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An extension of the collar lemma

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§1. WIDTH OF COLLAR

Let R be a Riemann surface with the hyperbolic distance d and γ a simple closed geodesic on R. A subdomain

$$C(\gamma,\omega)=\{p\in R \mid d(p,\gamma)<\omega\}$$

is called the collar of γ with width $\omega>0$ if it is an annular neighborhood of γ. The collar lemma asserts that if γ has the hyperbolic length ℓ_γ, then the collar $C(\gamma,\omega)$ of γ exists for the width

$$\omega=\operatorname{arcsinh}\frac{1}{\sinh(\ell_\gamma/2)}.$$

Actually, the collar lemma asserts more: if we take a family of mutually disjoint simple closed geodesics, then their collars of such width are mutually disjoint.

This lemma is mainly used for the investigation of the thin part of a hyperbolic surface, that is, the case where ℓ_γ is close to zero. On the contrary, when ℓ_γ is large, the width ω of the collar is small in general and there is few information obtained from the lemma.

In this note, we refine the collar lemma so as to be used in the case where the injectivity radius of R is bounded away from zero and the geodesic length ℓ_γ is close to the minimal possible length. However, the stronger assertion as above concerning disjointness of the collars is not taken into account in our refinement.

Theorem 1. *Let R be a hyperbolic Riemann surface such that the hyperbolic length of every non-trivial closed curve on R is not less than $\ell_0\geq 0$. Then, for any simple closed geodesic γ with length ℓ_γ, the collar $C(\gamma,\omega)$ of γ exists for width ω satisfying*

$$\sinh\omega=\max\left\{\frac{1}{\sinh(\ell_\gamma/2)}\ ,\ \sqrt{\left(\frac{\cosh(\ell_0/2)}{\cosh(\ell_\gamma/4)}\right)^2-1}\right\}.$$

The maximum of the right hand side is larger than a uniform constant ($\fallingdotseq 0.89$) *if* ℓ_γ *is sufficiently close to* ℓ_0. *Moreover, if* γ *is a dividing loop of* R, *then the collar exists for width* ω *satisfying*

