\,\mathrm{I}_{n,1}$ and $\sigma(g)=\mathrm{S}\,g\,\mathrm{S}$ respectively. Here $\mathrm{I}_{n,1}:=\mathrm{diag}(\mathrm{I}_n,-1)$ and $\mathrm{S}=\mathrm{diag}(\mathrm{I}_{n-r},-\mathrm{I}_r,1)$. Then $K:=\{g\in G|\,\theta(g)=g\}\cong\mathsf{U}(n)\times\mathsf{U}(1)$ is a maximal compact subgroup in G and $H:=\{g\in G|\,\sigma(g)=g\}\cong\mathsf{U}(n-r,1)\times\mathsf{U}(r)$ is a symmetric subgroup of G such that $K_H:=H\cap K\cong\mathsf{U}(n-r)\times\mathsf{U}(r)\times\mathsf{U}(1)$ is maximally compact in H.

The Lie group G acts transitively on the complex hyperball

$$\mathfrak{D} = \{z = {}^t(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |z_i|^2 < 1\}$$

by the fractional linear transformation $g.z = \frac{g_{11}z+g_{12}}{g_{21}z+g_{22}}$, $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in G, \quad z \in \mathbb{C}^n$. (Here the matrix $g \in \mathrm{GL}_{n+1}(\mathbb{C})$ is partitioned into blocks so that g_{11} is an $n \times n$-matrix and g_{22} is a scalar.) Since K is the stabilizer of the origin $0 \in \mathfrak{D}$, we have the identification $G/K \cong \mathfrak{D}$ of G-manifolds assigning the point $z = g.0$ to $g \in G$. Then $H/K_H \subset G/K$ corresponds to the H-orbit of 0 in $\mathfrak{D}$, that is $\mathfrak{D}^H := \{z \in \mathfrak{D} \mid z_{n-r+1} = \dots = z_n = 0\}$. In particular the real codimension of H/K_H in G/K is $2r$.
The Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ is realized in its complexification $\mathfrak{g}_\mathbb{C} = \mathfrak{gl}_{n+1}(\mathbb{C})$ as an $\mathbb{R}$-subalgebra of all $X \in \mathfrak{gl}_{n+1}(\mathbb{C})$ such that $X^* \, \mathrm{I}_{n,1} + \mathrm{I}_{n,1} X = 0_{n+1}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k} := \mathrm{Lie}(K)$ in $\mathfrak{g}$ with respect to the G-invariant, non-degenerate bi-linear form $\langle X, Y\rangle = 2^{-1}\mathrm{tr}(XY)$ on $\mathfrak{g}$. For $1 \leqslant i, j \leqslant n+1$, let $\mathrm{E}_{i,j} := (\delta_{ui}\delta_{vj})_{uv} \in \mathfrak{gl}_{n+1}(\mathbb{C})$ be the matrix unit. The operator $J := \mathrm{ad}(\tilde{Z}_0)|\mathfrak{p}$ with $\tilde{Z}_0 := \frac{\sqrt{-1}}{n+1}(\sum_{i=1}^n \mathrm{E}_{i,i} - n\mathrm{E}_{n+1,n+1})$ gives a K-invariant complex structure of $\mathfrak{p}$, which induces the K-invariant decomposition $\mathfrak{p}_\mathbb{C} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with $\mathfrak{p}_\pm$ the $(\pm\sqrt{-1})$-eigenspace of J in $\mathfrak{p}_\mathbb{C}$. Since $\mathfrak{p}$ is identified with the tangent space of G/K at K, we can extend J to the G-invariant complex structure of G/K making the identification $G/K \cong \mathfrak{D}$ bi-holomorphic. Put $X_i := \mathrm{E}_{i,n+1}$ $(1 \leqslant i \leqslant n-1)$, $X_0 := \mathrm{E}_{n,n+1}$. Then $\mathfrak{p}_+ = \sum_{i=0}^n \mathbb{C}X_i$, $\mathfrak{p}_- = \sum_{i=0}^n \mathbb{C}\bar{X}_i$ with $\bar{X}_i = \mathrm{E}_{n+1,i}$, $\bar{X}_0 = \mathrm{E}_{n+1,n}$. Let $\{\omega_i\}$ and $\{\bar{\omega}_i\}$ be the basis of $\mathfrak{p}_+^*$ and $\mathfrak{p}_-^*$ dual to $\{X_i\}$ and $\{\bar{X}_i\}$ respectively.

The exterior algebra $\bigwedge \mathfrak{p}_\mathbb{C}^*$ is decomposed to the direct sum of subspaces $\bigwedge^{p,q} \mathfrak{p}_\mathbb{C}^* := (\bigwedge^p \mathfrak{p}_+^*) \bigwedge (\bigwedge^q \mathfrak{p}_-^*)$ $(p, q \in \mathbb{N})$. Put

$$\omega := \frac{\sqrt{-1}}{2} \sum_{i=0}^{n-1} \omega_i \wedge \bar{\omega}_i \quad (\in \bigwedge^{1,1} \mathfrak{p}_\mathbb{C}^* \cap \bigwedge \mathfrak{p}^*), \qquad \mathrm{vol} := \frac{1}{n!}\omega^n \quad (\in \bigwedge^{n,n} \mathfrak{p}_\mathbb{C}^* \cap \bigwedge \mathfrak{p}^*).$$

The inner product $\langle X, Y\rangle$ on $\mathfrak{p}$ yields the Hermitian inner product $(\cdot|\cdot)$ of $\bigwedge \mathfrak{p}_\mathbb{C}^*$ in the standard way. Then the Hodge star operator $*$ is defined to be the $\mathbb{C}$-linear automorphism of $\bigwedge \mathfrak{p}_\mathbb{C}^*$ such that $*\bar{\alpha} = \overline{*\alpha}$ and such that $(\alpha|\beta)\,\mathrm{vol} = \alpha \wedge *\bar{\beta}$, $(\forall \alpha, \beta \in \bigwedge \mathfrak{p}_\mathbb{C}^*)$. For $\alpha \in \bigwedge \mathfrak{p}_\mathbb{C}^*$, let us define the endomorphism $e(\alpha) : \bigwedge \mathfrak{p}_\mathbb{C}^* \to \bigwedge \mathfrak{p}_\mathbb{C}^*$ by $e(\alpha)\beta = \alpha \wedge \beta$. As usual, we have the Lefschetz operator $L := e(\omega)$ and its adjoint operator Λ acting on the finite dimensional Hilbert space $\bigwedge \mathfrak{p}_\mathbb{C}^*$ ([8, Chap. V]).

