数理解析研究所講究録 1345 巻 2003 年 166-181



# A generalization of the sectional genus and the $\Delta$-genus of polarized varieties

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## 1 Introduction.

Let $X$ be a projective variety of $\dim X=n$ over the complex number field, and let $L$ be an ample line bundle on $X$. Then the pair $(X, L)$ is called a *polarized variety.* Moreover if $X$ is smooth, then $(X, L)$ is called a polarized *manifold.*

When we study polarized varieties, it is useful to use their invariants. The following invariants are well-known.

(1) The degree $L^{n}$.

(2) The sectional geuns $g(L)$.

(3) The $\Delta$-genus $\Delta(L)$.

Many authors studied polarized varieties by using these invariants. In particular, P. Ionescu classified polarized manifolds $(X, L)$ for the case where $L$ is very ample and $L^{n}\leq 8$, and T. Fujita classified polarized manifolds with low sectional genera and low $\Delta$-genera.

In order to study polarized varieties more deeply, in [7] and [10] the author introduced the notion of the *i-th sectional geometric genus* $g_{i}(X, L)$ and the *i-th* $\Delta$*-genus* $\Delta_{i}(X, L)$ *of* $(X, L)$ for every integer $i$ with $0\leq i\leq n$. The $i$-th sectional geometric genus is a generalization of the degree and the sectional genus of $(X, L)$, and the $i$-th $\Delta$-genus is a generalization of the $\Delta$-genus of $(X, L)$. Namely $g_{0}(X, L)=L^{n}$, $g_{1}(X, L)=g(L)$, and $\Delta_{1}(X, L)=\Delta(L)$. (See Remark 2.1 and Remark 2.2 below.) In Section 3, we give fundamental results of these invariants. In particular, if $\mathrm{Bs}|L|=\emptyset$, then $g_{i}(X, L)$ is the geometric genus of the $i$-dimensional manifold which is obtained by a general $(n-i)$ members of $|L|$ (see Theorem 3.1). Moreover there are some relations between $g_{i}(X, L)$ and $\Delta_{i}(X, L)$ (see Theorem 3.2 or [10]). So we find that the $i$-th sectional geometric genus and the $i$-th $\Delta$-genus are expected to satisfy results which are analogous to results of "i-dimensional geometry". (It has already been known that the first sectional geometric genus and the first $\Delta$-genus, that is, the sectional genus and the $\Delta$-genus reflect some properties of geometry of curves.)

Since the sectional genus and the $\Delta$-genus have been studied deeply (see e.g. [5]), the next step we should consider is the case where $i = 2$, and our main goal at present is to construct the theory of the second sectional geometric genus and the second $\Delta$-genus of polarized varieties. Under these consideration, in Section 4, we classify $(X, L)$ by the second sectional geometric genus and the second $\Delta$-genus for the case where $L$ is spanned or very ample. By the above philosophy, the second sectional geometric genus and the second $\Delta$-genus are expected to satisfy results which are analogous to theorems in the theory of projective surfaces. In order to propose some problems, first we define the *i-th sectional H-arithmetic genus* $\chi_i^H(X, L)$ *of* $(X, L)$. We note that when $i = 2$, this invariant corresponds to the Euler-Poincaré characteristic of the structure sheaf of surfaces and $\chi_2^H(X, L) = 1 - h^1(\mathcal{O}_X) + g_2(X, L)$ (see Remark 5.1). Hence we can propose some problems which are analogous to results about the Euler-Poincaré characteristic of the structure sheaf of surfaces. So in Section 5, we will propose some conjectures about the second sectional H-arithmetic genus and the second sectional geometric genus, and we get partial results about these conjectures. Here we note that Conjecture 4 is analogous to the Bogomolov-Miyaoka-Yau theorem.

On account of limited space, we cannot state all facts which are known at present. For the reader who wants to know these topics, see [7], [8], [9], [10], and [11].

This is a survey of my talk of the symposium "Local invariants of families of algebraic curves" at the RIMS (Kyoto). The author would like to thank Professor Kazuhiro Konno for giving the opportunity to talk about this topic.

## 2 Definition of the $i$-th sectional geometric genus and the $i$-th $\Delta$-genus of polarized varieties.

**Notation 2.1** Let $X$ be a projective scheme of $\dim X = n$ and let $L$ be a line bundle on $X$. Then we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L)\frac{t^{[j]}}{j!},$$

where

$$t^{[j]} = \begin{cases} t(t+1)\cdots(t+j-1), & \text{if } j > 0, \\ 1, & \text{if } j = 0. \end{cases}$$

**Definition 2.1** (See Definition 2.1 in [7].) Let $(X, L)$ be a polarized variety of $\dim X = n$. Then for any integer $i$ with $0 \leq i \leq n$ the *i-th sectional geometric genus of* $(X, L)$ is defined by the following:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i}(-1)^{n-i-j}h^{n-j}(\mathcal{O}_X).$$

(Here we use Notation 2.1.)

**Remark 2.1** (1) If $i = 0$ (resp. $i = 1$), then $g_i(X, L)$ is equal to the degree (resp. the sectional genus) of $(X, L)$.

(2) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$ and $g_n(X, L)$ is independent of $L$.

(3) If $i = 2$ and $X$ is smooth, then by the Hirzebruch-Riemann-Roch theorem, we get that

$$\begin{aligned} g_2(X, L) &= -1 + h^1(\mathcal{O}_X) + \frac{1}{12}(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} \\ &\quad + \frac{1}{12}c_2(X)L^{n-2} + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1}. \end{aligned}$$

**Definition 2.2** (See [10].) Let $(X, L)$ be a polarized variety of $\dim X = n$. For every integer $i$ with $0 \leq i \leq n$, the *i-th* $\Delta$*-genus of* $(X, L)$ is defined by the following formula:

$$\Delta_i(X, L) = \begin{cases} 0, & \text{if } i = 0, \\ g_{i-1}(X, L) - \Delta_{i-1}(X, L) & \\ \quad + (n - i + 1)h^{i-1}(\mathcal{O}_X) - h^{i-1}(L), & \text{if } 1 \leq i \leq n. \end{cases}$$

**Remark 2.2** (1) If $i = 1$, then $\Delta_1(X, L)$ is equal to the $\Delta$-genus of $(X, L)$. (See [5].)

(2) If $i = n$, then $\Delta_n(X, L) = h^n(\mathcal{O}_X) - h^n(L)$ (see [10]).

