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A NEW SIMULATION METHOD OF DIFFUSION PROCESSES APPLIED TO FINANCE

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ABSTRACT. The authors apply a new simulation method of diffusion processes to finance problems and show this new method realizes extremely fast calculation in some cases.

0. INTRODUCTION

Numerical calculation of $E[f(X(T,x))]$, where $X(t,x)$ denotes a diffusion process that is a solution of some SDE and f is a function with some regularity, is very important in many fields. Pricing of derivative securities is a typical example. Many studies have been done in this field (cf. [1] [3]). In [4] and [5], a new simulation scheme is proposed. The method is based on Malliavin calculus and higher order stochastic expansion and we can consider that it belongs to the higher order scheme. [6] and [7] show that this method achieves very fast calculation when it is applied to some finance problems. In [11], Talay and Tubaro reported that the higher order scheme does not have practical advantages over Euler-Maruyama scheme with acceleration methods. In this paper, we give some numerical examples and show that this new method with the proposed sampling methods and acceleration methods works in practice and outperforms existing methods in some cases.

0.1. **Notation.** $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ are sets of all natural numbers, all integers, and all real numbers, respectively. We also set

$$\mathbb{R}_{>a} = \{r \in \mathbb{R} \mid r > a\}$$
$$\mathbb{Z}_{\geq n} = \{i \in \mathbb{Z} \mid i \geq n\}$$
$$\vdots$$

and so on. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x, and $\mathrm{fp}(x) = x - \lfloor x \rfloor$.

1. A NEW APPROXIMATION SCHEME

We introduce the new approximation scheme following [4] and [5].

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1.1. **The Problem.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. In this paper, we consider the numerical calculation of $E[f(X(T,x))]$ where $T \in \mathbb{R}_{>0}$ and X is a diffusion process determined by the following stochastic differential equation:

$$X(t,x) = x + \sum_{j=0}^{d} \int_0^t V_j(X(s,x)) \circ dB^j(s) \tag{1}$$
$$V_j \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N),$$

where $\circ dB^j(s)$ denotes the Stratonovich integral, $B^0(t) = t$, $(B^1(t), \ldots, B^d(t))$ is the d-dimensional standard Brownian Motion, and f is an $\mathbb{R}$-valued function defined on $\mathbb{R}^n$. We often identify $V_j \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ with a section of the tangent bundle $T\mathbb{R}^N$ in the following manner: $V_j(x) = (V_j^1(x), \ldots, V_j^N(x)) \approx \sum_{i=1}^N V_j^i(x)(\partial/\partial x_i)$. Here $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ denotes the set of $\mathbb{R}^N$ valued smooth functions defined over $\mathbb{R}^N$ whose derivatives of any order are bounded. The following equation (2) is the Ito stochastic differential equation corresponding to (1).

$$X(t,x) = x + \sum_{j=0}^{d} \int_0^t \hat{V}_j(X(s,x))\, dB^j(s). \tag{2}$$

Let $\{P_t\}_{t \geq 0}$ be the semigroup of linear operators defined by

$$(P_t f)(x) = E[f(X(t,x))] \tag{3}$$

for $t \geq 0$ and $f \in C_b^\infty(\mathbb{R}^N)$.

1.2. **The Euler-Maruyama approximation.** The Euler-Maruyama approximation $X^{(n)}(t,x)$ of the process $X(t,x)$ is constructed by a difference equation $X^{(n)}(t_{i+1}, x) = \sum_{j=0}^d \hat{V}_j\left(X^{(n)}(t_i,x)\right)\sqrt{T/n}Z$, $X^{(n)}(0,x) = x$, where $t_i = iT/n\,(i = 0, \ldots, n-1)$, $Z = (Z^0, Z^1, \ldots, Z^d)$, $Z^0 = \sqrt{T/n}$, and $Z^i\,(i = 1, \ldots, d)$ are independent random variables that follow normal distribution $N(0,1)$. It is well known that when f is of class C_b^4,

$$\left|(P_T f)(x) - E\left[f\left(X^{(n)}(T,x)\right)\right]\right| = O\left(n^{-1}\right). \tag{4}$$

1.3. **The new approximation scheme.** We describe the new approximation scheme by using the notions of free Lie algebra [9].

1.3.1. *Free Lie algebra.* Let $A = \{v_0, v_1, \ldots, v_d\}$, $(d \geq 1)$ be an alphabet, that is to say, a set of letters. A^* denotes the free monoid on A, that is to say, the set of all words on A with concatenation product. The empty word is the identity of A^* and denoted by 1. For an arbitrary multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$, $(\alpha_i \in \{0, 1, \ldots, d\},\ i = 1, \ldots, k)$, v_α denotes the word $v_{\alpha_1} \cdots v_{\alpha_k}$ and $|v_\alpha|$ is defined as k. We also define $\|v_\alpha\|$ to be $|v_\alpha| + \mathrm{card}(\{i | \alpha_i = 0\})$. A_m^* denotes $\{w \in A^* |\ |w| = m\}$ and $A_{\leq m}^*$ denotes $\{w \in A^* |\ |w| \leq m\}$. Let K be a field, $K\langle A \rangle$ be the K-coefficients free algebra with basis A^*, and $K\langle\langle A \rangle\rangle$ be the set of all K-coefficients formal series with basis A^*. $K\langle A \rangle$ is a sub K-algebra of $K\langle\langle A \rangle\rangle$. We call the element of $K\langle A \rangle$, non-commutative polynomial. For all $P \in K\langle\langle A \rangle\rangle$, we write them as

$$P = \sum_{w \in A^*} (P, w)\, w \quad \text{or} \quad \sum_{w \in A^*} a_w w,$$

where $(P, w) = a_w \in K$ denotes the coefficient of w. The algebra structure is defined as usual, that is to say,

$$\left(\sum_{w\in A^*} a_w w\right)\left(\sum_{w\in A^*} b_w w\right) = \sum_{\substack{w=uv\\ w\in A^*}} a_u b_v w.$$