Put $\mathfrak{h} = \mathrm{Lie}(H)$. Then θ restricts to a Cartan involution of $\mathfrak{h}$ giving the decomposition $\mathfrak{h} = (\mathfrak{h}\cap\mathfrak{k}) \oplus (\mathfrak{h}\cap\mathfrak{p})$. The complex structure J of $\mathfrak{p}$ induces that of $\mathfrak{h}\cap\mathfrak{p}$ by restriction giving the decomposition $(\mathfrak{h}\cap\mathfrak{p})_\mathbb{C} = (\mathfrak{h}\cap\mathfrak{p})_+ \oplus (\mathfrak{h}\cap\mathfrak{p})_-$ with $(\mathfrak{h}\cap\mathfrak{p})_+ = \mathfrak{h}_\mathbb{C} \cap \mathfrak{p}_+ = \sum_{i=1}^{n-r} \mathbb{C}X_i$ and $(\mathfrak{h}\cap\mathfrak{p})_- = \mathfrak{h}_\mathbb{C} \cap \mathfrak{p}_- = \sum_{i=1}^{n-r} \mathbb{C}\bar{X}_i$. We introduce two tensors ω_H and η as

$$\omega_H := \frac{\sqrt{-1}}{2} \sum_{i=1}^{n-r} \omega_i \wedge \bar{\omega}_i, \qquad \eta := \frac{\sqrt{-1}}{2} \sum_{j=n-r+1}^{n-1} \omega_i \wedge \bar{\omega}_i = \omega - \omega_H - \frac{\sqrt{-1}}{2}\omega_0 \wedge \bar{\omega}_0.$$

The coadjoint representation of K on $\mathfrak{p}^*$ is extended to the unitary representaion $\tau : K \to \mathrm{GL}(\bigwedge \mathfrak{p}_\mathbb{C}^*)$ in such a way that $\tau(k)(\alpha \wedge \beta) = \tau(k)\alpha \wedge \tau(k)\beta$ holds for all $\alpha, \beta \in \bigwedge \mathfrak{p}_\mathbb{C}^*$ and $k \in K$. The differential of τ is also denoted by τ.

The irreducible decomposition of the K-invariant subspaces $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^*$ is well-known.

Lemma 1. *Let p, q be non-negative integers such that $p+q \leqslant n$. Put*

$$F_{p,q} := \{\alpha \in \bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^* | \Lambda(\alpha) = 0\}.$$

Then $F_{p,q}$ is an irreducible K-invariant subspace of $\bigwedge \mathfrak{p}_{\mathbb{C}}^*$. The K-homomorphism L induces a linear injection $\bigwedge^{p-1,q-1}\mathfrak{p}_{\mathbb{C}}^* \to \bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^*$ whose image is the orthogonal complement of $F_{p,q}$ in $\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^*$,i.e.,

$$\bigwedge^{p,q}\mathfrak{p}_{\mathbb{C}}^* = F_{p,q} \bigoplus L(\bigwedge^{p-1,q-1}\mathfrak{p}_{\mathbb{C}}^*).$$

The $\mathbb{R}$-subspace $\mathfrak{a}$ of $\mathfrak{g}$ generated by the element $Y_0 := X_0 + \bar{X}_0 \in \mathfrak{p}$ is a maximal abelian subalgebra in $\mathfrak{q} \cap \mathfrak{p}$ with $\mathfrak{q}$ the (-1)-eigenspace of $d\sigma$, the differential of σ. Since (G, H) is a symmetric pair, by the general theory, the group G is a union of double cosets $Ha_tK\,(t \geqslant 0)$ with

$$a_t := \exp(tY_0) = \operatorname{diag}\left(\mathrm{I}_{n-1}, \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}\right), \qquad t \in \mathbb{R}.$$

Put $A = \{a_t | t \in \mathbb{R}\}$. Let M be the group of all the elements $k \in H \cap K$ such that $\mathrm{Ad}(k)Y_0 = Y_0$ and put $M = M_0 \cap H$. Then

$$M = \{\operatorname{diag}(u_1, u_2, u_0, u_0) | \, u_1 \in \mathsf{U}(n-r),\, u_2 \in \mathsf{U}(r-1),\, u_0 \in \mathsf{U}(1)\}.$$

Propostion 1. *Let p be an integer such that $0 < p < r$. Put*

$$\mathsf{v}_0^{(p)} = \frac{1}{n-p+1}\sum_{j=0}^{p} c_{p-j}^{(p)}\, L^{p-j}\left((n-p-j+1)\eta^j + \frac{\sqrt{-1}}{2} j(r-j)\,\omega_0 \wedge \bar{\omega}_0 \wedge \eta^{j-1}\right),$$

$$\mathsf{v}_1^{(p)} = \frac{-1}{p(n-2p+1)}\sum_{j=0}^{p} c_{p-j}^{(p)}\, L^{p-j}\left((p-j)\eta^j + \frac{\sqrt{-1}}{2} j(r-j)\,\omega_0 \wedge \bar{\omega}_0 \wedge \eta^{j-1}\right)$$

with

$$c_{p-j}^{(p)} = (-1)^j \binom{p}{j}\binom{n-p+1}{j}\binom{r-1}{j}^{-1}, \quad 0 \leqslant j \leqslant p.$$

Then $F_{p,p}^M$ is a two dimensional space generated by $\mathsf{v}_0^{(p)}$ and $\mathsf{v}_1^{(p)}$.

For convenience, we put $\mathsf{v}_0^{(0)} = 1$, $\mathsf{v}_1^{(0)} = 0$; these are elements of $F_{0,0} = \mathbb{C}$.

2. Secondary spherical functions

Before we state the main theorem of this section, we put a lemma which is important not only here but in the 'global theory' to be developed in §4.

Lemma 2. *For each integer p with $1 \leqslant p \leqslant r$, there exists a unique holomorphic function $s \mapsto \nu_s^{(p)}$ on the domain $\mathbb{C} - L_p$ with*

(1) $$L_p = \{s \in \sqrt{-1}\mathbb{R} |\, |\mathrm{Im}(s)| \leqslant 2\sqrt{(r-p)(n-p-r+2)}\}$$

which takes a positive real value for $s > 0$ and such that

$$\{\nu_s^{(p)}\}^2 = s^2 + 4(r-p)(n-p-r+2).$$

We have the functional equation $\nu_{-s}^{(p)} = -\nu_s^{(p)}$, $(s \in \mathbb{C} - L_p)$. If $\mathrm{Re}(s) > 0$, then we have $\mathrm{Re}(\nu_s^{(p)}) > \mathrm{Re}(\nu_s^{(p+1)}) > |\mathrm{Re}(s)|$.

For convenience, we put

$$\mu = r - 1, \qquad \lambda = n - 2r + 2.$$

Consider the holomorphic function

$$d(s) := \prod_{p=1}^{r} \Gamma(\nu_s^{(p)})^{-1}\Gamma(2^{-1}(\nu_s^{(p)} - \lambda) + 1)^{-1}, \quad s \in \mathbb{C} - L_1$$

and put

$$D = \{s \in \mathbb{C} - L_1 |\ d(s) \neq 0\}, \quad \tilde{D} = \bigcap_{p=1}^{\mu}\{s \in D |\ \mathrm{Re}(\nu_s^{(p)}) + \mathrm{Re}(\nu_s^{(p+1)}) > 4\}.$$

Theorem 1. *There exists a unique family of C^∞-functions $\phi_s : G - HK \to \bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*$ $(s \in \tilde{D})$ with the following conditions.*

(i) *For each $g \in G - HK$, the function $s \mapsto \phi_s(g)$ is holomorphic.*
(ii) *ϕ_s has the (H, K)-equivariance*

$$\phi_s(hgk) = \tau(k)^{-1}\phi_s(g), \quad h \in H,\ k \in K,\ g \in G - HK.$$

(iii) *ϕ_s satisfies the differential equation*

$$\Omega\phi_s(g) = (s^2 - \lambda^2)\phi_s(g), \quad g \in G - HK$$

(iv) *We have*

$$\lim_{t \to +0} t^{2\mu}\phi_s(a_t) = (\omega - \omega_H)^\mu.$$

(v) *If $\mathrm{Re}(s) > n$, then $\phi_s(a_t)$ decays exponentially as $t \to +\infty$.*