Here we define the notion of $k$-ladder, which is used later.

**Definition 2.3** Let $(X, L)$ be a polarized variety of $\dim X = n$. Then $L$ has a *k-ladder* if there exists an irreducible and reduced subvariety $X_i$ of $X_{i-1}$ such that $X_i \in |L_{i-1}|$ for $1 \leq i \leq k$, where $X_0 := X$, $L_0 := L$, and $L_i := L_{i-1}|_{X_i}$ for $1 \leq i \leq k$.

**Notation 2.2** Let $(X, L)$ be a polarized variety of $\dim X = n$. Assume that $L$ has a $k$-ladder. We put $X_0 := X$ and $L_0 := L$. Let $X_i \in |L_{i-1}|$ be an irreducible and reduced member, and $L_i := L_{i-1}|_{X_i}$ for every integer $i$ with $1 \leq i \leq k$. Let $r_{p,q} : H^p(X_q, L_q) \to H^p(X_{q+1}, L_{q+1})$ be the natural map. If $h^0(L_k) > 0$, then we take an element $X_{k+1} \in |L_k|$ and we put $L_{k+1} = L_k|_{X_{k+1}}$.

Finally we define the notion of a reduction of polarized manifolds.

**Definition 2.4** (1) Let $X$ (resp. $Y$) be an $n$-dimensional projective manifold, and let $L$ (resp. $A$) be an ample line bundle on $X$ (resp. $Y$). Then $(X, L)$ is called a *simple blowing up of* $(Y, A)$ if there exists a birational morphism $\pi : X \to Y$ such that $\pi$ is a blowing up at a point of $Y$ and $L = \pi^*(A) - E$, where $E$ is the $\pi$-exceptional effective reduced divisor.

(2) Let $X$ (resp. $Y$) be an $n$-dimensional projective manifold, and let $L$ (resp. $A$) be an ample line bundle on $X$ (resp. $Y$). Then we say that $(Y, A)$ is a *reduction of* $(X, L)$ if there exists a birational morphism $\mu : X \to Y$ such that $\mu$ is a composite of simple blowing ups and $(Y, A)$ is not obtained by a simple blowing up of any polarized manifold. The morphism $\mu$ is called the *reduction map*.

**Remark 2.3** Let $(X, L)$ be a polarized manifold and let $(M, A)$ be a reduction of $(X, L)$. Let $\mu : X \to M$ be the reduction map.

(1) We obtain that $g_i(X, L) = g_i(M, A)$ for any integer $i$ with $1 \leq i \leq n$ (see Proposition 2.6 in [7]).

(2) Assume that $\mathrm{Bs}|L| = \emptyset$. Then for a general member $D$ of $|L|$, $D$ and $\mu(D) \in |A|$ are smooth.

(3) $\Delta_1(X, L) \leq \Delta_1(M, A)$ and $\Delta_i(X, L) = \Delta_i(M, A)$ for every integer $i$ with $2 \leq i \leq n$ (see [10]).

(4) If $(X, L)$ is not obtained by a simple blowing up of another polarized manifold, then $(X, L)$ is a reduction of itself.

(5) A reduction of $(X, L)$ always exists (see Chapter II, (11.11) in [5]).

## 3 Fundamental properties of $g_i(X, L)$ and $\Delta_i(X, L)$ of polarized manifolds.

**Theorem 3.1** *Let $X$ be a projective variety of $\dim X = n \geq 2$ and let $L$ be an ample Cartier divisor on $X$. Assume that $h^t(-sL) = 0$ for every integers $t$ and $s$ with $0 \leq t \leq n-1$ and $1 \leq s$.*

(A) *If $|L|$ has an $(n-i)$-ladder for an integer $i$ with $1 \leq i \leq n$, then $g_i(X, L) = g_i(X_1, L_1) = \cdots = g_i(X_{n-i}, L_{n-i}) = h^i(\mathcal{O}_{X_{n-i}}) \geq h^i(\mathcal{O}_{X_{n-i-1}}) = \cdots = h^i(\mathcal{O}_X)$.*

(B) *If $|L|$ has an $(n-i)$-ladder and $h^0(L_{n-i}) > 0$ for an integer $i$ with $1 \leq i \leq n$, then*

$$\Delta_i(X, L) = \sum_{j=0}^{n-i} \dim \mathrm{Coker}(r_{i-1,j}).$$

*In particular, $\Delta_i(X, L) \geq \Delta_i(X_1, L_1) \geq \cdots \geq \Delta_i(X_{n-i}, L_{n-i}) \geq 0$.*

*(Here we use Notation 2.2. )*

*A sketch of the proof.* (A) (See also [9].) We note that for every integer $k$ with $0 \leq k \leq n-i-1$

$$\chi_{n-k-i}(X_k, L_k) = \chi_{n-k-1-i}(X_{k+1}, L_{k+1}). \tag{1}$$

By the assumption we obtain that

$$\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_X) = \cdots = \sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_{X_{n-i}}), \tag{2}$$

$$h^i(\mathcal{O}_X) = \cdots = h^i(\mathcal{O}_{X_{n-i-1}}) \leq h^i(\mathcal{O}_{X_{n-i}}). \tag{3}$$

