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GROUP CONFIGURATIONS IN SIMPLE THEORIES (PART.2)

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Our goal is to make a gradedly almost hyperdefinable group from a group configuration in a simple theory. In a stable case, we get a group and a group operation from germs and a function on germs. But, in a simple case, since our hyperdefinable multifunction is not a function, we only have a polygroup. So, by using a core equivalence relation and a blow-up construction, we construct a group from a polygroup. In the following parts of this paper, our aim is to introduce necessary tools and new technique for simple version of grōup configuration theorem and illustrate these ideas. For the complete proof, see [4] and [5].

1. Ultraimaginaries and almost hyperimaginaries

To make a group from a polygroup, a core equivalence relation will be needed. But core equivalence relation is not type-definable but almost type-definable. So, we define new sorts called an ultraimaginary and a almost hyperimaginary. We will show that we can define a type of ultarimaginary over a hyperimaginary and two ultarimaginaries being independent over a hyperimaginary.

Definition 1.1 (Ultraimaginaries and an Almost hyperimaginaries). Let $(I, \leq)$ be a directed partial order, and X a sort.

1. An equivalence relation on X is *invariant* if it is automorhism-invariant.
2. A *graded equivalence relation* (g.e.r) R on X is the direct limit of reflexive symmetric type-definable relations $(R_i : i \in I)$ on X, such that:
   (a) If $i \leq j$ then R_j is coaser than R_i.
   (b) For every i, j there is k (which can be taken to be $\geq i, j$) such that $xR_iyR_jz \Rightarrow xR_kz$.
   We then note $R = R_I = \bigvee_{i\in I} R_i$, which is an invariant equivalence relation, and say that the R_i give a *grading* of R. If we want to emphasize I, we say I-graded and I-grading.
3. The class of a modulo R is noted a_R. Even when R is just a reflexive symmteric relation we note $a_R = \{x : xRa\}$ and call this the R-class of a. For a set A we may also note $A_R = \bigcup_{a\in A} a_R$. We also write $x \in_i A$ instead of $x \in A_{R_i}$, and $\pi(x_{R_i})$ for $\exists y[xR_iy \wedge \pi(y)]$, where π is a partial type. If there are too many indices, we may occasionally use a/R instead.
4. An invariant equivalence relation R is *almost = type-definable* if there is a type-difinable symmetric and reflexive relation R' finer than R such that any R-class can be covered by boundedly many R'-classes. If in addition R is graded and R' is finer than some R_i, then we say that it is *gradedly almost type-definable (above i)*.
5. A class modulo a (graded) invariant equivalence relation is called a *(graded) ultraimaginary*. A class modulo a (gradedly) almost type-definable equivalence relation is called a *(graded) almost hyperimaginary*.

There is an ultraimaginary which is not a almost hyperimaginary (i.e. there is an invariant equivalence relation which is not almost type-definable).

Example 1.2. Let $E_i (i \in \omega)$ be equivalence relations such that E_i is a refinement of E_{i+1} with infinitely many E_{i+1}-classes and every E_0 class has infinite elements. Put a language $L = \{E_i(i \in \omega)\}$ and consider the above structure. Then an equivalence relation $E = \bigcup_{i \in \omega} E_i$ is automorphism invariant but not almost type-definable.

The following lemma shows that we can define a type and a Lacar strong type of ultraimaginary over a hyperimaginary.

Lemma 1.3. *For two ultraimaginaries a_R and b_R and a hyperimaginary c, the following are equivalent:*

1. *There are $a' \in a_R$ and $b' \in b_R$, such that $a' \equiv_c b'$ in the usual sense.*
2. *There is an automorphism fixing c sending a_R to b_R.*
3. *For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_c b'$.*

And the following are also equivalent:

1. *There are $a' \in a_R$ and $b' \in b_R$, such that $a' \equiv_c^{Ls} b'$.*
2. *a_R and b_R are equivalent modulo any bounded c-invariant equivalence relation.*
3. *For every $a' \in a_R$ there is $b' \in b_R$ such that $a' \equiv_c^{Ls} b'$.*

So we define types and Lascar strong types and independence relation for ultraimaginaries.