We call the function ϕ_s *the secondary spherical function.*

2.1. **Construction of ϕ_s.** We set

$$c(s) := \frac{\Gamma(s+1)\,\Gamma(\mu+2)}{\Gamma((s+n)/2+1)\,\Gamma((s-\lambda)/2+1)},$$

and

$$h_s(z) := {}_2F_1\Big(-\frac{s-n}{2}+1, -\frac{s+\lambda}{2}+1\,; \mu+2\,; z\Big),$$
$$H_s(z) := {}_2F_1\Big(\frac{s-n}{2}, \frac{s+\lambda}{2}\,; s+1\,; 1-z\Big).$$

Propostion 2. *Let $\{\gamma_p\}_{p=0}^{\mu}$ be the sequence of real numbers defined by the reccurence relation:*

$$\gamma_\mu = \frac{1}{c_0^{(\mu)}}, \quad \gamma_j\, c_0^{(j)} = -\sum_{p=j+1}^{\mu} \gamma_p\, c_{p-j}^{(p)}, \ (0 \leqslant j < \mu).$$

Then we have

$$\phi_s(ha_tk) = \mu\, r\Bigg\{\sum_{p=1}^{\mu} \frac{\gamma_p\,(n-p-r+1)p}{c(\nu_s^{(p+1)})\, c(\nu_s^{(p)})}\tau(k)^{-1}\Big(\tilde{f}_{01}^{(p)}(s\,;\tanh^2 t)\, \mathrm{v}_0^{(p)} + \tilde{f}_{11}^{(p)}(s\,;\tanh^2 t)\, \mathrm{v}_1^{(p)}\Big)$$

$$+\frac{\gamma_0}{c(\nu_s^{(1)})}\tilde{f}_{01}^{(0)}(s\,;\tanh^2 t)\, \mathrm{v}_0^{(0)},\Bigg\} \qquad \forall (h,t,k) \in H \times (0,\infty) \times K.$$

Here the functions $\tilde{f}_{ij}^{(p)}$ are given as follows.

- *For $p > 0$,*

$$\tilde{f}_{01}^{(p)}(s\,;z) = f_{00}^{(p)}(s\,;z)\, a_{01}^{(p)}(s\,;z) + f_{01}^{(p)}(s\,;z)\, a_{11}^{(p)}(s\,;z),$$
$$\tilde{f}_{11}^{(p)}(s\,;z) = f_{10}^{(p)}(s\,;z)\, a_{01}^{(p)}(s\,;z) + f_{11}^{(p)}(s\,;z)\, a_{11}^{(p)}(s\,;z)$$

with

$$a_{01}^{(p)}(s\,;z) = -z^{-\mu}(1-z)^{(\nu_s^{(p+1)}+\nu_s^{(p)})/2-1} H_{\nu_s^{(p+1)}}(z) H_{\nu_s^{(p)}}(z)$$
$$+ \int_1^z w^{-(\mu+1)}(1-w)^{(\nu_s^{(p+1)}+\nu_s^{(p)})/2-2}(1+w)\, H_{\nu_s^{(p+1)}}(w) H_{\nu_s^{(p)}}(w)\, dw,$$

$$a_{11}^{(p)}(s\,;z) = z(1-z)^{(-\nu_s^{(p+1)}+\nu_s^{(p)})/2-1} h_{\nu_s^{(p+1)}}(z) H_{\nu_s^{(p)}}(z)$$
$$- \int_0^z (1-w)^{(-\nu_s^{(p+1)}+\nu_s^{(p)})/2-2}(1+w)\, h_{\nu_s^{(p+1)}}(w) H_{\nu_s^{(p)}}(w)\, dw$$

and

$$f_{10}^{(p)}(s\,;z) = (1-z)^{(-\nu_s^{(p+1)}+n)/2+1}\, h_{\nu_s^{(p+1)}}(z),$$

$$f_{11}^{(p)}(s\,;z) = z^{-(\mu+1)}(1-z)^{(\nu_s^{(p+1)}+n)/2+1}\, H_{\nu_s^{(p+1)}}(z),$$

$$f_{00}^{(p)}(s\,;z) = -\frac{(1-z)^{(-\nu_s^{(p+1)}+n)/2}}{(n-p-r+1)p}$$
$$\times \left(z(1-z)\frac{d}{dz} + \frac{\nu_s^{(p+1)}+n-2p}{2}z + \frac{(r-p)(n-p+1)}{n-2p+1}(1-z)\right) h_{\nu_s^{(p+1)}}(z),$$

$$f_{01}^{(p)}(s\,;z) = -\frac{z^{-(\mu+1)}(1-z)^{(\nu_s^{(p+1)}+n)/2}}{(n-p-r+1)p}$$
$$\times \left(z(1-z)\frac{d}{dz} + \frac{-\nu_s^{(p+1)}+n-2p}{2}z - \frac{p(n-p-r+1)}{n-2p+1}(1-z)\right) H_{\nu_s^{(p+1)}}(z).$$

- *For $p = 0$,*

$$\tilde{f}_{01}^{(0)}(s\,;z) = \frac{2\, z^{-\mu}(1-z)^{(\nu_s^{(1)}+n)/2}}{\nu_s^{(1)}+n}\, {}_2F_1\left(\frac{\nu_s^{(1)}-n}{2}+1, \frac{\nu_s^{(1)}+\lambda}{2}\,; \nu_s^{(1)}+1\,; 1-z\right).$$

2.2. **Some properties of the secondary spherical function.**

Theorem 2. *Let ϕ_s $(s \in \tilde{D})$ be the secondary spherical function constructed in Theorem 1.*

- *There exist μ polynomial functions $\mathsf{a}_\alpha(s)$ with values in $(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*)^M$, positive number ϵ and $(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*)^M$-valud holomorphic functions $\mathsf{b}_i(s,z)$ $(i = 0,1,2)$ on $\{(s,z) | s \in \tilde{D},\ |z| < \epsilon\}$ such that*

$$\mathsf{a}_0(s) = (\omega - \omega_H)^\mu,$$
$$\mathsf{a}_\alpha(-s) = \mathsf{a}_\alpha(s), \quad \deg(\mathsf{a}_\alpha(s)) \leqslant 2\alpha$$

and such that

$$\phi_s(a_t) = \sum_{\alpha=0}^{\mu-1} \frac{\mathsf{a}_\alpha(s)}{z^{\mu-\alpha}} + \mathsf{b}_0(s\,;z) + \mathsf{b}_1(s\,;z)\,\log z + \mathsf{b}_2(s\,;z)\,z^{\mu+2}(\log z)^2,$$
$$s \in \tilde{D},\ z = \tanh^2 t \in (0,\epsilon).$$

- *There exists a positive number ϵ', $(\bigwedge^{\mu,\mu}\mathfrak{p}_{\mathbb{C}}^*)^M$-valued holomorphic functions $\mathsf{f}^{(p)}(s\,;y)$ $(0 \leqslant p \leqslant \mu)$ on $\{(s,y) | \, |y| < \epsilon',\ \mathrm{Re}(s) > n\}$ such that*

$$\phi_s(a_t) = \sum_{p=0}^{\mu} y^{(\nu_s^{(p)}+n)/2}\,\mathsf{f}^{(p)}(s\,;y), \quad \mathrm{Re}(s) > n,\ y = \frac{1}{\cosh^2 t} \in (0,\epsilon')$$

2.3. **The function ψ_s.** For each $s \in \tilde{D}$, let us define the function $\psi_s : G - HK \to \bigwedge^{r,r}\mathfrak{p}_{\mathbb{C}}^*$ by

$$\psi_s(g) = \sum_{i,j=0}^{n-1} R_{X_i\bar{X}_j}\phi_s(g) \wedge \omega_i \wedge \bar{\omega}_j, \quad g \in G - HK. \tag{2}$$