By the definition of the $i$-th sectional geometric genus of $(X, L)$, and by (1), (2), and (3), we obtain the assertion.
(B) (See also [10].) If $i = n$, then $\Delta_n(X, L) = h^n(\mathcal{O}_X) - h^n(L)$ by Remark 2.2 (2). By the exact sequence

$$H^{n-1}(L) \to H^{n-1}(L_1) \to H^n(\mathcal{O}_X) \to H^n(L) \to 0,$$

we get that $\Delta_n(X, L) = \dim \operatorname{Coker}(r_{n-1,0})$.
If $1 \leq i \leq n-1$, then by [10], we obtain that

$$\begin{aligned}\Delta_i(X, L) &= (-1)^{i-1}\sum_{j=0}^{i-1}\chi_{n-j}(X, L) + (n-i+1)(-1)^{i-1}\Big(\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_X)\Big) \\ &\quad +(-1)^i\Big(\sum_{k=0}^{i-1}(-1)^k h^k(L)\Big).\end{aligned}$$

Here we note that

$$\begin{aligned}(-1)^{i-1}\sum_{j=0}^{i-1}\chi_{n-j}(X, L) &= (-1)^{i-1}\sum_{j=0}^{i-1}\chi_{i-j}(X_{n-i}, L_{n-i}) \\ &= (-1)^{i-1}(\chi(X_{n-i}, L_{n-i}) - \chi(\mathcal{O}_{X_{n-i}})).\end{aligned}$$

By (2) and the following exact sequence

$$\begin{aligned}0 &\to H^0(\mathcal{O}_{X_j}) \to H^0(L_j) \to H^0(L_{j+1}) \\ &\to H^1(\mathcal{O}_{X_j}) \to \cdots \\ &\to H^{i-1}(\mathcal{O}_{X_j}) \to H^{i-1}(L_j) \to H^{i-1}(L_{j+1}).\end{aligned}$$

we get that

$$\begin{aligned}&(n-i+1)(-1)^{i-1}\Big(\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_X)\Big) + (-1)^i\Big(\sum_{k=0}^{i-1}(-1)^k h^k(L)\Big) \\ &= (-1)^{i-1}\Big(\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_{X_{n-i}})\Big) + (-1)^i\Big(\sum_{k=0}^{i-1}(-1)^k h^k(L_{n-i})\Big) \\ &\quad + \sum_{j=0}^{n-i-1}\dim \operatorname{Coker}(r_{i-1,j}).\end{aligned}$$

Since

$$(-1)^{i-1}(\chi(X_{n-i}, L_{n-i}) - \chi(\mathcal{O}_{X_{n-i}})) + (-1)^{i-1}(\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_{X_{n-i}}))$$
$$+(-1)^i(\sum_{k=0}^{i-1}(-1)^k h^k(L_{n-i}))$$
$$= h^i(\mathcal{O}_{X_{n-i}}) - h^i(L_{n-i})$$
$$= \dim \mathrm{Coker}(r_{i-1,n-i}),$$

we obtain the assertion. □

If $X$ is smooth and $L$ is ample and spanned, then $L$ has an $(n-1)$-ladder, $h^0(L_{n-1}) > 0$, and $h^t(-sL) = 0$ for every integers $t$ and $s$ with $0 \leq t \leq n-1$ and $1 \leq s$. Hence by using Theorem 3.1, we get the following.

**Corollary 3.1** *Let $(X, L)$ be a polarized manifold of* $\dim X = n$. *Assume that* $\dim \mathrm{Bs}|L| = \emptyset$. *Then $g_i(X, L) \geq h^i(\mathcal{O}_X)$ and $\Delta_i(X, L) \geq 0$ for every integer $i$ with $1 \leq i \leq n$.*

By the above observation, we propose the following problem.

**Problem 3.1** *Let $(X, L)$ be a polarized manifold of* $\dim X = n$.

(1) *Does an inequality $g_i(X, L) \geq h^i(\mathcal{O}_X)$ hold for every integer $i$ with $0 \leq i \leq n$ ?*

(2) *Does an inequality $\Delta_i(X, L) \geq 0$ hold for every integer $i$ with $0 \leq i \leq n$ ?*

Here we note the following.

(a) If $i = 0$, then (1) is true.

(b) There exists an example of $(X, L)$ with $\Delta_i(X, L) < 0$ in general. In detail, see [10].

**Theorem 3.2** *Let $(X, L)$ be a polarized manifold of* $\dim X = n$, *and let $i$ be an integer. Assume that* $\mathrm{Bs}|L| = \emptyset$.

(1) *If $1 \leq i \leq n$, then $\Delta_i(X, L) = 0$ if and only if $g_i(X, L) = 0$.*

(2) *If $\Delta_i(X, L) = 0$ for an integer $i$ with $1 \leq i \leq n-1$, then $\Delta_{i+1}(X, L) = 0$.*

(3) *If $\Delta_i(X, L) = 1$ for an integer $i$ with $2 \leq i \leq n$, then $g_i(X, L) = 1$.*

*A sketch of the proof.* (In detail, see [10].) (1) Assume that $g_i(X,L)=0$. Then by Theorem 3.1 (A) we have $h^i(\mathcal{O}_{X_j})=0$ for every integer $j$ with $0\le j\le n-i$. Hence $\dim \mathrm{Coker}(r_{i-1,j})=0$ for every $j$ with $0\le j\le n-i$. Therefore $\Delta_i(X,L)=0$.

Assume that $\Delta_i(X,L)=0$. Then $\Delta_i(X_j,L_j)=0$ for every integer $j$ with $1\le j\le n-i$ by Theorem 3.1 (B). In particular, $H^{i-1}(L_{n-i})\to H^{i-1}(L_{n-i+1})$ is surjective. Hence $h^i(\mathcal{O}_{X_{n-i}})=h^i(L_{n-i})$. Then we can show that $h^i(\mathcal{O}_{X_{n-i}})=0$. Therefore $g_i(X,L)=0$.