The Lie bracket is defined as $[x, y] = xy - yx$ for $x, y \in K\langle\langle A\rangle\rangle$. For $w = v_{i_1}\cdots v_{i_n} \in A^*$, $l(w)$ denotes $[[\ldots[v_{i_1}, v_{i_2}], \ldots, v_{i_{n-1}}], v_{i_n}]$. We define $\mathcal{L}_K(A)$ to be the set of Lie polynomials in $K\langle A\rangle$ and $\mathcal{L}_K((A))$ to be the set of Lie series. That is to say, $\mathcal{L}_K(A)$ is the smallest sub K-module of $K\langle A\rangle$ that includes A and is closed under the Lie bracket, and $\mathcal{L}_K((A))$ is the set of elements of $K\langle\langle A\rangle\rangle$ whose every homogeneous components belong to $\mathcal{L}_K(A)$. For $m \in \mathbb{Z}_{\geq 0}$, we define $K\langle A\rangle_m$, $K\langle A\rangle_{\leq m}$, and $j_m : K\langle\langle A\rangle\rangle \longrightarrow K\langle A\rangle_{\leq m}$ as follows:

$$K\langle A\rangle_m = \{P \in K\langle A\rangle \mid (P, w) = 0, \text{ if } \|w\| \neq m\}$$
$$K\langle A\rangle_{\leq m} = \{P \in K\langle A\rangle \mid (P, w) = 0, \text{ if } \|w\| > m\}$$
$$j_m\left(\sum_{w\in A^*} a_w w\right) = \sum_{\|w\|\leq m} a_w w.$$

For arbitrary $P, Q \in K\langle A\rangle$, the inner product $\langle P, Q\rangle$is defined as follows:

$$\langle P, Q\rangle = \sum_{w\in A^*} (P, w)(Q, w).$$

For an arbitrary $P \in K\langle\langle A\rangle\rangle$ whose constant term equals to 0, that is to say, $(P, 1) = 0$, we can define its exponential $\exp(P)$ to be $1 + \sum_{k=1}^{\infty} P^k/k!$. We can also define $\log(Q)$ to be $\sum_{k=1}^{\infty}(-1)^{k-1}(Q-1)^k/k$ for $Q \in K\langle\langle A\rangle\rangle$ if its constant term equals to 1, that is to say, $(Q, 1) = 1$. The following relations hold:

$$\log(\exp(P)) = P, \quad \text{and} \quad \exp(\log(Q)) = Q.$$

In the following part of this paper, we consider only the case in which $K = \mathbb{R}$. We define $\|P\|_2 = (\langle P, P\rangle)^{1/2}$, for $P \in \mathbb{R}\langle A\rangle$.

1.3.2. *Stochastic Taylor expansion.* For the process X defined in (1) and an arbitrary $f \in C_b^\infty(\mathbb{R}^N)$, we obtain the following expansion:

$$\begin{aligned} f(X(t,x)) &= f(x) + \sum_{i=0}^{d}\int_0^t (V_i f)(X(s,x)) \circ dB^i(s) \\ &\vdots \\ &= f(x) + \sum_{j=1}^{k}\sum_{i_1,\ldots,i_j=0}^{d} \left(V_{i_1}\ldots V_{i_j} f\right)(x) B(t; i_1\ldots i_j) + R_k^f(t,x), \end{aligned} \tag{5}$$

where

$$\begin{aligned} B(t; v_i) &= B^i(t), \quad \text{for } i \in \{0, 1, \ldots, d\}, \\ B(t; uv_i) &= \int_0^t B(s; u) \circ dB^i(s), \quad \text{for } u \in A^*,\ i \in \{0, 1, \ldots, d\}. \end{aligned} \tag{6}$$

We extend the definition of $B(t;P)$ to the case in which $P \in \mathbb{R}\langle A\rangle$ by

$$B\left(t; \sum_{w\in A^*} a_w w\right) = \sum_{w\in A^*} a_w B(t;w).$$

1.3.3. *$\mathbb{R}\langle\langle A\rangle\rangle$-valued processes.* By the natural identification $\mathbb{R}\langle\langle A\rangle\rangle \approx \mathbb{R}^\infty$, we can induce the direct product topology into $\mathbb{R}\langle\langle A\rangle\rangle$. $\mathbb{R}\langle\langle A\rangle\rangle$ becomes a Polish space by the topology and we can consider its Borel σ-algebra $\mathcal{B}(\mathbb{R}\langle\langle A\rangle\rangle)$, $\mathbb{R}\langle\langle A\rangle\rangle$-valued random variables, their expectations, and other notions as usual. $\mathbb{R}\langle\langle A\rangle\rangle$-valued continuous semi-martingale is a map $\mathcal{X} : \mathbb{R}_{\geq 0} \times \Omega \longrightarrow \mathbb{R}\langle\langle A\rangle\rangle$ which satisfies the following condition that there exist continuous semi-martingale $\mathcal{X}_w : \mathbb{R}_{\geq 0} \times \Omega \longrightarrow \mathbb{R}$ and satisfy $\mathcal{X}(t) = \sum_{w\in A^*} \mathcal{X}_w(t)w$. For $\mathbb{R}\langle\langle A\rangle\rangle$-valued continuous semi-martingales $\mathcal{X}$ and $\mathcal{Y}$, we define $\mathbb{R}\langle\langle A\rangle\rangle$-valued continuous semi-martingale $\int_0^t \mathcal{X}(s) \circ d\mathcal{Y}(s)$ and $\int_0^t \circ d\mathcal{X}(s)\mathcal{Y}(s)$ as follows:

$$\int_0^t \mathcal{X}(s) \circ d\mathcal{Y}(s) = \sum_{u,w\in A^*} \left(\int_0^t \mathcal{X}_u(s) \circ d\mathcal{Y}_w(s)\right) uw,$$

$$\int_0^t \circ d\mathcal{X}(s)\mathcal{Y}(s) = \sum_{u,w\in A^*} \left(\int_0^t \mathcal{Y}_w(s) \circ d\mathcal{X}_u(s)\right) uw,$$

where

$$\mathcal{X}(t) = \sum_{u\in A^*} \mathcal{X}_u(t)u, \text{ and } \mathcal{Y}(t) = \sum_{w\in A^*} \mathcal{Y}_w(t)w.$$

The Ito formula

$$\mathcal{X}(t)\mathcal{Y}(t) - \mathcal{X}(0)\mathcal{Y}(0) = \int_0^t \mathcal{X}(s) \circ d\mathcal{Y}(s) + \int_0^t \circ d\mathcal{X}(s)\mathcal{Y}(s) \tag{7}$$

holds.

We consider the stochastic differential equation

$$\mathcal{X}(t) = 1 + \sum_{i=0}^{d} \int_0^t \mathcal{X}(s)v_i \circ dB^i(s), \tag{8}$$

in $\mathbb{R}\langle\langle A\rangle\rangle$, which corresponds to the stochastic differential equation (1). From (7), it is easy to see that (8) has a unique solution

$$\mathcal{X}(t) = \sum_{w\in A^*} B(t;w)w. \tag{9}$$

We have the following theorem.

Theorem 1.1 ([5][10]). $\log(\mathcal{X}(t)) \in \mathcal{L}_{\mathbb{R}}((A))$.

This theorem is proved in [5] and [10] by using Friedrich's theorem.

1.3.4. *The new approximation scheme.* We construct a family of linear operators in $C_b(\mathbb{R}^N)$ that approximate $\{P_t\}_{t\in\mathbb{R}_{\geq 0}}$ defined in (3). We define Φ to be a homomorphism between $\mathbb{R}\langle A\rangle$ and the $\mathbb{R}$-algebra which consists of smooth differential operators over $\mathbb{R}^N$ defined by:

$$\begin{aligned} &\Phi(1) = \mathrm{Id}, \\ &\Phi(v_{i_1}\cdots v_{i_n}) = V_{i_1}\cdots V_{i_n},\ i_1,\ldots,i_n \in \{0,1,\ldots,d\}. \end{aligned} \tag{10}$$

In this paper, we assume that the stochastic differential equation (1) satisfies the following condition **(UFG)**:

(UFG): There exist an integer l, and $\varphi_{u,u'} \in C_b^\infty(\mathbb{R}^N)$ which satisfy

$$\Phi(\mathfrak{l}(u)) = \sum_{u' \in A^*_{\leq l} \setminus \{1, v_0\}} \varphi_{u,u'} \Phi(\mathfrak{l}(u')),$$

for any $u \in A^* \setminus \{1, v_0\}$.

Definition 1.1. *A family $\{Q_{(s)}\}_{0 \leq s < 1}$ of linear operators in $C_b(\mathbb{R}^N)$ is called m-similar ($m \in \mathbb{Z}_{\geq 1}$), if there are constants $C \in \mathbb{R}_{>0}$ and $M \in \mathbb{R}_{\geq m+1}$ which satisfy*

$$\left\| P_s f - Q_{(s)} f \right\|_\infty \leq C \left(\sum_{k=m+1}^{M} s^{k/2} \|f\|_{V,k} + s^{(m+1)/2} \| \operatorname{grad}(f) \|_\infty \right),$$

$$\left\| Q_{(s)} f - P_s f \right\|_\infty \leq C s^{1/2} \| \operatorname{grad}(f) \|_\infty,$$

and

$$\left\| Q_{(s)} f \right\|_\infty \leq \exp(Cs) \|f\|_\infty$$

for any $s \in (0, 1]$ and $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$, where

$$\|f\|_{V,n} = \sum_{k=1}^{n} \sum_{\substack{u_1, \ldots, u_k \in A^* \setminus \{1, v_0\} \\ \|u_1 \cdots u_k\| = n}} \|\Phi(\mathfrak{l}(u_1) \cdots \mathfrak{l}(u_k)) f\|_\infty.$$

Definition 1.2. *An $\mathcal{L}_{\mathbb{R}}((A))$-valued random variable ξ is called m-$\mathcal{L}$-moment similar ($m \in \mathbb{Z}_{\geq 2}$), if it satisfies the following conditions:*

(i) $E[\|j_n \xi\|_2^p] < +\infty$ *for any* $n \in \mathbb{Z}_{\geq 1}$ *and* $p \in \mathbb{R}_{>1}$,
(ii) $(\xi, v_0) = 1$ *with probability* 1,
(iii) $E[j_m(\exp(\xi))] = E[j_m(\mathcal{X}(1))]$,

where $\mathcal{X}$ is the $\mathbb{R}\langle\langle A \rangle\rangle$-valued random variable defined by (9).