Definition 1.4.
1. Two ultraimaginaries a_R and b_R have the same type over a hyperimaginary c, denoted $a_R \equiv_c b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c b'$ in the usual sense.
2. Two ultraimaginaries a_R and b_R have the same Lascar strong type over a hyperimaginary c, denoted $a_R \equiv_c^{Ls} b_R$, if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \equiv_c^{Ls} b'$ in the usual sense.

Definition 1.5. We say that $a_R \downarrow_c b_R$ if there are $a' \in a_R$ and $b' \in b_R$ such that $a' \downarrow_c b'$.

When we define types and independence as above, we have some desired properties as follows:

1. Exept finite character, ordinary properties of independence (symmetry, transitivity and etc.) hold for ultraimaginaries.
2. Two almost hyperimaginaries being independent over a hyperimarinary is type-definable.

2. Polygroups

In a stable theory, we construct an interdefinable group configuration from a group configuration. And we have a group and a group operation from germs and a function on germs in an interdefinable group configuration. But, in a simple theory, we only have an interbounded group configuration and have only a polygroup.

Definition 2.1. A polygroup is axiomatized in the language $\{\cdot, ^{-1}\}$ by the following axioms:

1. $t \in (x \cdot y) \cdot z \leftrightarrow t \in x \cdot (y \cdot z)$,
2. $t \in x \cdot y^{-1} \leftrightarrow x \in t \cdot y$,

3. $t \in x^{-1} \cdot y \leftrightarrow y \in x \cdot t$,
4. $t \in x \cdot e \leftrightarrow t \in x \leftrightarrow t \in e \cdot x$.

Since a group operateion and an inverse function are multifunctions in a polygroup, $x \cdot y$ and x^{-1} are not elements but sets. So $(x \cdot y) \cdot z$ represents $\bigcup\{u \cdot z : u \in x \cdot y\}$. Now we give some examples of a polygroup.

Example 2.2 (Double coset algebras). Let G and H be groups such that H is a subgroup of G. Put $M = \{HgH : g \in G\}$ and define a group operation and inverse on M as follows:

1. $(Hg_1H) \cdot (Hg_2H) = \{Hg_1hg_2H : h \in H\}$;
2. $(HgH)^{-1} = Hg^{-1}H$;
3. identity is the $H(= HeH)$.

Then a double coset algebra $< M, \cdot, ^{-1}, H >$ is clearly a polygroup and denoted by $G//H$.

Example 2.3 (Prenowitz algebras). Let P be a set of points, L a set of lines and $I \subseteq P \times L$ a incidence relation. An incidence system (P, L, I) is projective geometry if it satisfy the following axioms:

1. any line contains at least three points;
2. two distinct points a, b are contained in a unique line denoted by $L(a, b)$;
3. if a, b, c, d are distinct points and $L(a, b)$ intersect $L(b, d)$,
   then $L(a, c)$ must intersetcts $L(c, d)$ (Pasch axiom).

Choose $e \notin P$ and put $P' = P \cup \{e\}$. We define $\circ$ (group operation), $^{-1}$ (inverse) and e (identity) as follows:

1. $a^{-1} = a$ and $e \circ a = a = a \circ e$ for all $a \in P'$;
2. $a \circ b = L(a, b) \setminus \{a, b\}$ for all $a \neq b \in P$;
3. $a \circ a = \{a, e\}$ for all $a \in P'$.

By Pasche axiom a group operation $\circ$ is associative, so $(P', \circ, ^{-1}, e)$ is a polygroup and we called it a Prenowitz algebra.

3. Core equivalence relation

In a polygroup, inverse and identity are not unique. To have a unique inverse and a unique identity, we construct a group modulo an equivalence relation. This equivalence relation is called core equivalence. Our goal is to construct a group which has some kind of definability, so we will show that core equivalence is (only) gradedly almost type-definable.