(2) Assume that $\Delta_i(X,L)=0$. Then $\Delta_i(X_{n-i},L_{n-i})=0$ by Theorem 3.1 (B). In particular $h^0(K_{X_{n-i}})-h^0(K_{X_{n-i}}-L_{n-i})=0$ by Remark 2.2 (2) and the Serre duality. Since $\mathrm{Bs}|L_{n-i}|=\emptyset$, we get that $h^0(K_{X_{n-i}})=h^0(K_{X_{n-i}}-L_{n-i})=0$. Therefore $h^0(K_{X_{n-i-1}}+L_{n-i-1})=0$. Since $\mathrm{Bs}|L_{n-i-1}|=\emptyset$, we get that $h^0(K_{X_{n-i-1}})=0$. Hence $\Delta_{i+1}(X_{n-i-1},L_{n-i-1})=0$ and $h^{i+1}(\mathcal{O}_{X_{n-i-1}})=0$. Here we note that

$$0=h^{i+1}(\mathcal{O}_{X_{n-i-1}})\ge h^{i+1}(\mathcal{O}_{X_{n-i-2}})=\cdots=h^{i+1}(\mathcal{O}_X).$$

Hence by Theorem 3.1 (B)

$$\Delta_{i+1}(X,L)=\cdots=\Delta_{i+1}(X_{n-i-1},L_{n-i-1})=0.$$

(3) If $1=\Delta_i(X,L)>g_i(X,L)$, then $g_i(X,L)=0$ and by (1) we get that $\Delta_i(X,L)=0$. But this is impossible. Therefore we find that $g_i(X,L)\ge\Delta_i(X,L)$. If $h^0(K_{X_{n-i}}-L_{n-i})\ne 0$, then we can prove that $\Delta_i(X,L)\ge i\ge 2$ and this is a contradiction. Hence $h^0(K_{X_{n-i}}-L_{n-i})=0$. By Theorem 3.1 (B), we get that

$$\begin{aligned}\Delta_i(X,L) &\ge \Delta_i(X_{n-i},L_{n-i})\\ &= h^i(\mathcal{O}_{X_{n-i}})-h^i(L_{n-i})\\ &= h^i(\mathcal{O}_{X_{n-i}})\\ &= g_i(X,L).\end{aligned}$$

Therefore $g_i(X,L)=\Delta_i(X,L)=1$.

These complete the proof of Theorem 3.2. □

Before we study behavior of the $i$-th sectional geometric genus and the $i$-th $\Delta$-genus of polarized manifolds under deformation, we define the notion of deformation family.

**Definition 3.1** If $f:\mathcal{X}\to T$ is a proper surjective smooth morphism onto a connected but possibly non-compact manifold $T$ together with an $f$-ample line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $f^{-1}(0)=X$ and $\mathcal{L}|_{f^{-1}(0)}=L$, then we say that *$(f:\mathcal{X}\to T,\mathcal{L})$ is a deformation family of $(X,L)$*.

**Proposition 3.1** *Let $(X,L)$ be a polarized manifold of $\dim X=n$. For every integer $i$ with $0\le i\le n$, $g_i(X,L)$ is a deformation invariant.*

*Proof.* Let $s$ be an indeterminate. Then $\chi(sL)$ is a deformation invariant (see Chapter III, §7 in [4] or Chapter III 12.9 in [12]). Hence $\chi_{n-i}(X,L)$ is a deformation invariant. On the other hand $h^k(\mathcal{O}_X)$ is also a deformation invariant for any integer $k$ (see Part I, 10.5 in [3]). Therefore by definition we obtain that for every integer $i$ with $0 \leq i \leq n$, $g_i(X,L)$ is a deformation invariant. □

**Proposition 3.2** *Let $(f : \mathcal{X} \to T, \mathcal{L})$ be a deformation family of $(X,L)$. For every integer $i$ with $0 \leq i \leq n$, $\Delta_i(X_t, \mathcal{L}_t)$ is a lower semicontinuous function on $t \in T$.*

*Proof.* As in the proof of Theorem 3.1 (B), we obtain that

$$\begin{aligned}\Delta_i(X_t,L_t) &= (-1)^{i-1}\sum_{j=0}^{i-1}\chi_{n-j}(X_t,L_t) + (n-i+1)(-1)^{i-1}(\sum_{k=0}^{i-1}(-1)^k h^k(\mathcal{O}_{X_t})) \\ &\quad +(-1)^i(\sum_{k=0}^{i-1}(-1)^k h^k(L_t)).\end{aligned}$$

We note that $\chi_k(X_t,L_t)$ and $h^k(\mathcal{O}_{X_t})$ are deformation invariants. On the other hand $(-1)^q\sum_{j=0}^{q}(-1)^j h^j(L_t)$ is an upper semi-continuous function on $t \in T$. (For a proof, see, e.g., Part I, 10.4 in [3].) Hence $(-1)^i(\sum_{k=0}^{i-1}(-1)^k h^k(L_t))$ is a lower semi-continuous function on $t \in T$. Therefore we get the assertion. □

# 4 Classification of polarized manifolds by the second sectional geometric genus and the second $\Delta$-genus.

**Theorem 4.1** *Let $(X,L)$ be a polarized manifold of* $\dim X = n \geq 3$. *Assume that* $\mathrm{Bs}|L| = \emptyset$. *Then $g_2(X,L) = h^2(\mathcal{O}_X)$ if and only if $(X,L)$ is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth curve.*

(4) $K_X \sim -(n-1)L$, *that is, $(X,L)$ is a Del Pezzo manifold.*

(5) *A quadric fibration over a smooth curve.*

(6) *A scroll over a smooth surface $S$.*

(7) *Let $(M,A)$ be a reduction of $(X,L)$.*
(7-1) $n=4$, $(M,A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
(7-2) $n=3$, $(M,A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
(7-3) $n=3$, $(M,A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
(7-4) $n=3$, *$M$ is a $\mathbb{P}^2$-bundle over a smooth curve $C$ with $(F, A|_F) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ for any fiber $F$ of it.*