Theorem 3. • *The function ψ_s is C^∞ on $G - HK$ and satisfies*

$$\psi_s(hgk) = \tau(k)^{-1}\psi_s(g), \quad \forall h \in H, \forall g \in G - HK, \forall k \in K.$$

- *There exist μ $(\bigwedge^{r,r}\mathfrak{p}_{\mathbb{C}}^*)^M$-valued polynomial functions $\tilde{\mathsf{c}}_\alpha(s)$, positive number ϵ and $(\bigwedge^{r,r}\mathfrak{p}_{\mathbb{C}}^*)^M$-valud holomorphic functions $\mathsf{d}_i(s,z)$ $(i = 0,1,2)$ on $\{(s,z) | s \in \tilde{D},\ |z| < \epsilon\}$ such that*

$$\tilde{\mathsf{c}}_0(s) = -\frac{\sqrt{-1}}{2}\frac{(r-1)!}{(n-r)!}(\omega - \omega_H)^r,$$
$$\tilde{\mathsf{c}}_\alpha(-s) = \tilde{\mathsf{c}}_\alpha(s), \quad \deg(\tilde{\mathsf{c}}_\alpha(s)) \leqslant 2\alpha$$

and

$$\psi_s(a_t) = (s^2 - \lambda^2)\sum_{\alpha=0}^{\mu-1} \frac{\tilde{\mathsf{c}}_\alpha(s)}{z^{\mu-\alpha}} + \mathsf{d}_0(s\,;z) + \mathsf{d}_1(s\,;z)\log z + \mathsf{d}_2(s\,;z)\,z^\mu(\log z)^2,$$
$$s \in \tilde{D},\ z = \tanh^2 t \in (0,\epsilon).$$

- *There exists a positive number ϵ', $(\bigwedge^{r,r}\mathfrak{p}_{\mathbb{C}}^*)^M$-valued holomorphic functions $\mathrm{g}^{(p)}(s\,;y)$ $(0 \leqslant p \leqslant r)$ on $\{(s,y)|\,\mathrm{Re}(s)>n,\ |y|<\epsilon'\}$ such that*

$$\psi_s(a_t)=\sum_{p=0}^{r} y^{(\nu_s^{(p)}+n)/2}\,\mathrm{g}^{(p)}(s\,;y),\quad \mathrm{Re}(s)>n,\ y=\frac{1}{\cosh^2 t}\in(0,\epsilon')$$

3. Poincaré series

Let Γ be a discrete subgroup of G. We assume that (G,H,Γ) is arranged as follows. There exists a connected reductive $\mathbb{Q}$-group G, a $\mathbb{Q}$-subgroup H of G and an arithmetic subgroup Δ of $\mathsf{G}(\mathbb{Q})$ such that there exists a morphism of Lie groups from $\mathsf{G}(\mathbb{R})$ onto G with compact kernel which maps $\mathsf{H}(\mathbb{R})$ onto H and Δ onto Γ.

3.1. **Invariant measures.** Let dk and dk_0 be the Haar measures of compact groups K and K_H with total volume 1. Then we can take a unique Haar measure dg (resp. dh) of G (resp. H) such that the quotient measure $\frac{dg}{dk}$ (resp. $\frac{dh}{dk_0}$) corresponds to the measure on the symmetric space G/K (resp. H/K_H) determined by the invariant volume form vol (resp. vol_H).

Lemma 3. *For any measurable functions f on G we have*

$$\int_G f(g)\,dg=\int_H dh\int_K dk\int_0^\infty f(ha_tk)\varrho(t)\,dt$$

with dt the usual Lebesgue measure on $\mathbb{R}$ and

$$\varrho(t)=2\,c_r\,(\sinh t)^{2r-1}(\cosh t)^{2n-2r+1},\quad c_r=\frac{\pi^r}{\mu!}.$$

3.2. **Currents defined by Poincaré series.** Let $\mathfrak{F}$ denote the set of the families of functions $\{\varphi_s\}_{s\in\tilde{D}}$ such that $\varphi_s=\partial_s\phi_s$ $(s\in\tilde{D})$ or $\varphi_s=\partial_s\psi_s$ $(s\in\tilde{D})$ with some differential operator ∂_s with holomorphic coefficient on $\tilde{D}$.

For $\{\varphi_s\}\in\mathfrak{F}$, let us introduce the Poincare seires

$$\tilde{P}(\varphi_s)(g)=\sum_{\gamma\in\Gamma_H\backslash\Gamma}\varphi_s(\gamma g)\qquad g\in G, \tag{3}$$

which is the most basic object in our investigation. First of all, we discuss its convergence in a weak sense. Note that φ_s takes its values in the finite dimensional Hilbert space $\bigwedge\mathfrak{p}_{\mathbb{C}}^*$ with the norm $\|\alpha\|=(\alpha|\alpha)^{1/2}$.

Theorem 4. *The function in s defined by the integral*

$$\tilde{P}(\|\varphi_s\|)(g):=\int_{\Gamma\backslash G}\Big(\sum_{\gamma\in\Gamma_H\backslash\Gamma}\|\varphi_s(\gamma g)\|\Big)\,d\dot{g}$$

is locally bounded on $\mathrm{Re}(s)>n$. For each s with $\mathrm{Re}(s)>n$, the series (3) *converges absolutely almost everywhere in $g\in G$ to define an L^1-function on $\Gamma\backslash G$.*

If Γ is neat, then the quotient space $\Gamma\backslash G/K$ acquires a structure of complex manifold from the one on $G/K\cong\mathfrak{D}$. Let $\pi: G/K\to\Gamma\backslash G/K$ be the natural projection. Let $A(\Gamma\backslash G/K)$ denote the space of C^∞-differential forms on $\Gamma\backslash G/K$ and $A_c(\Gamma\backslash G/K)$ the subspace of compactly supported forms. Given $\alpha \in A(\Gamma\backslash G/K)$, we have a unique C^∞-function $\tilde{\alpha} : G \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ such that $\tilde{\alpha}(\gamma g k) = \tau(k)^{-1}\tilde{\alpha}(g)$, $(\gamma \in \Gamma,\ k \in K)$ and such that

$$\langle (\pi^*\alpha)(gK), (\wedge dL_g)(\xi_o)\rangle = \langle \tilde{\alpha}(g), \xi_o\rangle, \quad g \in G,\ \xi_o \in \bigwedge \mathfrak{p} = \bigwedge \mathrm{T}_o(G/K)$$

holds. Here L_g denotes the left translation on G/K by the element g and we identify $\mathfrak{p}$ with $\mathrm{T}_o(G/K)$, the tangent space of G/K at $o = eK$.