$$\sinh \omega = \frac{\cosh(\ell_0/2)}{\sinh(\ell_\gamma/4)}.$$

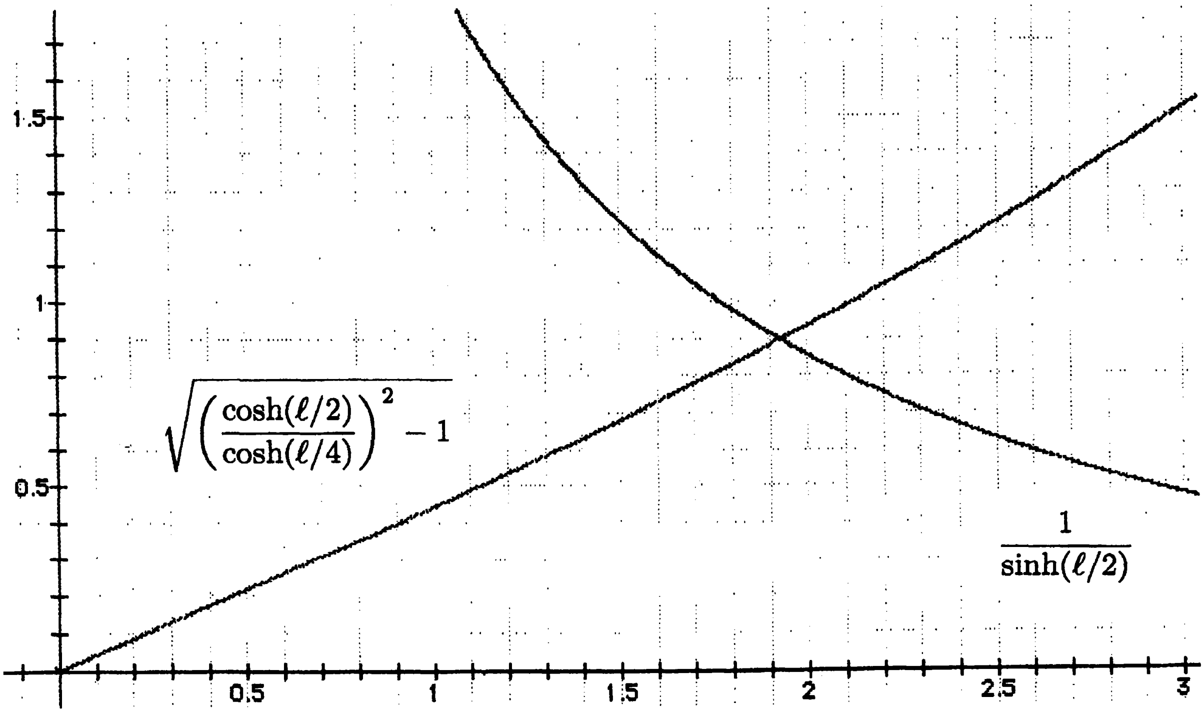

FIGURE 1. Maximum function

Proof. We take the shortest geodesic segment δ connecting two points in γ that is not contained in γ. There are two cases: δ returns to γ either in the same side or in the opposite side.

In the first case (Figure 2), there exists a pair of pants Y with three geodesic boundary components α, β and γ such that the inclusion map $\gamma \cup \delta \to Y$ is a homotopy equivalence. Remark that the lengths of α and β are not less than ℓ_0. A pair of pants is made from two congruent right-angled hexagons. The length of δ is the same as twice the distance t between the side of the hexagon corresponding to γ and its opposite side. By trigonometry on the hexagon (see [1, Ch.2]), we have

$$t \geq \operatorname{arcsinh} \frac{\cosh(\ell_0/2)}{\sinh(\ell_\gamma/4)}.$$

In the second case (Figure 3), there exists a once-holed torus Q such that the inclusion map $\gamma \cup \delta \to Q$ is a homotopy equivalence. We consider two congruent triangles made of δ, the shorter subarc γ' in γ connecting the end points of δ and the simple closed geodesic η freely homotopic to $\delta \cup \gamma'$. Remark that the lengths

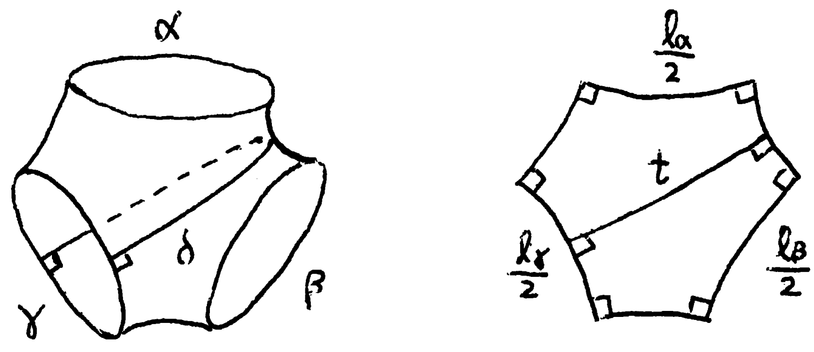

FIGURE 2. Y-piece and hexagon

of η is not less than ℓ_0. The length of δ is the same as twice the length t of the corresponding side of the triangle. By the Pythagorean theorem on the triangle, we have