*A sketch of the proof.* (In detail, see Theorem 3.3, Corollary 3.4, and Remark 3.4.1 in [7].) Here we note that

$$\begin{aligned} g_2(X,L) &= g_2(X_{n-3},L_{n-3}) \\ &= h^0(K_{X_{n-3}}+L_{n-3}) - h^0(K_{X_{n-3}}) + h^2(\mathcal{O}_{X_{n-3}}) \\ &= h^0(K_{X_{n-3}}+L_{n-3}) - h^0(K_{X_{n-3}}) + h^2(\mathcal{O}_X). \end{aligned}$$

So if $g_2(X,L) = h^2(\mathcal{O}_X)$, then $h^0(K_{X_{n-3}}+L_{n-3}) = 0$ by $\mathrm{Bs}|L_{n-3}| = \emptyset$. Hence $h^0(K_X+(n-2)L) = 0$ . Therefore by a Sommese's result (Proposition 13.2.4 in [1]) and the adjunction theory, we get the assertion. □

**Theorem 4.2** *Let $(X,L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $L$ is very ample and $g_2(X,L) = h^2(\mathcal{O}_X)+1$. Let $(M,A)$ be a reduction of $(X,L)$. Then $(M,A)$ is one of the following.*

(1) *$(M,A)$ is a Mukai manifold.*

(2) *$(M,A)$ is a Del Pezzo fibration over a smooth curve $C$. Let $f: M \to C$ be its morphism. Then there exists an ample line bundle $H$ on $C$ such that $K_M+(n-2)A = f^*(H)$. In this case $(g(C), \deg H) = (1,1)$.*

(3) *$(M,A)$ is a quadric fibration over a smooth surface $S$. Let $f: M \to S$ be its morphism. Then there exists an ample line bundle $H$ on $S$ such that $K_M + (n-2)A = f^*(K_S+H)$. In this case $(S,H)$ is one of the following types:*
*(3.1) $S$ is a $\mathbb{P}^1$-bundle, $p: S \to B$, over a smooth elliptic curve $B$, and $H = 3C_0 - F$, where $C_0$ (resp. $F$) denotes the minimal section of $S$ with $C_0^2 = 1$ (resp. a fiber of $p$).*
*(3.2) $S$ is an abelian surface, $H^2 = 2$, and $h^0(H) = 1$.*
*(3.3) $S$ is a hyperelliptic surface, $H^2 = 2$, and $h^0(H) = 1$.*

(4) *$(M,A) = (X,L)$, $n = \dim X \geq 4$, and $(X,L)$ is a scroll over a normal projective variety $Y$ of $\dim Y = 3$. If $\dim X \geq 5$, then $Y$ is smooth and there exists an ample vector bundle $\mathcal{E}$ of rank $n-2$ on $Y$ such that $X = \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$, where $H(\mathcal{E})$ is the tautological line bundle on $X$. In this case $(Y, c_1(\mathcal{E}))$ is one of the following.*

   (4.1) *$(Y,c_1(\mathcal{E}))$ is a Mukai manifold. In this case, $(Y,\mathcal{E})$ is one of the following:*
      (4.1.1) $(Y,\mathcal{E}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4})$.
      (4.1.2) $(Y,\mathcal{E}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2})$.
      (4.1.3) $(Y,\mathcal{E}) \cong (\mathbb{P}^3, T_{\mathbb{P}^3})$, *where $T_{\mathbb{P}^3}$ is the tangent bundle of $\mathbb{P}^3$.*
      (4.1.4) $(Y,\mathcal{E}) \cong (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1)^{\oplus 3})$.

   (4.2) *$(Y,c_1(\mathcal{E}))$ is a Del Pezzo fibration over a smooth curve such that $(Y,c_1(\mathcal{E}))$ is the type (2) above. In this case $\dim X = 5$.*

*A sketch of the proof.* (In detail, see Theorem 3.6 in [7].) By the same argument as the proof of Theorem 4.1, we get that $h^0(K_{X_{n-3}}) = 0$ and $h^0(K_{X_{n-3}} + L_{n-3}) = 1$. By using a Beltrametti-Sommese's result (Remark 3.4 in [2]), we find that the nef value of $(M, A)$ is greater than or equal to $n-2$. By the adjunction theory, we can pick up possible types of $(X, L)$. By calculating $g_2(X, L)$ in each case, we get the assertion. □

**Theorem 4.3** *Let $(X, L)$ be a polarized manifold of* $\dim X = n \geq 3$. *Assume that* $Bs|L| = \emptyset$. *Then* $\Delta_2(X, L) = 0$ *if and only if* $g_2(X, L) = 0$.

*Proof.* By Theorem 3.2 (1) we get the assertion. □

**Theorem 4.4** *Let $(X, L)$ be a polarized manifold of* $\dim X = n \geq 3$ *and let* $(M, A)$ *be a reduction of* $(X, L)$. *Assume that L is very ample. If* $\Delta_2(X, L) = 1$, *then* $(X, L)$ *is one of the types* (1), (2), (3.1), (3.3), *and* (4) *in Theorem* 4.2. *Furthermore if* $(X, L)$ *is one of the types* (1), (2), (3.1), (3.3), (4.1.1), (4.1.2), (4.1.3), (4.1.4), *and* (4.2) *in Theorem* 4.2, *then* $\Delta_2(X, L) = 1$.

*A sketch of the proof.* (In detail, see [10].) By Theorem 3.2 (3), we get that $\Delta_2(X, L) = 1$ implies $g_2(X, L) = 1$. Hence $h^2(\mathcal{O}_X) \leq g_2(X, L) \leq h^2(\mathcal{O}_X) + 1$.