Definition 3.1. Let $P = P_0/R_I$ be a gradedly almost hyperdefinable polygroup.

1. For $a, b \in P_0$ and $i \in I$, we say that $a \sim_{i1} b$ if there is a generic $g \downarrow ab$ such that $a, b \in_i g \cdot h$ for some h (which must also be generic). $\sim_{in}$ is the n-closure of $\sim_{i1}$, and $\bigvee_{in} \sim_{in}$. We shall show that $\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation, which we call the *core equivalence*.
2. We define the *core* N of P as follows: $N_{i1} \subseteq P_0$ is the set of all a such that $a \in_i g \cdot g^{-1}$ for some generic $g \downarrow a$, and $N_{in} = N_{i1}^n$. One verifies that $\bigcup_i N_{in}$ is a union of R-classes closed under inverse for all $n < \omega$, so we can put $N_n = (\bigcup_i N_{in})/R = N_1^n$, and $N = \bigcup_n N_n \leq P$, the sub-polygroup generated by N_1.
3. P is *coreless* if the core equivalence is the same as R, that is for every $(i, n) \in I \times \omega$ there is $j \in I$ such that R_j is coaser than $\sim_{in}$.

By using a stratified rank and boundedness of the product, we show that the core equivalence is almost type-definable. And by the definition of core equivalence, inverse is unique modulo core equivalence.

Lemma 3.2. *Let $P = P_0/R$ be an I-gradedly almost hyperdefinable polygroup.*

1. *$\sim$ is an $(I \times \omega)$-gradedly almost type-definable equivalence relation on P coarser than R, and every $\sim$-class contains boundedly many R-classes (that is, if $a_R \sim b_R$ then a_R and b_R are interbounded as almost hyperimaginaries).*
2. *$P/\sim$ is coreless. Any almost hyperdefinable group is coreless.*
3. *If P is coreless, then inverses are unique, and a unique identity exists. This is to say that there are $i \in I$ and $e \in P_0$ such that $(a^{-1})^{-1}, e \cdot a, a \cdot e \subseteq a_{R_i}$ for every $a \in P_0$.*

4. Blowing up generic chunks

Our group operation is still a multifuncion. By a blow-up construction, we construct a new group and a new group operation from a coreless polygroup. Then the group operation is a function. Exactly speaking, by a blow-up construcion, we construct a gradedly almost hyperdefinable generic group chunk from a coreless gradedly almost hyperdefinable genric polygroup chunk.

Lemma 4.1. *For every $i \in I$ there is $j \in I$ such that whenever $a_1, a_2, b_1, b_2, d_1 \in S_0$, the triplet $\{a_{1R}, b_{1R}, b_{2R}\}$ is independent, and $d_1 \in (a_1^{-1} \cdot b_1)_{R_i} \cap (a_2 \cdot b_2^{-1})_{R_i}$, then there is $f \in a_1 \cdot a_2 \cap (b_1 \cdot b_2)_{R_j}$.*
Moreover, if we have also $c_1, c_2, d_2 \in S_0$ such that $c_{1R} \downarrow c_{2R}, = a_{1R} \downarrow b_{1R} b_{2R} c_{1R} c_{2R}$ and $d_2 \in (a_1^{-1} \cdot c_1)_{R_i} \cap (a_2 \cdot c_2^{-1})_{R_i}$, and we take $f' \in a_1 \cdot a_2 \cap (c_1 \cdot c_2)_{R_j}$, then $f \in_1 f'$ for some $1 \in I$ dependent only on i. In particular, f is unique up to R_1.

By the previous lemma, if we choose copies of $a_i (i = 1, 2)$, say b_i and c_i, and define $(a_1, b_1, c_2) \cdot (a_2, b_2, c_2) = f$ as in the previous lemma, the product is unique modulo R_1. So we define a triplet $\tilde{a} = (a, a', a'')$ from $a \in S_0$ and the product of two triplets $\tilde{a} \cdot \tilde{b}$. Then we have a group from a polygroup as desired.

