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A boundary value problem and crack propagation in an infinite (visco)elastic strip with a semi-infinite crack

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Abstract

In this paper we study a boundary value problem for an infinite elastic strip with a semi-infinite crack. By using the single and double layer potentials this problem is reduced to a singular integral equation, which is uniquely solved in the Hölder spaces by the Fredholm alternative.

And we also study a quasi-stationary model of crack propagation in an infinite elastic strip with a semi-infinite crack and how to determine the real crack propagation from virtual crack extension by applying maximum energy release rate criterion at the crack tip. Then we prove that the crack propagates the direction only given by surface force.

1 Introduction

Theory of elasticity has been thoroughly developed (see for example, [17], [18], [19]). Mathematical existence theorems in a linear elastic theory were established by Fichera [6]. Recently, Constanda studied the boundary value problems for the system of equilibrium equations of plane elasticity in [2]–[5]. By means of elastic single and double layer potentials he reduced the boundary value problems mentioned above to the integral equations. Then applying the theory of integral equations lead to the solvability of the interior and exterior Dirichlet and Neumann problems. However, the problems considered in [2]–[5] are those in a compact domain without any cracks.

On the other hand, for boundary value problems in a planar domain with cracks, Airy's stress function is, in general, used so that the system of partial differential equations is transformed into a biharmonic equation (see, for example

[8]). Although the stress tensor is uniquely determined by this transformation, the boundary conditions seem to be inequivalent. Recently, Chudinovich and Constanda [1] investigated plate problems for both an infinite and a finite plates with a finite crack and proved a unique solvability in Sobolev spaces. Krutitskiĭ [14]–[16] studied the Dirichlet and Neumann problems for Laplace and Helmholtz equations in a connected plane region with cuts. The problems were reduced to Fredholm integral equations of second and first kind, which were uniquely solvable with the help of a nonclassical angular potential.

In the present paper we consider a problem in a two-dimensional infinite elastic strip with a semi-infinite crack. This problem leads to a singular integral equation by the potential theory. By proving the compactness of singular integral operator and using the results in [13], [20], [25], the existence of a unique solution is proved by the Fredholm alternative.

And propagation of cracks is a phenomenon which leads to the brittle failure of materials. Analysis of the crack growth has been a major subject of fracture mechanics from the mathematical viewpoint since Griffith's celebrated work [11]. Two types of fracture criteria have been advanced for defining the condition of crack instability. The first one assumes that the onset of crack propagation is governed by the local stresses, while the second one by energy consideration of the crack system. Of these, the latter has been misinterpreted in [11]. To clarify this [24] investigated the correct version of the Griffith energy treatment. And [23] dealt with the application of linear elasticity to fracture and discussed dynamic running crack problems, the energy rate computations and the stress concentrations at smooth-ended notches. [8] described the energy release rate at the crack tip following [23] and [7]. [9] analyzed an asymptotic solution of fields near the moving crack tip. The coefficients of leading terms in this solution is called stress intensity factors. When a crack propagates in an elastic medium, the stress intensity factors evolve with the crack tip. Then, [10] derived formulae which describe the evolution of these stress intensity factors for a homogeneous isotropic medium under plane strain conditions. At present, it is well known that there are many criteria which determine the crack extension. Ohtsuka [22] introduced the three famous criteria in homogeneous isotropic elastic plates and showed the crack extension is described by the stress intensity factor. In the present paper we only apply the maximum energy release rate criterion of them, (see for example [26]). For virtual crack extension, using the results of [21], [22], an energy release rate due to non-smooth crack growth can be represented by calculating the potential energy function. And in our situation we show the direction of kinked crack extension can be given only by the surface force without using the stress intensity factor.