If $g_2(X, L) = h^2(\mathcal{O}_X)$, then $(X, L)$ is a scroll over a smooth surface $S$ with $h^2(\mathcal{O}_S) = 1$. But by calculating $\Delta_2(X, L)$, we find that this case is impossible.

If $g_2(X, L) = h^2(\mathcal{O}_X) + 1$, then we can pick up possible types of $(X, L)$ by using Theorem 4.2. By calculating $\Delta_2(X, L)$ in each case, we get the assertion. □

## 5 Problems of polarized manifolds which are analogous to theorems of projective surfaces.

First we define the following.

**Definition 5.1** (See [11].) Let $(X, L)$ be a polarized variety of $\dim X = n$, and let $i$ be an integer with $0 \leq i \leq n$. Then the *i-th sectional H-arithmetic genus* $\chi_i^H(X, L)$ *of* $(X, L)$ is defined by the following.

$$\chi_i^H(X, L) := \chi_{n-i}(X, L).$$

**Remark 5.1** (1) $\chi_i^H(X, L) = 1 - h^1(\mathcal{O}_X) + \cdots + (-1)^{i-1}h^{i-1}(\mathcal{O}_X) + (-1)^i g_i(X, L)$ for every integer $i$ with $1 \leq i \leq n$.

(2) If $X$ is smooth and $\mathrm{Bs}|L| = \emptyset$, then $\chi_i^H(X, L) = \chi(\mathcal{O}_{X_{n-i}})$. (Here we use Notation 2.2.) Namely $\chi_i^H(X, L)$ is the arithmetic genus of $X_{n-i}$ in the sense of Hirzebruch ([13]).

(3) Let $(M, A)$ be a reduction of $(X, L)$. By Remark 2.3 (1) and Remark 5.1 (1), we obtain that $\chi_i^H(X, L) = \chi_i^H(M, A)$ for every integer $i$ with $1 \leq i \leq n$.

(4) I called this invariant the *i-th sectional Todd genus* $\mathrm{Td}_i(X, L)$ before. But from now on, I call this invariant like the above.

By Theorem 3.1 and Remark 5.1 (2), we can expect that the second sectional geometric genus $g_2(X, L)$ and the second sectional H-arithmetic genus $\chi_2^H(X, L)$ reflect the "2-dimensional geometry". So it is natural to consider the following. "Can we get results which are analogous to theorems related to the geometric genus and the Euler-Poincaré characteristic of the structure sheaf of projective surfaces ?"

In this section, we consider this.

First we consider the case where $\mathrm{Bs}|L| = \emptyset$ and we use Notation 2.2.

(A) In this case by Theorem 3.1 (resp. the Lefschetz theorem, Remark 5.1 (2), and the adjunction formula) we get that $g_2(X, L) = h^2(\mathcal{O}_{X_{n-2}})$ (resp. $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_{n-2}})$, $\chi_2^H(X, L) = \chi(\mathcal{O}_{X_{n-2}})$, and $(K_X + (n-2)L)^2 L^{n-2} = K_{X_{n-2}}^2$).

(B) Moreover if $(X, L)$ is not a scroll over a smooth surface, then there is the following correspondence between $\kappa(X_{n-2})$ and $\kappa(K_X + (n-2)L)$ (see [11]).

| Value of $\kappa(X_{n-2})$ | $\Leftrightarrow$ | Value of $\kappa(K_X + (n-2)L)$ |
|---|---|---|
| $-\infty$ | $\Leftrightarrow^*$ | $-\infty$ |
| 0 | $\Leftrightarrow^{**}$ | 0 |
| 1 | $\Leftrightarrow^{**}$ | 1 |
| 2 | $\Leftrightarrow^{**}$ | $\geq 2$ |

(We note that the direction $\Leftarrow$ in $(*)$ and the direction $\Rightarrow$ in $(**)$ need the assumption that $(X, L)$ is not a scroll over a smooth surface.)

(C) Let $(X, L)$ be a polarized manifold which is not a scroll over a smooth surface, let $(M, A)$ be a reduction of $(X, L)$, and we put $M_{n-2} := \mu(X_{n-2})$, where $\mu : X \to M$ is the reduction map. Then $M_{n-2}$ is smooth and $K_{M_{n-2}} = (K_M + (n-2)A)|_{M_{n-2}}$. Assume that $\kappa(X_{n-2}) \geq 0$. (We note that this condition is equivalent to the condition that $\kappa(K_X + (n-2)L) \geq 0$ by above.) Then $\kappa(K_M + (n-2)A) \geq 0$. Hence by the adjunction theory $K_M + (n-2)A$ is nef. In particular, $K_{M_{n-2}}$ is nef. Hence $\mu|_{X_{n-2}} : X_{n-2} \to M_{n-2}$ is the minimalization of $X_{n-2}$.

From (A), (B), and (C), we infer that there are the following correspondence between invariants of smooth projective surfaces $S$ and invariants of $(X, L)$.

| Invariants of $S$ | $\Leftrightarrow$ | Invariants of $(X, L)$ |
|---|---|---|
| $h^2(\mathcal{O}_S)$ | $\Leftrightarrow$ | $g_2(X, L)$ |
| $h^1(\mathcal{O}_S)$ | $\Leftrightarrow$ | $h^1(\mathcal{O}_X)$ |
| $\chi(\mathcal{O}_S)$ | $\Leftrightarrow$ | $\chi_2^H(X, L)$ |
| $K_S^2$ | $\Leftrightarrow$ | $(K_X + (n-2)L)^2 L^{n-2}$ |
| $K_{\widetilde{S}}^2$ | $\Leftrightarrow^*$ | $(K_M + (n-2)A)^2 A^{n-2}$ |
| $\kappa(S) = k$ | $\Leftrightarrow^{**}$ | $\kappa(K_X + (n-2)L) = k$ |
| $\kappa(S) = 2$ | $\Leftrightarrow^{***}$ | $\kappa(K_X + (n-2)L) \geq 2$ |

(In $(*)$, we assume that $\kappa(K_X + (n-2)L) \geq 0$ and let $\widetilde{S}$ (resp. $(M, A)$) be the minimalization of $S$ (resp. a reduction of $(X, L)$). In $(**)$ $k$ is an integer with

$k = -\infty, 0$, or 1, and we assume that $(X, L)$ is not a scroll over a smooth surface. In $(***)$ we assume that $(X, L)$ is not a scroll over a smooth surface. )

By considering these correspondences, we can propose some problems which are analogous to the case of smooth projective surfaces. For example there are the following five theorems of projective surfaces.