For any left Γ-invariant continuous function f on G, put

$$\mathfrak{I}_H(f\,;g) = \int_{\Gamma_H\backslash H} f(hg)\,dh, \quad g \in G.$$

We already discussed the convergence problem of this integral in [7, 3.2]. For convenience we recall the result. If Γ is co-compact, we take a compact fundamental domain $\mathfrak{S}^1$ for Γ in G and $t_{\mathfrak{S}^1}$ the constant function 1. Hence $G = \Gamma\mathfrak{S}^1$ in this case. If Γ is not co-compact, then one can fix a complete set of representatives P^i $(1 \leqslant i \leqslant h)$ of Δ-conjugacy classes of $\mathbb{Q}$-parabolic subgroups in G together with $\mathbb{Q}$-split tori $\mathbb{G}_m \cong \mathsf{A}^i$ in the radical of P^i such that an eigencharacter of $\mathrm{Ad}(t)$ $(t \in \mathbb{G}_m)$ in the Lie algebra of P^i is one of t^j $(j = 0, 1, 2)$. For each i, let T^i and N^i be the images in G of $\mathsf{A}^i(\mathbb{R})$ and the unipotent radical of $\mathsf{P}^i(\mathbb{R})$ respectively. Then we can choose a Siegel domain $\mathfrak{S}^i$ in G with respect to the Iwasawa decomposition $G = N^iT^iK$ for each i such that G is a union of $\Gamma\mathfrak{S}^i$ $(1 \leqslant i \leqslant h)$. Let $t_{\mathfrak{S}^i} : \mathfrak{S}^i \to (0, \infty)$ be the function $t_{\mathfrak{S}^i}(n_i\,\underline{t}_i\,k) = t$, $(n_i\,\underline{t}_i\,k \in \mathfrak{S}^i)$. Here $\underline{t}_i$ denote the image of $t \in \mathbb{G}_m(\mathbb{R}) \cong \mathsf{A}^i(\mathbb{R})$ in T^i.

Given $\delta \in (2rn^{-1}, 1)$, let $\mathfrak{M}_\delta$ be the space of all left Γ-invariant C^∞-functions $f : G \to \bigwedge \mathfrak{p}_{\mathbb{C}}^*$ with the K-equivariance $f(gk) = \tau(k)^{-1}f(g)$ such that for any $\epsilon \in (0, \delta)$ and $D \in U(\mathfrak{g}_{\mathbb{C}})$ the estimation

$$\|R_D\varphi(g)\| \prec t_{\mathfrak{S}^i}(g)^{(2-\epsilon)n}, \quad \forall g \in \mathfrak{S}^i,\ \forall i$$

holds.

Propostion 3. *Let $f \in \mathfrak{M}_\delta$ with $\delta \in (2rn^{-1}, 1)$ and $D \in U(\mathfrak{g}_{\mathbb{C}})$.*

- *We have*

$$\mathfrak{I}_H(\|R_D f\|\,; a_t) \prec e^{(2-\epsilon)nt} \quad t \geqslant 0$$

for any $\epsilon \in (2rn^{-1}, \delta)$. The function $\mathfrak{I}_H(f\,;g)$ is of class C^∞, belongs to C_τ^∞ and

$$\mathfrak{I}_H(R_D f\,;g) = R_D\mathfrak{I}_H(f\,;g), \quad g \in G.$$

- *For any $\{\varphi_s\} \in \mathfrak{F}$, the integral*

$$\int_{\Gamma\backslash G} |(\tilde{P}(\varphi_s)(g)|R_D f(g))|\,d\dot{g}$$

is finite if $\mathrm{Re}(s) > 3n - 2r$. We have

$$\int_{\Gamma\backslash G} (\tilde{P}(\varphi_s)(g)|R_D f(g))\,d\dot{g} = \int_0^\infty \varrho(t)\,(\varphi_s(a_t)|R_D\mathfrak{I}_H(f\,;a_t))\,d\dot{g}.$$

Propostion 4. *There exists a unique current $P(\varphi_s)$ on $\Gamma\backslash G/K$ such that*

$$\begin{aligned}\langle P(\varphi_s), *\overline{\alpha}\rangle &= \int_{\Gamma\backslash G} (\tilde{P}(\varphi_s)(g)|\tilde{\alpha}(g))d\dot{g} \\ &= \int_0^\infty \varrho(t)\,(\varphi_s(a_t)|\mathfrak{I}_H(\tilde{\alpha}\,;a_t))\,dt, \quad \alpha \in A_c(\Gamma\backslash G/K)\end{aligned}$$

Let ∂_s be a holomorphic differential operator on $\tilde{D}$. Then for any $\alpha \in A_c(\Gamma\backslash G/K)$, the function $s \mapsto \langle P(\varphi_s), \alpha\rangle$ is holomorphic on $\tilde{D}$ and $\partial_s\langle P(\varphi_s), \alpha\rangle = \langle P(\partial_s\varphi_s), \alpha\rangle$.

Definition

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n$, we put

$$\begin{aligned}\tilde{G}_s &:= \tilde{P}(\phi_s), \quad \tilde{\Psi}_s := \tilde{P}(\psi_s), \\ G_s &:= P(\psi_s), \quad \Psi_s := P(\psi_s).\end{aligned}$$

The current G_s and Ψ_s on $\Gamma\backslash G/K$ are of type $(r-1, r-1)$ and of type (r, r) respectively.

4. Spectral expansion

In this section we investigate the spectral expansion of the functions $\boldsymbol{\delta}_{j,s}\tilde{G}_s$ with

$$\boldsymbol{\delta}_{j,s} := \frac{1}{j!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^j, \quad j \in \mathbb{N}$$

to obtain a meromorphic continuation of the current-valued function $s \mapsto G_s$, which is already holomorphic on the half plane $\mathrm{Re}(s) > n$.

4.1. **Spectral expansion.** In order to describe the spectral decomposition of the function $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$, we need some preparations.
For $q > 0$, let $\mathcal{L}^q_\Gamma(\tau)$ denote the Banach space of all measurable functions $f : G \to \bigwedge \mathfrak{p}^*_{\mathbb{C}}$ such that $f(\gamma g k) = \tau(k)^{-1}f(g)$, $(\forall\gamma \in \Gamma, \forall k \in K)$ and $\int_{\Gamma\backslash G} \|f(g)\|^q\,d\dot{g} < \infty$. For $0 \leqslant d \leqslant n$, let $\mathcal{L}^q_\Gamma(\tau)^{(d)}$ denote the subspace of those functions $f \in \mathcal{L}^q_\Gamma(\tau)$ with values in $\bigwedge^{d,d}\mathfrak{p}^*_{\mathbb{C}}$. The inner product of two functions f_1 and f_2 in $\mathcal{L}^2_\Gamma(\tau)$ is given as $\langle f_1|f_2\rangle = \int_{\Gamma\backslash G}(f_1(g)|f_2(g))\,d\dot{g}$. Let $\tilde{\Delta}$ be the operator on $\mathcal{L}^2_\Gamma(\tau)$ whose action on the smooth functions in $\mathcal{L}^2_\Gamma(\tau)$ is induced by $-R_\Omega$. For each $0 \leqslant d \leqslant n$, let $\{\lambda^{(d)}_n\}_{n\in\mathbb{N}}$ be the increasing sequence of the eigenvalues of the bidegree (d,d)-part of $\tilde{\Delta}$ such that each eigenvalue occurs with its multiplicity. Choose an orthonormal system $\{\tilde{\alpha}^{(d)}_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}^2_\Gamma(\tau)^{(d)}$ consisting of automorphic forms such that $\tilde{\Delta}\tilde{\alpha}^{(d)}_n = \lambda^{(d)}_n\tilde{\alpha}^{(d)}_n$ for each n and put $\mathcal{L}^2_{\Gamma,\mathrm{dis}}(\tau)^{(d)}$ to be the closed span of the functions $\tilde{\alpha}^{(d)}_n$ in $\mathcal{L}^2_\Gamma(\tau)^{(d)}$. When Γ is co-compact we have $\mathcal{L}^2_{\Gamma,\mathrm{dis}}(\tau)^{(d)} = \mathcal{L}^2_\Gamma(\tau)^{(d)}$. Otherwise we need the Eisenstein series to describe the orthogonal complement of $\mathcal{L}^2_{\Gamma,\mathrm{dis}}(\tau)^{(d)}$.