Example 1.1. *$\xi = \log(\mathcal{X}(1))$ is a trivial example of m-$\mathcal{L}$-similar random variable.*

Example 1.2. *In the case of $d = 1$, an $\mathbb{R}\langle\langle A \rangle\rangle$-valued random variable ξ defined as:*

$$(\xi, v_0) = 1, \quad (\xi, v_1) = \eta, \quad (\xi, v_1 v_1) = \frac{\eta^2}{2},$$

$$(\xi, v_1 v_0) = (\xi, v_0 v_1) = \frac{\eta}{2}, \quad (\xi, v_1 v_1 v_1) = \frac{\eta^3}{6},$$

$$(\xi, v_1 v_1 v_0) = (\xi, v_0 v_1 v_1) = \frac{1}{4}, \quad (\xi, v_0 v_0) = \frac{1}{2}, \quad (\xi, v_1 v_1 v_1 v_1) = \frac{1}{8},$$

$$(\xi, w) = 0 \quad \textit{otherwise},$$

where η is a random variable which satisfies $P(\eta = 0) = 2/3$, $P(\eta = \pm\sqrt{3}) = 1/6$ becomes a 5-$\mathcal{L}$-similar random variable.

Example 1.3. *In the case of $d = 2$, an $\mathbb{R}\langle\langle A\rangle\rangle$-valued random variable ξ defined as:*

$$(\xi, v_0) = 1, \quad (\xi, v_i) = \eta_i,$$

$$(\xi, v_1v_2) = \frac{1}{2}(\eta_1\eta_2 + \sigma), \quad (\xi, v_2v_1) = \frac{1}{2}(\eta_1\eta_2 - \sigma), \quad (\xi, v_iv_i) = \frac{\eta_i^2}{2},$$

$$(\xi, v_iv_0) = (\xi, v_0v_i) = \frac{\eta_i}{2}, \quad (\xi, v_iv_iv_i) = \frac{\eta^3}{6},$$

$$(\xi, v_1v_1v_2) = (\xi, v_2v_1v_1) = \frac{\eta_2}{4}, \quad (\xi, v_2v_2v_1) = (\xi, v_1v_2v_2) = \frac{\eta_1}{4}, \quad (\xi, v_0v_0) = \frac{1}{2},$$

$$(\xi, v_1v_1v_2v_2) = (\xi, v_2v_2v_1v_1) = \frac{1}{8}, \quad (\xi, v_0v_iv_i) = (\xi, v_iv_iv_0) = \frac{1}{4}, \quad i \in \{1, 2\},$$

$$(\xi, w) = 0 \quad \textit{otherwise},$$

where σ, η_1, and η_2 are independent random variables which satisfy $P(\sigma = \pm 1) = 1/2$, $P(\eta_i = 0) = 2/3$, and $P(\eta_i = \pm\sqrt{3}) = 1/6$ $(i = 1, 2)$ becomes a 5-$\mathcal{L}$-similar random variable.

For $s \in \mathbb{R}_{>0}$, we define $\Psi_s : \mathbb{R}\langle\langle A\rangle\rangle \longrightarrow \mathbb{R}\langle\langle A\rangle\rangle$ as follows:

$$\Psi_s\left(\sum_{m=0}^{\infty} P_m\right) = \sum_{m=0}^{\infty} s^{m/2} P_m, \text{ where } P_m \in \mathbb{R}\langle A\rangle_m.$$

Following two theorems give the new simulation scheme.

Theorem 1.2. *Let $m \in \mathbb{Z}_{\geq 1}$ and ξ be an m-$\mathcal{L}$-similar random variable. We also let $Y : (0, 1] \times \mathbb{R}^N \times \Omega \longrightarrow \mathbb{R}^N$ be a measurable map that satisfies the following conditions:*

(i) $Y(s, \cdot, \omega) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous for any $s \in (0, 1]$ and $\omega \in \Omega$,

(ii)

$$\sup_{0<s\leq 1, x\in\mathbb{R}^N} s^{(m+1)/2} E\left[|Y(s, x)|\right] < \infty.$$

Then, the family of linear operators $\{Q_{(s)}\}_{0\leq s<1}$ defined by

$$\left(Q_{(s)} f\right)(x) = E\left[f\left(\exp\left(\Phi\left(j_m\left(\Psi_s(\xi)\right)\right)\right)(x) + Y(s, x)\right)\right],$$

where $f \in C_b(\mathbb{R}^N)$, becomes m-similar.

This theorem shows how to construct m-similar operators and the following theorem shows the new approximation scheme.