Definition 4.2.
1. We fix some $e \in S_0$, and set $S_0' = \{a \in S_0 : a_R \downarrow e_R\}$.
2. Define $\tilde{S} = \{(a, a', a'') \in S_0', a' \in e^{-1} \cdot a \text{ and } a'' \in a \cdot e\}$ and $\tilde{S} = \tilde{S}_0/R$. (We follow a tacit understanding that R may also stand for $R \times R \times R$, where this is clear from the context.)
3. A triplet $\tilde{a} = (a, a', a'') \in \tilde{S}_0$ is called a *blow-up* of a. Conversely, we define the *blow-up* map $\pi : \tilde{S}_0 \to S_0'$ by $\pi(a, a', a'') = a$, where a is sometimes referred to as the *axis* of (a, a', a'').
4. Given $\tilde{a}_R \downarrow_e \tilde{b}_R$, we wish to define $\tilde{a} \cdot \tilde{b}$. First, we know that $e \in (a^{-1} \cdot a'')_{R_1} \cap (b \cdot b'^{-1})_{R_1}$ for some $1 \in I$. By Lemma4.1 there is $c \in a \cdot b \cap (a'' \cdot = b')_{R_2}$, for some $2 \in I$. Again by Lemma4.1 there is $c \in e^{-1} \cdot c \cap (a' \cdot b)_{R_2}$ and $c'' \in c \cdot e \cap (a \cdot b'')_{R_2}$. Set $\tilde{a} \cdot \tilde{b}$ to be the set of all $\tilde{c} = (c, c', c'')$ obtained in this manner.
5. Recall that the inverse is a gradedly definable map, so it is only defined up to some R_i. Thus, for $\tilde{a} = (a, a', a'') \in \tilde{S}_0$, we can define its inverse as:
$\tilde{a}^{-1} = \{(b, b', b'') \in \tilde{S}_0 : b \in a^{-1}, b' \in_j a''^{-1}, b'' \in_j a'^{-1}\}$
for $j \in I$ big enough to make sure that $\tilde{a}^{-1}$ cannot be empty; re-arranging previous choices we may assume that $j \leq 0$.

Theorem 4.3. *Let $S = S_0/R$ be a coreless gradedly almost hyperdefinable (over $\emptyset$) generic polygroup chunk, and $e \in S_0$. Let $\tilde{S}_0$ be as above. Then $\tilde{S} = \tilde{S}_0/R$ is a gradedly almost hyperdefinable generic group chnk over e.*

5. Constructing an almost hyperdifinable group

Strictly speaking, we have a gradedly almost hyperdefinable generic group chunk from a group configuration in a simple theory. So we need the Weil-Hrushovski group chunk theorem for almost hyperdefinable group chunk to get a group.

Theorem 5.1. *Let $< S_0/R, \cdot, ^{-1} >$ be an I-gradedly almost hyperdefinable group chunk. Then there is an I-g.e.r. R' on S_0^2, such that $G = S_0^2/R'$ is a gradedly almost hyperdefinable goup. Moreover, there is a gradedly type-definable map $\sigma : S \mapsto G$ whose image generates S, and the couple (G, σ) is gradedly unique as such, up to a unique graded isomorphism (i.e., for every other couple (G', σ'), there is a unique isomorphism, up to graded equality of maps, rendering σ and σ' gradedly qual).*

References

[1] Stephen D. Comer, *Polygroups derived from cogroups*, Journal of Algebra, 89:397-405

[2] Frank Wagner, *Stable Groups*, London Mathematical Society Lecture Note Series 240

[3] Frank Wagner, *Simple Theories*, Kluwer Academic Publishers

[4] Itay Ben-Yaacov, Ivan Tomasic and Frank Wagner, *The group configuration in simple theories and its applications*, The Bulletion of Symbolic Logic Vol.8 Number2 June 2002

[5] Itay Ben-Yaacov, Ivan Tomasic and Frank Wagner, *Constructing an almost hyperdefinable group*, preprint