$$t \geq \operatorname{arccosh} \frac{\cosh(\ell_0/2)}{\cosh(\ell_\gamma/4)}.$$

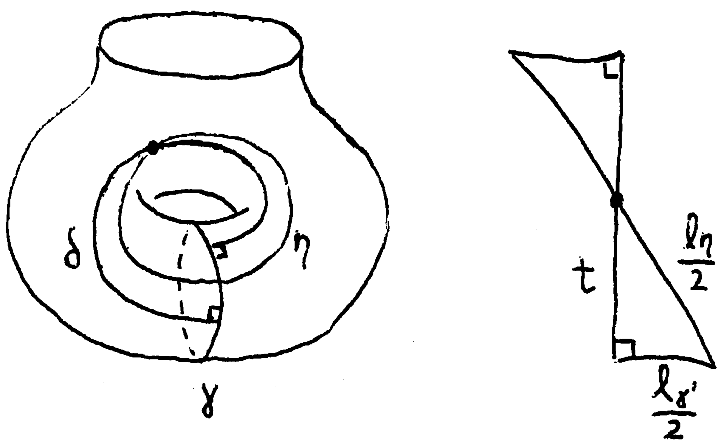

FIGURE 3. Q-piece and triangle

Since

$$\operatorname{arcsinh} \frac{\cosh(\ell_0/2)}{\sinh(\ell_\gamma/4)} \geq \operatorname{arccosh} \frac{\cosh(\ell_0/2)}{\cosh(\ell_\gamma/4)} = \operatorname{arcsinh} \sqrt{\left(\frac{\cosh(\ell_0/2)}{\cosh(\ell_\gamma/4)}\right)^2 - 1},$$

the second estimate is applied to a simple closed geodesic γ in general. Taking the maximum between this and the width coming from the collar lemma, we have the former statement. However, for a dividing loop γ, the second case does not occur and only the first estimate is applied. Since this is always larger than the width of the collar lemma, we have the latter statement. □

§2. Hyperbolic Riemann surfaces with large rings

We consider a certain class of hyperbolic Riemann surfaces of infinite topological type with arbitrarily large injectivity radius. In conformal geometry, the modulus m_γ of a collar of a simple closed geodesic γ is bounded by the inverse of the length ℓ_γ and hence the product of $m_\gamma \times \ell_\gamma$ is uniformly bounded. This affects a fact that deformation of the conformal geometry within a collar requires large dilatation if ℓ_γ is large. On the other hand, in hyperbolic geometry, even if ℓ_γ is large, the width ω of the collar can be also large. Namely, the size of a collar can be enlarged in proportion.

A half-collar of a simple closed geodesic γ with width $\omega > 0$ is an annular component of $\{p \in R \mid 0 < d(p, \gamma) < \omega\}$. This is a ring domain in R whose size we measure instead of a full collar.

Definition. We say that a hyperbolic Riemann surface R has arbitrarily large rings if there exists a positive constant $a > 0$ satisfying the following property: there exists a sequence of simple closed geodesics $\{\gamma_n\}$ in R such that ℓ_{γ_n} tends to the infinity as $n \to \infty$ and and the width ω_n of a half-collar of γ_n can be taken as $\omega_n \geq a\ell_{\gamma_n}$ for every n.

As an application of Theorem 1, we prove the following.

Theorem 2. *There exists a hyperbolic Riemann surface R that has arbitrarily large rings.*

Proof. We consider a planer Riemann surface R, where every simple closed geodesic is dividing. It is easy to make R having the following property: there exists a sequence of simple closed geodesics $\{\gamma_n\}$ such that $\ell_{\gamma_n} \to \infty$ as $n \to \infty$ and each γ_n is the shortest closed geodesic in one of the two geodesic subdomains of R divided by γ_n. For example, R can be made of pairs of pants with geodesic boundaries of suitable lengths by gluing them to be a planer surface. Then, by the latter assertion of Theorem 1, we have a half-collar of γ_n with width

$$\omega_n = \operatorname{arcsinh} \frac{\cosh(\ell_{\gamma_n}/2)}{\sinh(\ell_{\gamma_n}/4)} > \operatorname{arcsinh} \exp \frac{\ell_{\gamma_n}}{4} > \frac{\ell_{\gamma_n}}{4}.$$

This shows that R has arbitrarily large rings. □

References

1. P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, 1992.

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