Theorem 1.3. *We assume (1) satisfies the condition **(UFG)**. Let $m \geq 1$ and $\{Q_{(s)}\}_{0\leq s<1}$ be an m-similar family of linear operators as defined in Definition 1.1. Let $t_k^{(n)} = k^\gamma n^{-\gamma} T$ where $n \in \mathbb{N}$, $\gamma \in \mathbb{R}_{>0}$ and $k = 0, 1, \ldots, n$, and let $s_k = t_k^{(n)} - t_{k-1}^{(n)}$. For $f \in C_b^\infty(\mathbb{R}^N)$, we define*

$$\epsilon(f) = \left\| P_T f - Q_{(s_n)} Q_{(s_{n-1})} \cdots Q_{(s_1)} f \right\|_\infty.$$

Then we have the following statements. If $0 < \gamma < m - 1$, there exists a constant $C > 0$ which satisfies

$$\epsilon(f) \leq C n^{-\gamma/2} \left\|\mathrm{grad}(f)\right\|_\infty \quad \textit{for all } f \in C_b^\infty(\mathbb{R}^N) \textit{ and } n \in \mathbb{N}.$$

If $\gamma = m - 1$, there exists a constant $C > 0$ which satisfies

$$\epsilon(f) \leq C n^{-(m-1)/2} \log(n + 1) \left\|\mathrm{grad}(f)\right\|_\infty \quad \textit{for all } f \in C_b^\infty(\mathbb{R}^N) \textit{ and } n \in \mathbb{N}.$$

If $m-1<\gamma$, *there exists a constant* $C>0$ *which satisfies*

$$\epsilon(f) \leq Cn^{-(m-1)/2} \|\mathrm{grad}(f)\|_\infty \quad \textit{for all } f \in C_b^\infty(\mathbb{R}^N) \textit{ and } n \in \mathbb{N}.$$

2. Numerical examples: Application to Finance

Taking Theorem 1.3 and (4) into account, we can reasonably expect that the operator $Q_{(s_n)}Q_{(s_{n-1})}\cdots Q_{(s_1)}$ realizes more accurate approximation than the Euler-Maruyama approximation does. We confirm this expectation in the following. We give some numerical examples and compare the new approximation scheme with Euler-Maruyama scheme. In the following, we call the new approximation scheme based on the results in the previous section K-method.

2.1. Comparison to Euler-Maruyama approximation (1): 1 dimensional case. We calculate the price of an Asian call option with maturity T and strike price K written on a stock. The payoff at T of this option is

$$\max\left(0, \frac{1}{T}\int_0^T X_1(t,x)\,dt - K\right)$$

where $X_1(t,x)$ denotes the price process of the underlying stock. We assume that $X_1(t,x)$ follows the Black-Scholes equation $dX_1(t,x) = X_1(t,x)(r'\,dt + \sigma\,dB(t))$ where r' and σ are constants and $B(t)$ is the 1-dimensional standard Brownian motion with respect to the equivalent martingale measure. In this example, $r' = 0.05$, $\sigma = 0.3$, $T = 0.25$ and $K = 1.05$. Let $X_2(t,x) = \int_0^t X_1(s,x)\,ds$ and $X(t,x) = (X_1(t,x), X_2(t,x))$. The price of this Asian option is $e^{-r'T}E[f(X(T,x))]$. We use the 5-$\mathcal{L}$-similar random variable defined in Example 1.2. Figure 1 shows the relation between the number of partition and the approximation error that comes from the discretization. We compare following four methods:

(i.) Euler-Maruyama approximation.
(ii.) Euler-Maruyama approximation with Romberg extrapolation [11][3].
(iii.) K-method.
(iv.) K-method with Romberg extrapolation.

Here we set $\gamma = 2$. Theorem 1.3 suggests that greater γ is better as far as convergence order is concerned. In reality, the constant term is very important and in this case, $\gamma = 2$ realizes the best results. In the case of Euler-Maruyama approximation, we calculate the expectation by using Monte Carlo method. In the case of K-method, we proceed exhaustive calculation. Overall results from the point of CPU time required for 4-digits accuracy are shown in Figure 2.1 and Figure 2.1.

2.2. Partial sampling problem. As we see in Examples 1.2 and 1.3, we construct ξ, the m-$\mathcal{L}$-similar random variable included in the new approximation scheme to be discrete. In the cases of Example 1.2 and Example 1.3, the numbers of states of ξ are 3 and $2 \times 3 \times 3 = 18$, respectively. Then, if we discretize the time by partitioning the simulation period $[0,T]$ into n parts, the number of points to be calculated becomes 3^n in the case of Example 1.2 and 18^n in the case of Example 1.3. In the latter case, $n = 7$ is enough for us to giving up the exhaustive calculation. Even in the former case, it is not possible to calculate all points in practice when n is 15 or 16. As a result, there are many cases in which we cannot do without partial sampling from finite set of points.