Recall the parabolic subgroups P^i used to construct the Siegel domains $\mathfrak{S}^i$ (see 3.2). Let $P^i = M^i_0T^iN^i$ be its Langlands decomposition with $M^i_0 := Z_K(T^i)$. For each i let $\Gamma_{P^i} = \Gamma \cap P^i$ and $\Gamma_{M^i_0} = M^i_0 \cap (\Gamma_{P^i}N^i)$. Then $\Gamma_{M^i_0}$ is just a finite subgroup of the compact group M^i_0.

For a vector $\mathsf{u} \in \mathsf{V}_i^{(d)} := (\bigwedge^{d,d} \mathfrak{p}_{\mathbb{C}}^*)^{\Gamma_{M_0^i}}$ and a complex number s, let us define the function $\varphi_s^i(\mathsf{u}\,;g)$ on G using the Iwasawa decomposition $G = N^i T^i K$ by

$$\varphi_s^i(\mathsf{u}\,; n_i \underline{t}_i\, k) = t^{s+n}\tau(k)^{-1}\mathsf{u}, \quad n_i \in N^i,\ t > 0,\ k \in K.$$

Then the Eisenstein series associated with u is defined by the infinite series

$$\mathsf{E}^i(s\,;\mathsf{u}\,;g) = \sum_{\gamma \in \Gamma_{P^i}\backslash\Gamma} \varphi_s^i(\mathsf{u}\,;\gamma g), \quad g \in G \tag{4}$$

By the general theory, the series is convergent in $\mathrm{Re}(s) > n$ normally and the function $g \mapsto \mathsf{E}^i(s\,;\mathsf{u}\,;g)$ is an automorphic form on $\Gamma\backslash G$. Moreover there exists a family of linear maps $\mathsf{E}^i(s)$ from $\mathsf{V}_i^{(d)}$ to the space of automorphic forms on $\Gamma\backslash G$, which depends meromorphically on $s \in \mathbb{C}$ and is holomorphic on the imaginary axis, such that $(\mathsf{E}^i(s)(\mathsf{u}))(g) = \mathsf{E}^i(s\,;\mathsf{u}\,;g)$ coincides with (4) when $\mathrm{Re}(s) > n$. For each $1 \leqslant i \leqslant h$, let $\Omega_{M_0^i}$ be the Casimir element of M_0^i corresponding to the invariant form $\langle X, Y\rangle$ on its Lie algebra. Then if $\mathsf{u} \in \mathsf{V}_i^{(d)}$ is an eigenvector of $\tau(\Omega_{M_0^i})$ with eigenvalue $c \in \mathbb{C}$, then $R_\Omega \mathsf{E}(s\,;\mathsf{u}) = (s^2 - n^2 + c)\,\mathsf{E}^i(s,\ \mathsf{u})$ for any $s \in \mathbb{C}$ where $\mathsf{E}^i(s)$ is regular.

Lemma 4. *For $0 \leqslant p \leqslant d$ and $\epsilon \in \{0,1\}$, let $\mathsf{W}_i^{(d)}(p\,;\epsilon)$ be the eigenspace of $\tau(\Omega_{M_0^i})$ on $\mathsf{V}_i^{(d)}$ corresponding to the eigenvalue $(2p-\epsilon)(2n-2p+\epsilon)$. Then we have the orthogonal decomposition*

$$\mathsf{V}_i^{(d)} = \bigoplus_{p=0}^{\mu} \bigoplus_{\epsilon \in \{0,1\}} \mathsf{W}_i^{(d)}(p\,;\epsilon).$$

For each index (d, i, p, ϵ), fix an orthonormal basis $\mathcal{B}_i^{(d)}(p\,;\epsilon)$ of the space $\mathsf{W}_i^{(d)}(p\,;\epsilon)$.

4.2. Some properties of Eisenstein period.

Propostion 5.
- *For $1 \leqslant i \leqslant h$ and $\mathsf{u} \in \mathsf{V}_i^{(d)}$, there exists a unique $\bigwedge^{d,d}\mathfrak{p}_{\mathbb{C}}^*$-valued meromorphic function $\mathcal{P}_H^i(s\,;\mathsf{u})$ on $\mathbb{C}$ which is regular and has the value given by the absolutely convergent integral $\mathcal{I}_H(\mathsf{E}^i(s\,;\mathsf{u})\,;e)$ at any regular point $s \in \mathbb{C}$ of $\mathsf{E}^i(s\,;\mathsf{u})$ in $|\mathrm{Re}(s)| < 1 - 2rn^{-1}$.*
- *Let $1 \leqslant i \leqslant h$ and $1 \leqslant p \leqslant d$. Then for any $\mathsf{u} \in \mathsf{W}_i^{(d)}(p\,;1)$, we have $\mathcal{P}_H^i(s\,;\mathsf{u}) = 0$ identically.*

4.3. Meromorphic continuation and functional equations.

Put $\mathsf{w} := (\omega - \omega_H)^\mu$.

Theorem 5. *Let $\mathrm{Re}(s) > 3n - 2r$. Then there exists $\epsilon > 0$ such that the function $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s(g)$ belongs to the space $\mathcal{L}_\Gamma^{2+\epsilon}(\tau)^{(\mu)}$. The spectral expansion of $\boldsymbol{\delta}_{\mu,s}\tilde{G}_s$ is given as*

$$\boldsymbol{\delta}_{\mu,s}\tilde{G}_s = \sum_{m=0}^{\infty} \frac{4\,(\mathsf{w}|\mathcal{I}_H(\tilde{\alpha}_m^{(\mu)}\,;e))}{\mu!\,(\lambda^2 - \lambda_m^{(\mu)} - s^2)^r}\,\tilde{\alpha}_m^{(\mu)}$$
$$+ \sum_{p=0}^{\mu} \frac{1}{4\pi\sqrt{-1}} \int_{\sqrt{-1}\mathbb{R}} \sum_{i=1}^{h} \sum_{\mathsf{u} \in \mathcal{B}_i^{(\mu)}(p\,;0)} \frac{4\,(\mathsf{w}|\mathcal{I}_H(\mathsf{E}^i(\zeta\,;\mathsf{u})\,;e))}{\mu!\,(\zeta^2 - (\nu_s^{(p+1)})^2)^r}\,\mathsf{E}^i(\zeta\,;\mathsf{u})\,d\zeta,$$

where the summations in the right-hand side of this formula are convergent in $\mathcal{L}_\Gamma^2(\tau)^{(\mu)}$.

Let $\mathcal{K}_\Gamma(\tau)$ be the space of C^∞-functions $\tilde\beta : G \to \bigwedge \mathfrak{p}_\mathbb{C}^*$ with compact support modulo Γ such that $\tilde\beta(\gamma g k) = \tau(k)^{-1}\tilde\beta(g)$ $(\forall\gamma \in \Gamma, \forall k \in K)$.