**Theorem 1 (Castelnuovo's theorem)** *Let $S$ be a smooth projective surface. Assume that $\kappa(S) \geq 0$ (resp. $\kappa(S) = 2$). Then $\chi(\mathcal{O}_S) \geq 0$ (resp. $\chi(\mathcal{O}_S) > 0$).*

**Theorem 2 (Noether's inequality)** *Let $S$ be a smooth projective surface of general type and let $\widetilde{S}$ be the minimal model of $S$. Then $K_{\widetilde{S}}^2 \geq 2p_g(\widetilde{S}) - 4$.*

**Theorem 3 (Debarre's inequality)** *Let $S$ be a smooth projective surface of general type with $q(S) > 0$, and let $\widetilde{S}$ be the minimal model of $S$. Then $K_{\widetilde{S}}^2 \geq 2p_g(\widetilde{S})$.*

**Theorem 4 (Bogomolov-Miyaoka-Yau's inequality)** *Let $S$ be a smooth projective surface of general type. Then $9\chi(\mathcal{O}_S) \geq K_S^2$.*

**Theorem 5 (Inequality of Castelnuovo-Beauville)** *Let $S$ be a smooth projective surface of general type. Then $p_g(S) \geq 2q(S) - 4$ (that is, $\chi(\mathcal{O}_S) \geq q(S) - 3$ ).*

By using the above correspondences, we can give the following conjedctures. We note that for $k = 1, \ldots, 5$, Conjecture $k$ corresponds Theorem $k$ above.

**Conjecture 1** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 0$ (resp. $\geq 2$). Then $\chi_2^H(X, L) \geq 0$ (resp. $> 0$).*

**Conjecture 2** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$. Let $(M, A)$ be a reduction of $(X, L)$. Then $(K_M + (n-2)A)^2 A^{n-2} \geq 2g_2(M, A) - 4$.*

**Conjecture 3** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$ and $q(X) > 0$. Let $(M, A)$ be a reduction of $(X, L)$. Then $(K_M + (n-2)A)^2 A^{n-2} \geq 2g_2(M, A)$.*

**Conjecture 4** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$. Then $9\chi_2^H(X, L) \geq (K_X + (n-2)L)^2 L^{n-2}$.*

**Conjecture 5** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$. Assume that $\kappa(K_X + (n-2)L) \geq 2$. Then $g_2(X, L) \geq 2q(X) - 4$ (that is, $\chi_2^H(X, L) \geq q(X) - 3$).*

For Conjecture 1 we get the following result.

**Theorem 5.1** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 4$. Assume that $\kappa(X) \geq 0$. Then $\chi_2^H(X, L) > 0$.*

For the proof, see Corollary 3.5.2 in [8] □.

Here we note that in my preprint [11], we study Conjecture 1 and Conjecture 4. Furthermore we propose more stronger conjecture than Conjecture 4. See my preprint [11] in detail.

Next we consider the following situation. Let $(X, L)$ be a polarized manifold and let $f : X \to C$ be a fiber space over a smooth projective curve $C$. Then we consider a polarized version of the following theorems.

**Theorem 6 (Beauville's inequality)** *Let $S$ be a smooth projective surface and let $f : S \to C$ be a fiber space over a smooth projective curve $C$. Then $\chi(\mathcal{O}_S) \geq (g(F) - 1)(g(C) - 1)$, where $F$ is a general fiber of $f$.*

**Theorem 7 (Arakelov's inequality)** *Let $S$ be a smooth projective surface and let $f : S \to C$ be a fiber space over a smooth projective curve $C$. Let $S'$ be a relatively minimal model of $S$. Assume that $g(F) \geq 2$, where $F$ is a general fiber of $f$. Then $K_{S'}^2 \geq 8(g(F) - 1)(g(C) - 1)$.*

First we prove a polarized version of Theorem 7.

**Theorem 5.2** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$ and let $f : X \to C$ be a fiber space over a smooth projective curve $C$. Let $(M, A)$ be a reduction of $(X, L)$. (Then there exists a fiber space $h : M \to C$ such that $f = h \circ \pi$, where $\pi : X \to M$ is the reduction map.) Assume that $g(A_{F_h}) \geq 2$ and $(F_h, A_{F_h})$ is not a scroll over a smooth curve, where $F_h$ is a general fiber of $h$. Then*

$$(K_M + (n-2)A)^2 A^{n-2} \geq 8(g(A_{F_h}) - 1)(g(C) - 1).$$

Proof. First we calculate $(K_M + (n-2)A)^2 A^{n-2}$.

$$\begin{aligned}
&(K_M + (n-2)A)^2 A^{n-2} \\
&= (K_{M/C} + (n-2)A)(K_M + (n-2)A)A^{n-2} \\
&\quad +(2g(C) - 2)(K_{F_h} + (n-2)A_{F_h})A_{F_h}^{n-2} \\
&= (K_{M/C} + (n-2)A)^2 A^{n-2} + 2(2g(C) - 2)(K_{F_h} + (n-2)A_{F_h})A_{F_h}^{n-2} \\
&= (K_{M/C} + (n-2)A)^2 A^{n-2} + 2(2g(C) - 2)(2g(A_{F_h}) - 2), \qquad (4)
\end{aligned}$$

where $F_h$ is a general fiber of $h$.