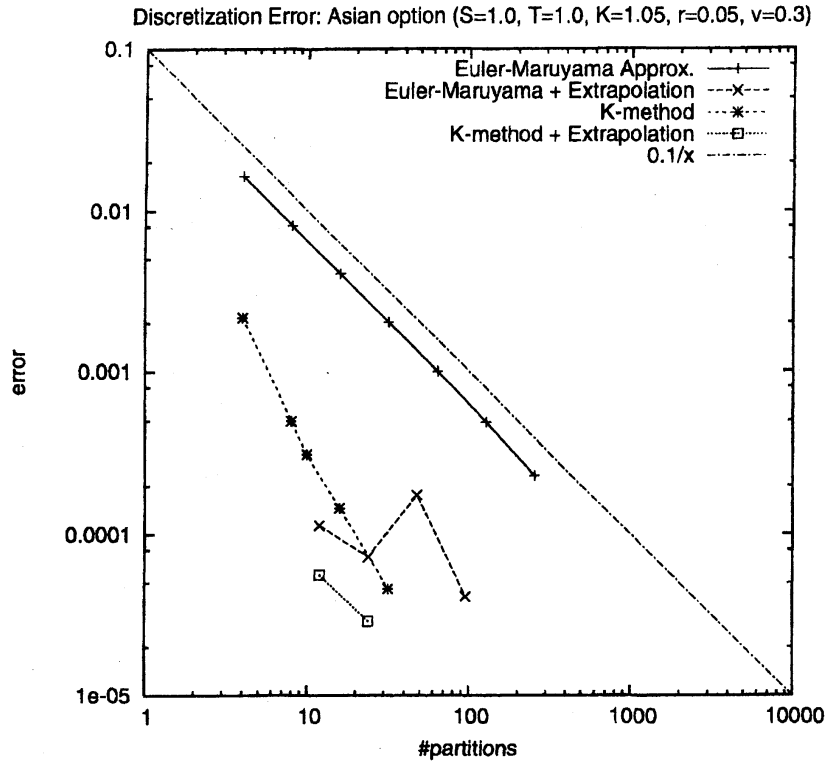

FIGURE 1. Approximation Error and number of partition. Asian option, BS-model

Method	#Partition	#Sample	CPU time (sec)
E-M	2000	3.6×10^7	20563.2
E-M + Extrpltn	$32 + 64$	3.24×10^8	856.6
K-method	20	5.76×10^6	573.69
K-method + Extrpltn	$4 + 8$	6642	0.0170

FIGURE 2. #Partition, #Sample, and CPU time required for 4-digits accuracy

E-M	E-M + Extrpltn	K-method	K-method + Extrpltn
0.0279	0.67	1.0	3.37×10^4

FIGURE 3. Performance ratio (K-method = 1, greater = faster)

2.2.1. *Tree based branching algorithm (TBBA).* The tree based branching algorithm (TBBA) is introduced by Crisan and Lyons in [2] and it is shown in the paper that the algorithm realized the best sampling in some sense. In [7], it is reported that when we apply this algorithm to the K-method, we can reduce the number of sampling points by using the algorithm called tree based branching algorithm (TBBA). Here we give a short explanation of TBBA. See [2] and [7] for details.

Let I be a countable set and $\mathcal{M}^P(I)$ be the set of probability measures on I with the product topology. $\mathcal{B}(\mathcal{M}^P(I))$ is the Borel σ-algebra on $\mathcal{M}^P(I)$ generated by the cylinder sets as usual. We prepare another probability space $(\Omega, \mathcal{F}, P)$ with no atoms. Let $a \in \mathcal{M}^P(I)$ be a probability with finite entropy $H(a) = \sum_{i\in I} a(i)\log a(i) < \infty$ and $\mathcal{A}_n^a$, $n \in \mathbb{N}$ be a set of $\mathcal{M}^P(I)$-valued random variables of $(\Omega, \mathcal{F}, P)$ defined as follows:

$$\mathcal{A}_n^a = \left\{ \tilde{a} : \Omega \to \mathcal{M}^P(I) \;\middle|\; \begin{array}{l} \exists V_k : \Omega \to I,\ \mathcal{F}\text{-measurable s.t.} \\ \tilde{a} = \frac{1}{n}\sum_{k=1}^n \delta_{V_k}, \quad E_\Omega^P[\tilde{a}] = a \end{array} \right\}, \tag{11}$$

where δ_x denotes the Dirac distribution concentrated at x. We can regard $\mathcal{A}_n^a$ to be a set of unbiased estimators that takes n samples, or a set of measures realized by n samples. What we want is how to find a $\tilde{a} \in \mathcal{A}_n^a$ which has small discrepancy with respect to a. For measuring the discrepancy between an element $\tilde{a}$ of $\mathcal{A}_n^a$ and a, we introduce $H(\tilde{a}|a)$, the mean relative entropy of $\tilde{a}$ with respect to a, as follows:

(12)

$$H(\tilde{a}|a) = E_\Omega^P\left[\sum_{i\in I}\tilde{a}(i)\log\left(\frac{\tilde{a}(i)}{a(i)}\right)\right] = \sum_{i\in I} E_\Omega^P[\tilde{a}(i)\log(\tilde{a}(i))] - \sum_{i\in I} a(i)\log(a(i)).$$

If, for any $i \in I$, $\tilde{a} \in \mathcal{A}_n^a$ satisfies the following condition:

$$\tilde{a}(i) = \begin{cases} \lfloor na(i)\rfloor / n & \text{with probability } 1 - \mathrm{fp}(na(i)), \\ (\lfloor na(i)\rfloor + 1)/n & \text{with probability } \mathrm{fp}(na(i)), \end{cases} \tag{13}$$

then we say that $\tilde{a}$ has property $MV(n)$.