Theorem 6. *Let L_1 be the interval on the imaginary axis defined by* (1). *Let $0 \leqslant j \leqslant \mu$. Then for each $\tilde\beta \in \mathcal{K}_\Gamma(\tau)$ the holomorphic function $s \mapsto \mathcal{G}_j(s, \tilde\beta) := \langle \boldsymbol{\delta}_{j,s}\tilde G_s|\tilde\beta\rangle$ on* $\mathrm{Re}(s) > n$ *has a meromorphic continuation to the domain $\mathbb{C} - L_1$. A point $s_0 \in \mathbb{C} - L_1$ with* $\mathrm{Re}(s_0) \geqslant 0$ *is a pole of the meromorphic function $\mathcal{G}_j(s, \beta)$ if and only if there exists an $m \in \mathbb{N}$ such that $(\mathsf{w}|\mathcal{I}_H(\tilde\alpha_m^{(\mu)}\,; e)) \neq 0$, $\langle \tilde\alpha_m^{(\mu)}|\tilde\beta\rangle \neq 0$ and $s_0^2 - \lambda^2 = -\lambda_m^{(\mu)}$. In this case, the function*

$$\mathcal{G}_j(s,\beta) - \sum_{m\in\mathbb{N};\lambda_m^{(\mu)}=\lambda^2-s_0^2} \frac{4(\mathsf{w}|\mathcal{I}_H(\tilde\alpha_m^{(\mu)}\,;e))\langle\tilde\alpha_m^{(\mu)}|\tilde\beta\rangle}{\mu!\,(s_0^2-s^2)^{j+1}}$$

is holomorphic at $s = s_0$. We have the functional equation

$$\mathcal{G}_j(-s,\tilde\beta) - \mathcal{G}_j(s,\tilde\beta) = (-1)^\mu\,\boldsymbol{\delta}_{j,s}\left(\sum_{p=0}^{\mu}\frac{\langle\tilde{\mathcal{E}}_p^{(\mu)}(\nu_s^{(p+1)})|\tilde\beta\rangle}{2\,\nu_s^{(p+1)}}\right).$$

with

$$\tilde{\mathcal{E}}_p^{(\mu)}(\nu\,;g) := \frac{4}{\mu!}\sum_{i=1}^{h}\sum_{\mathsf{u}\in\mathcal{B}_i^{(\mu)}(p\,;0)}(\mathsf{w}|\mathcal{P}_H^i(-\bar\nu\,;\mathsf{u}))\,\mathsf{E}^i(\nu\,;\mathsf{u}\,;g),\quad g\in G,\ \nu\in\mathbb{C}. \tag{5}$$

5. Green currents

We put the Kähler form $\boldsymbol{\omega}$ on $\Gamma\backslash G/K$ such that $\tilde{\boldsymbol{\omega}}(g) = \omega\,(\forall g \in G)$. The metric on $\Gamma\backslash G/K$ corresponding to $\boldsymbol{\omega}$ defines the Laplacian Δ, the Lefschetz operator and its adjoint Λ acting on the space of forms and currents on $\Gamma\backslash G/K$.

5.1. **Currents defined by modular cycles.** Let D be the image of the map $\Gamma_H\backslash H/K_H -$ $\Gamma\backslash G/K$ induced by the natural holomorphic inclusion $H/K_H \hookrightarrow G/K$. Then D, a closed complex analytic subset of $\Gamma\backslash G/K$, defines an (r,r)-current δ_D on $\Gamma\backslash G/K$ by the integration

$$\langle\delta_D,\alpha\rangle = \int_{D_{\mathrm{ns}}} j^*\alpha,\quad \alpha\in A_c(\Gamma\backslash G/K). \tag{6}$$

Here $j : D \hookrightarrow \Gamma\backslash G/K$ is the natural inclusion and D_{ns} is the smooth locus of D. Since δ_D is real and closed, it defines a cycle on $\Gamma\backslash G/K$ of real codimension $2r$ ([4, p.32–33]).

5.2. **Differential equations.**

Theorem 7. *Let* $\mathrm{Re}(s) > n$. *Then we have*

$$\begin{aligned}
&(\Delta + s^2 - \lambda^2)G_s = -4\Lambda\delta_D,\\
&\Delta\,\Psi_s = (\lambda^2 - s^2)(\Psi_s - 2\sqrt{-1}\delta_D),\\
&\partial\bar\partial G_s = \Psi_s - 2\sqrt{-1}\,\delta_D.
\end{aligned}$$

5.3. **Main theorem.** Let $A^{p,q}_{(2)}(\Gamma\backslash G/K)$ be the Hilbert space of the measurable (p,q)-forms on $\Gamma\backslash G/K$ with the finite L^2-norm $\|\alpha\| := \int_{\Gamma\backslash G/K} \alpha \wedge *\bar{\alpha}$. For each $c \in \mathbb{C}$, let $A^{p,q}_{(2)}(\Gamma\backslash G/K\,; c)$ be the c-eigenspace of the Laplacian Δ acting on $A^{p,q}_{(2)}(\Gamma\backslash G/K)$. In particular, $\mathcal{H}^{p,q}_{(2)}(\Gamma\backslash G/K) := A^{p,q}_{(2)}(\Gamma\backslash G/K\,;0)$ is the space of the harmonic L^2-forms of (p,q)-type. For each p, let $\mathcal{E}^{(\mu)}_p(\nu)$ be the C^∞-form of (μ,μ)-type on $\Gamma\backslash G/K$ corresponding to the function $\tilde{\mathcal{E}}^{(\mu)}_p(\nu)$ on G defined by (5). Then Theorem 6 immediately gives us the following theorem.

Theorem 8. *There exists a meromorphic family of (μ,μ)-currents $G_s\,(s \in \mathbb{C} - L_1)$ on $\Gamma\backslash G/K$ with the following properties.*

- *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n$, it is given by*

$$\langle G_s, *\bar{\alpha}\rangle = \frac{1}{(r-1)\pi^r}\int_0^\infty \varrho(t)\,(\phi_s(a_t)|\mathfrak{I}_H(\bar{\alpha}\,;a_t))\,dt, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

- *A point $s_0 \in \mathbb{C} - L_1$ with $\mathrm{Re}(s) \geqslant 0$ is a pole of G_s if and only if there exists an L^2-form $\alpha \in A^{r-1,r-1}_{(2)}(\Gamma\backslash G/K\,; (n-2r+2)^2 - s_0^2)$ such that*

$$\int_D j^*(\omega \wedge \bar{\alpha}) \neq 0.$$

  In this case s_0 is a simple pole with the residue

$$\mathrm{Res}_{s=s_0} G_s = \frac{2}{s_0}\sum_m \left(\int_D j^*(\omega \wedge \bar{\alpha}_m)\right)\cdot \alpha_m.$$

  Here $\{\alpha_m\}$ is an arbitrary orthonormal basis of $A^{r-1,r-1}_{(2)}(\Gamma\backslash G/K\,;(n-2r+2)^2 - s_0^2)$.
- *The functional equation*

$$G_{-s} - G_s = (-1)^{r-1}\sum_{p=0}^{r-1}\frac{\mathcal{E}^{(r-1)}_p(\nu^{(p+1)}_s)}{2\nu^{(p+1)}_s}, \quad s \in \mathbb{C} - L_1$$

  holds.

Theorem 9. *There exists a meromorphic family of (r,r)-currents $\Psi_s\,(s \in \mathbb{C} - L_1)$ on $\Gamma\backslash G/K$ with the following properties.*

- *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > n$, it is given by*

$$\langle \Psi_s, *\bar{\alpha}\rangle = \frac{1}{(r-1)\pi^r}\int_0^\infty \varrho(t)\,(\psi_s(a_t)|\mathfrak{I}_H(\bar{\alpha}\,;a_t))\,dt, \quad \alpha \in A_c(\Gamma\backslash G/K).$$

- *Ψ_s is holomorphic at $s = n - 2r + 2$.*

Definition

We define the $(r-1, r-1)$-current $\mathcal{G}$ on $\Gamma\backslash G/K$ to be the quarter of the constant term of the Laurent expansion of G_s at $s = \lambda$. Namely, if $\{\alpha_m\}$ is any orthonormal basis of $\mathcal{H}^{r-1,r-1}_{(2)}(\Gamma\backslash G/K)$, then we put

$$\mathcal{G}(x) = \frac{1}{4}\lim_{s\to\lambda}\left(G_s(x) - \frac{2}{n-2r+2}\sum_m \int_D j^*(\omega\wedge\bar{\alpha}_m)\,\frac{\alpha_m(x)}{s-(n-2r+2)}\right).$$

Theorem 10. *We have the equation*

$$dd_c\,\mathcal{G} = \frac{\sqrt{-1}}{2}\Psi_{n-2r+2} + \delta_D, \quad \triangle\Psi_{n-2r+2} = 0$$

The current Ψ_{n-2r+2} *is represented by an element of* $A^{r,r}(\Gamma\backslash G/K)$.