Since $g(A_{F_h}) \geq 2$ and $(F_h, A_{F_h})$ is not a scroll over a smooth curve, by Theorem 1.1.1, Theorem 1.1.2, and Theorem 1.1.3 in [6], we get that $K_{M/C} + (n-2)A$ is nef. Therefore $(K_{M/C} + (n-2)A)^2 A^{n-2} \geq 0$. So we get the assertion by (4). □

The following example shows that the assumption that $(F_h, A_{F_h})$ is not a scroll over a smooth curve is necessary.

**Example 5.1** Let $F$ and $C$ are smooth projective curves with $g(F) \geq 2$ and we put $S := F \times C$. Let $\pi : S \to C$ be the second projection. Let $\mathcal{E}$ be an ample vector bundle on $S$ of rank $n-1$. We put $X := \mathbb{P}_S(\mathcal{E})$, $L := H(\mathcal{E})$, and $f := \pi \circ p$, where $H(\mathcal{E})$ is the tautological line bundle on $X$ and $p : X \to S$ be the projection. Let $F_f$ be a general fiber of $f$. Then $(F_f, L_{F_f})$ is a scroll over $F$, $g(L_{F_f}) \geq 2$, $(X, L)$ is a reduction of itself, $K_S^2 = 8(g(F)-1)(g(C)-1) = 8(g(L_{F_f})-1)(g(C)-1)$, and $(K_X + (n-2)L)^2 L^{n-2} = K_S^2 - c_2(\mathcal{E}) < K_S^2$.

Next we give a conjecture which is a polarized version of Theorem 6.

**Conjecture 6** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 3$ and let $f : X \to C$ be a fiber space over a smooth projective curve $C$. Then $\chi_2^H(X, L) \geq (g(L|_F)-1)(g(C)-1)$, where $F$ is a general fiber of $f$.*

For Conjecture 6, we get the following result.

**Theorem 5.3** *Let $(X, L)$ be a polarized manifold of $\dim X = n \geq 4$. Assume that $\kappa(X) \geq 0$ and there exists a fiber space $f : X \to C$ over a smooth curve $C$. Let $F$ be a general fiber of $f$. Then*

$$\chi_2^H(X, L) \geq \frac{1}{3}(g(L|_F)-1)(g(C)-1) + \frac{n^2-5n+5}{12}L^n.$$

*Proof.* Let $(M, A)$ be a reduction of $(X, L)$. Then there exists a fiber space $h : M \to C$ such that $f = h \circ \pi$, where $\pi : X \to M$ is the reduction map. Here we note the following.

**Proposition 5.1** *Let $X$ be a smooth projective variety of $\dim X = n \geq 3$ such that $X$ is not uniruled. Let $L$ be an ample line bundle on $X$. Then*

$$c_2(X)L^{n-2} \geq -\binom{n}{2}L^n - (n-1)K_X L^{n-1}.$$

For the proof, see Proposition 3.4 in [8]. □

By Remark 2.1 (3), Remark 5.1 (3), and Proposition 5.1, we get that

$$\begin{aligned}
\chi_2^H(X, L) &= \chi_2^H(M, A) \\
&= \frac{1}{12}(K_M + (n-1)A)(K_M + (n-2)A)A^{n-2} + \frac{1}{12}c_2(M)A^{n-2} \\
&\quad + \frac{n-3}{24}(2K_M + (n-2)A)A^{n-1} \\
&\geq \frac{1}{12}K_M(K_M + (n-2)A)A^{n-2} + \frac{1}{12}(n-1)(\frac{n}{2}-2)A^n \\
&\quad + \frac{n-3}{24}(2K_M + (n-2)A)A^{n-1} \\
&= \frac{1}{12}K_{M/C}(K_M + (n-2)A)A^{n-2} + \frac{1}{12}h^*(K_C)(K_M + (n-2)A)A^{n-2} \\
&\quad + \frac{1}{12}(n-1)(\frac{n}{2}-2)A^n + \frac{n-3}{24}(2K_M + (n-2)A)A^{n-1}.
\end{aligned}$$

Since $\kappa(X) \geq 0$, we obtain that $K_M A^{n-1} \geq 0$, $K_M + (n-2)A$ is nef by the adjunction theory, and $h_*(K_{M/C}^{\otimes m}) \neq 0$ for sufficiently large $m$. Therefore $h_*(K_{M/C}^{\otimes m})$ is semipositive by a Kawamata's theorem [14]. Hence $K_{M/C}^{\otimes m}$ is pseudo-effective by Remark 1.3.2 in [6]. Therefore $K_{M/C}(K_M + (n-2)A)A^{n-2} \geq 0$. So we get that

$$\begin{aligned}
\chi_2^H(X,L) &= \chi_2^H(M,A) \\
&\geq \frac{1}{12}K_{M/C}(K_M+(n-2)A)A^{n-2} + \frac{1}{12}h^*(K_C)(K_M+(n-2)A)A^{n-2} \\
&\quad + \frac{1}{12}(n-1)(\frac{n}{2}-2)A^n + \frac{n-3}{24}(2K_M+(n-2)A)A^{n-1} \\
&\geq \frac{1}{12}(2g(C)-2)(K_{F_h}+(n-2)A|_{F_h})(A|_{F_h})^{n-2} + \frac{1}{12}(n-1)(\frac{n}{2}-2)A^n \\
&\quad + \frac{n-3}{24}(2K_M+(n-2)A)A^{n-1} \\
&\geq \frac{1}{3}(g(C)-1)(g(A|_{F_h})-1) + \frac{n^2-5n+5}{12}A^n,
\end{aligned}$$

where $F_h$ is a general fiber of $h$.

On the other hand, since $A^n \geq L^n$ and $g(L|_F) = g(A|_{F_h})$, we get the assertion. □

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