Theorem 2.1 ([2]). *Let $\tilde{a} \in \mathcal{A}_n^a$, then $H(\tilde{a}|a)$ is minimal if and only if $\tilde{a}$ has property $MV(n)$.*

According to this theorem, our object is deduced to construction of a $\tilde{a} \in \mathcal{A}_n^a$ which has the property $MV(n)$. In [2], Crisan and Lyons give an algorithm by which we can construct a random measure in $\mathcal{A}_n^a$ satisfying the property $MV(n)$. We sketch out this algorithm. In the first step, we construct a filtration $\{\mathcal{F}_k\}_{k\geq 0}$ of I which satisfy $\{\emptyset, I\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \cdots \subset 2^I$ and $\sigma(\cap_k \mathcal{F}_k) = 2^I$. Because I is discrete, we can consider a partitioning $\mathcal{P}_k^{\mathcal{F}}$ of I which corresponds to $\mathcal{F}_k$. For $i \in I$, $\mathcal{P}_k^{\mathcal{F}}(i)$ denotes the element of $\mathcal{P}_k^{\mathcal{F}}$ which includes i. In the second step, we construct a sequence $\{\tilde{a}^k\}_{k\geq 0}$ of elements of $\mathcal{A}_n^a$ inductively from $\tilde{a}^0$ in the way that the equality

$$\tilde{a}^k(i) = \sum_{j\in\mathcal{P}_k^{\mathcal{F}}(i)} \tilde{a}(j)$$

holds for all $i \in I$ and $k \geq 0$. We compare naive Monte Carlo sampling with TBBA under K-method. The problem is the same as before, that is, pricing of an Asian option. We use the natural filtration

$$\mathcal{F}_t = \sigma(Q_{(s)}; s \leq t) \tag{14}$$

as the filtration in the first step of TBBA. Figure 2.2.1 shows the square root of the variance of 10 Monte Carlo trials and 10 TBBA trials around the true value obtained by exhaustive calculation. From this figure we can see that TBBA achieves much faster calculation than the naive Monte Carlo sampling does. If we want to make the standard deviation smaller than 10^{-5}, TBBA needs about 2×10^5 samples and the naive Monte Carlo about 10^7 samples, therefore TBBA is about 50 times faster in this case. If we want more accuracy, this ratio becomes larger. Figure 2.2.1 shows