6. The current Ψ_s

We remark that $*\mathrm{vol}_H = \frac{1}{r!}(\omega - \omega_H)^r$ with $\mathrm{vol}_H = \frac{1}{(n-r)!}\omega_H^{n-r}$ the 'volume form' of H/K_H.

Theorem 11. *Let* $\mathrm{Re}(s) > 3n - 2r$. *Then there exists* $\epsilon > 0$ *such that the function* $\boldsymbol{\delta}_{\mu,s}((s^2-\lambda^2)^{-1}\tilde{\Psi}_s)$ *belongs to the space* $\mathcal{L}_\Gamma^{2+\epsilon}(\tau)^{(r)}$. *The spectral expansion of* $\boldsymbol{\delta}_{\mu,s}((s^2-\lambda^2)^{-1}\tilde{\Psi}_s)$ *is given as*

$$\boldsymbol{\delta}_{\mu,s}\left(\frac{\tilde{\Psi}_s}{s^2-\lambda^2}\right) = \sum_{m=0}^{\infty} \frac{-2\sqrt{-1}\,(*\mathrm{vol}_H|\mathfrak{I}_H(\tilde{\alpha}_m^{(r)}\,;\mathrm{e}))}{(\lambda^2-\lambda_m^{(r)}-s^2)^r}\tilde{\alpha}_m^{(r)}$$
$$+\sum_{p=0}^{r}\frac{1}{4\pi\sqrt{-1}}\int_{\sqrt{-1}\mathbb{R}}\sum_{i=1}^{h}\sum_{\mathrm{u}\in\mathcal{B}_i^{(r)}(p\,;0)}\frac{-2\sqrt{-1}\,(*\mathrm{vol}_H|\mathfrak{I}_H(\mathsf{E}^i(\zeta\,;\mathrm{u})\,;\mathrm{e}))}{(\zeta^2-(\nu_s^{(p+1)})^2)^r}\mathsf{E}^i(\zeta\,;\mathrm{u})\,d\zeta,$$

where the summations in the right-hand side of this formula are convergent in $\mathcal{L}_\Gamma^2(\tau)^{(r)}$.

Theorem 12. *Let* L_1 *be the interval on the imaginary axis defined by* (1). *Let* $0 \leqslant j \leqslant \mu$. *Then for each* $\tilde{\beta} \in \mathcal{K}_\Gamma(\tau)$ *the holomorphic function* $s \mapsto \mathfrak{F}_j(s,\tilde{\beta}) := \langle\boldsymbol{\delta}_{j,s}(s^2-\lambda^2)^{-1}\tilde{\Psi}_s|\tilde{\beta}\rangle$ *on* $\mathrm{Re}(s) > n$ *has a meromorphic continuation to the domain* $\mathbb{C} - L_1$. *A point* $s_0 \in \mathbb{C} - L_1$ *with* $\mathrm{Re}(s_0) \geqslant 0$ *is a pole of the meromorphic function* $\mathfrak{F}_j(s,\beta)$ *if and only if there exists an* $m \in \mathbb{N}$ *such that* $(*\mathrm{vol}_H|\mathfrak{I}_H(\tilde{\alpha}_m^{(r)}\,;\mathrm{e})) \neq 0$, $\langle\tilde{\alpha}_m^{(r)}|\tilde{\beta}\rangle \neq 0$ *and* $s_0^2 - \lambda^2 = -\lambda_m^{(r)}$. *In this case, the function*

$$\mathfrak{F}_j(s,\beta) - \sum_{m\in\mathbb{N};\lambda_m^{(r)}=\lambda^2-s_0^2}\frac{2\sqrt{-1}(*\mathrm{vol}_H|\mathfrak{I}_H(\tilde{\alpha}_m^{(r)}\,;\mathrm{e}))\langle\tilde{\alpha}_m^{(r)}|\tilde{\beta}\rangle}{(s_0^2-s^2)^{j+1}}$$

is holomorphic at $s = s_0$. *We have the functional equation*

$$\mathfrak{F}_j(-s,\tilde{\beta}) - \mathfrak{F}_j(s,\tilde{\beta}) = (-1)^\mu\,\boldsymbol{\delta}_{j,s}\left(\sum_{p=0}^{r}\frac{\langle\tilde{\mathcal{E}}_p^{(r)}(\nu_s^{(p+1)})|\tilde{\beta}\rangle}{2\,\nu_s^{(p+1)}}\right).$$

with

$$\tilde{\mathcal{E}}_p^{(r)}(\nu\,;g) := -2\sqrt{-1}\sum_{i=1}^{h}\sum_{\mathrm{u}\in\mathcal{B}_i^{(r)}(p\,;0)}(*\mathrm{vol}_H|\mathfrak{I}_H(\mathsf{E}^i(-\bar{\nu}\,;\mathrm{u})\,;\mathrm{e}))\,\mathsf{E}^i(\nu\,;\mathrm{u}\,;g), \quad g \in G.$$

Theorem 13. • *A point* $s_0 \in \mathbb{C} - L_1$ *with* $\mathrm{Re}(s) \geqslant 0$, $s_0 \neq n-2r+2$ *is a pole of* Ψ_s *if and only if there exists an* L^2*-form* $\alpha \in A_{(2)}^{r,r}(\Gamma\backslash G/K\,;(n-2r+2)^2 - s_0^2)$ *such that*

$$\int_D j^*\alpha \neq 0.$$

In this case s_0 is a simple pole with the residue

$$\mathrm{Res}_{s=s_0}\Psi_s = \frac{\sqrt{-1}(s_0^2 - (n-2r+2)^2)}{s_0}\sum_m \left(\int_D j^*\bar{\alpha}_m\right)\cdot \alpha_m.$$

Here $\{\alpha_j\}$ is an arbitrary orthonormal basis of $A_{(2)}^{r,r}(\Gamma\backslash G/K\,;(n-2r+2)^2 - s_0^2)$.

- *We have*

$$\Psi_{n-2r+2} = 2\sqrt{-1}\sum_m \left(\int_D j^*\bar{\beta}_m\right)\cdot \beta_m$$

with $\{\beta_m\}$ an arbitrary orthonormal basis of $\mathcal{H}_{(2)}^{r,r}(\Gamma\backslash G/K)$. In particular $\Psi_{n-2r+2} \in \mathcal{H}_{(2)}^{r,r}(\Gamma\backslash G/K)$.

The equations in Theorem 10 means the fundamental class $[\delta_D] \in \mathrm{H}^{r,r}(\Gamma\backslash G/K\,;\mathbb{C})$ of D has the harmonic L^2-representative Ψ_{n-2r+2}.

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Masao TSUZUKI
Department of Mathematics
Sophia University, Kioi-cho 7-1 Chiyoda-ku Tokyo, 102-8554, Japan
E-mail : tsuzuki@mm.sophia.ac.jp