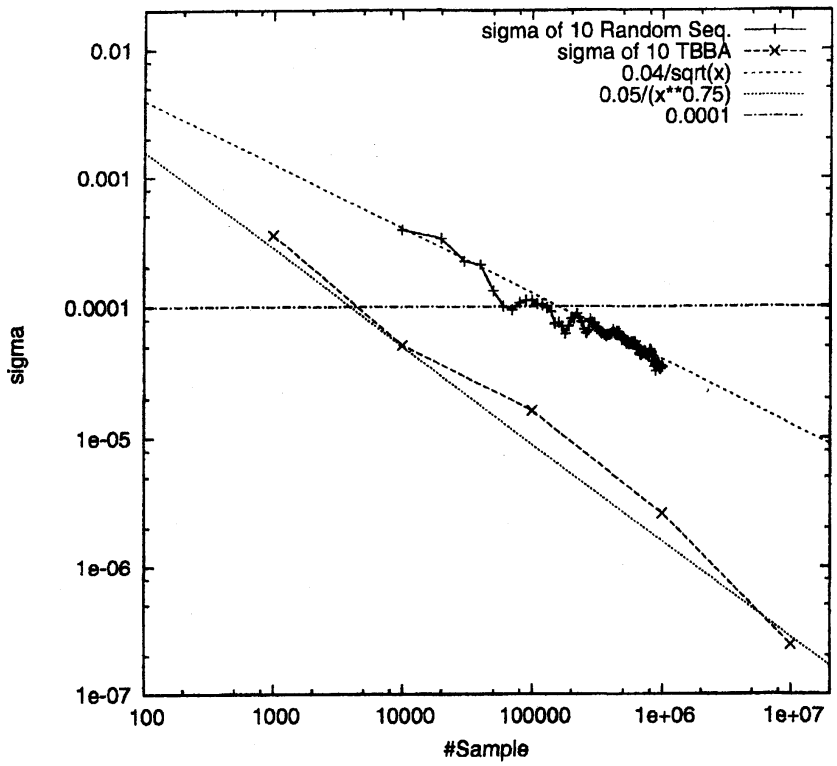

FIGURE 4. Convergence speed of MC and TBBA with K-method.: n=12

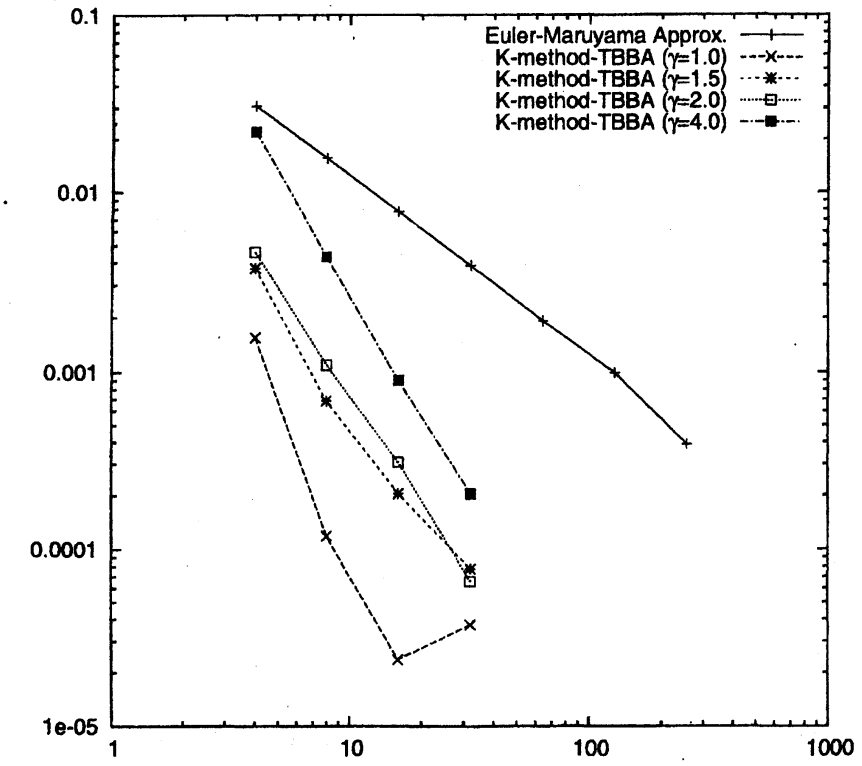

FIGURE 5. Approximation Error and number of partition. Levy area

another example. This is a example of $d = 2$ case. In this example, we calculate $E[\max(S(1), 0)]$, where $S(t)$ is the Levy area defined as follows:

$$S(t) = \frac{1}{2}\left(\int_0^t B^1(s) \circ dB^2(s) - \int_0^t B^2(s) \circ dB^1(s)\right).$$

We can see how γ effects in this case.

2.2.2. *Exotic filter with TBBA.* In the previous example, we see that TBBA realize very effective partial sampling. The property $MV(n)$ does not give enough explanation. If a is uniform over I, TBBA coincides with the random sampling without replacement, which behaves almost the same as the naive Monte Carlo provided that the number of samples is not so large as card(I). In the previous example, the discrete set I is constructed by connecting identical trinomial tree unit and we can regard this set to be a set with almost uniform probability measure. Therefore, $MV(n)$ property of $\tilde{a}$ dose not explain why TBBA show such superior performance to the naive Monte Carlo. The property $MV(n)$ relates only $\tilde{a}$ and the filtration $\{\mathcal{F}_k\}_{k\geq 0}$ used in the first step of TBBA is perfectly ignored. Then, we may consider that $\{\mathcal{F}_k\}_{k\geq 0}$ is the essential factor. In [8], it is reported that by changing the filtration used in the first step of TBBA, faster calculations are realized in some cases. It seems this report supports our hypothesis. In the following, we introduce the exotic filter method briefly and see some numerical examples. For details of the algorithm, see [8]. We consider an exotic filter $\{\mathcal{G}_k\}_{k\geq 0}$ of I defined by:

$$\begin{aligned} \mathcal{G}_0 &= \{\emptyset, I\}, \\ \mathcal{G}_k &= \sigma\left(B_{(1-r(n))}, \mathcal{G}_{k-1}\right), \end{aligned} \tag{15}$$

where $r(n)$ is the n-th element of the van der Corput sequence and B_s is the standard Brownian motion which drives the SDE (1). The van der Corput sequence is a sequence of real numbers defined as:

$$r(n) = \sum_{l=0}^{\infty} \alpha(n,l) 2^{-l-1},$$

$$n = \sum_{l=0}^{\infty} \alpha(n,l) 2^{l}, \quad \alpha(n,l) \in \{0,1\} \quad \text{for all } n, l \in \mathbb{Z}_{\geq 0}.$$

Figure 2.2.2, shows the result of calculation of the same Asian option pricing example ($d = 1$) as we see before. Here we compare the following three methods: K-method with natural filter TBBA, K-method with exotic filter TBBA, and K-method with Monte Carlo.

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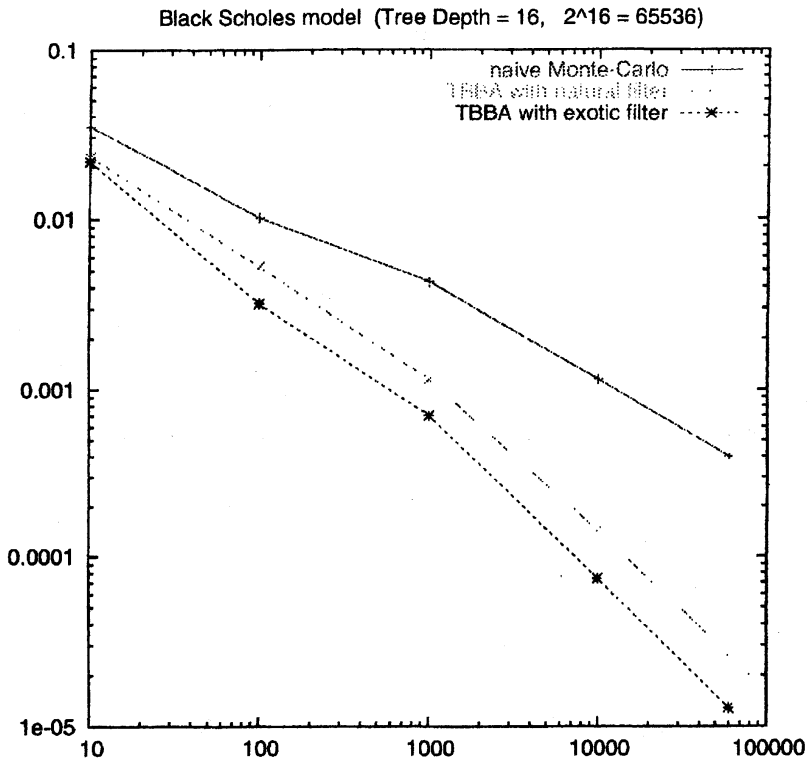

FIGURE 6. K-method with 3 sampling method: Convergence speed.

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