2 Preliminaries

By $u = (u_i)_{i=1,2,3}$, $\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}$ and $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ we denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for a homogeneous isotropic material consist of the constitutive law (Hooke's law)

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}, \quad i,j = 1,2,3 \tag{1}$$

and the equilibrium conditions without any body forces

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad i,j = 1,2,3. \tag{2}$$

Here and in what follows we use the summation convention. λ and μ are *Lamé* constants, δ_{ij} is the Kronecker's delta and the strain-displacement relation is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad u_{i,j} = \partial_j u_i, \quad i,j = 1,2,3. \tag{3}$$

In the state of a plane strain, the 3rd component u_3 of the displacement u is zero, while the components u_1 and u_2 are functions of x_1 and x_2 only, hence $\varepsilon_{i3} = 0$, $\sigma_{13} = \sigma_{23} = 0$. Let $\Omega = \{(x_1, x_2) \mid x_1 \in \mathbf{R}, -a < x_2 < a\}$ $(a > 0)$ be a strip in $\mathbf{R}^2$, representing a homogeneous elastic plate. Then (2) gives the system of equations

$$A\left(\partial_x\right) u = 0 \tag{4}$$

for $u = (u_1, u_2)^{\mathrm{T}}$, where $A(\partial_x) = A\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$,

$$A(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi^2 + (\lambda+\mu)\xi_1^2 & (\lambda+\mu)\xi_1\xi_2 \\ (\lambda+\mu)\xi_1\xi_2 & \mu\xi^2 + (\lambda+\mu)\xi_2^2 \end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2.$$

We assume that shearing strain $\mu > 0$, modulus of compression $3\lambda + 2\mu \geq 0$, in which case it is easy to see that the operator A is elliptic. Moreover we introduce the boundary stress operator $T(\partial_x) = T\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)$ defined by

$$T(\xi_1, \xi_2) = \begin{pmatrix} (\lambda+2\mu)\nu_1\xi_1 + \mu\nu_2\xi_2 & \mu\nu_2\xi_1 + \lambda\nu_1\xi_2 \\ \lambda\nu_2\xi_1 + \mu\nu_1\xi_2 & \mu\nu_1\xi_1 + (\lambda+2\mu)\nu_2\xi_2 \end{pmatrix},$$

where $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal to $\partial\Omega$. In the case of $\nu = (0,1)^{\mathrm{T}}$

$$T(\xi_1, \xi_2) = \begin{pmatrix} \mu\xi_2 & \mu\xi_1 \\ \lambda\xi_1 & (\lambda+2\mu)\xi_2 \end{pmatrix}.$$

We denote by $\Gamma = \{(x_1, 0) \mid -\infty < x_1 \le 0\}$ the crack in Ω. On the crack we assume the free traction condition

$$\sigma_{ij}^{+}\nu_j = \sigma_{ij}^{-}\nu_j = 0 \quad \text{on} \quad \Gamma^{\pm}, \tag{5}$$

where $\Gamma^{\pm}$ means both sides of Γ. Here for every $x \in \Gamma$ $\sigma_{ij}^{\pm} = \sigma_{ij}^{\pm}(x)$ means the limit of $(\nu_x, \sigma_{ij}(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \Gamma$ along the normal ν_x, in this case $\nu_x = (0, \mp 1)$. The limit values σ_{ij}^{+} and σ_{ij}^{-} may be different in general, therefore σ_{ij} may have a jump on Γ. At the end-point $(0,0)$ of Γ we assume

$$\lim_{x_1 \to -0} \sigma_{ij}^{\pm}\nu_j \Big|_{x \in \Gamma^{\pm} \setminus \{(0,0)\}} = 0.$$

On $\partial\Omega_+ = \{(x_1, a) \mid x_1 \in \mathbf{R}\}$, $\partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbf{R}\}$ $(a > 0)$ the boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega_-, \tag{6}$$

$$\sigma_{ij}\nu_j = p_i \quad \text{on} \quad \partial\Omega_+ \tag{7}$$

are imposed, where p_i are given continuous functions on $\partial\Omega_+$.

We introduce the class $\mathcal{K}$ of functions $u(x)$ with the properties (cf. [16]):

1) $u \in C^0(\overline{\Omega \setminus \Gamma}) \cap C^2(\Omega \setminus \Gamma)$,

2) $\nabla u \in C^0(\overline{\Omega \setminus \Gamma} \setminus \{(0,0)\})$,

3) in the neighborhood of $(0,0)$ there exist positive constant C and $\epsilon > -1$ such that

$$| \nabla u(x) | \le C \, | x |^{\epsilon} \quad \text{as} \quad x \to 0, \tag{8}$$

4) for every $x \in \partial\Omega_{\pm}$ there exists a uniform limit of $(\nu_x, \nabla_{\bar{x}} u(\bar{x}))$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \partial\Omega_{\pm}$ along the normal $-\nu_x$.

We define the internal energy density by

$$E(u,u) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\left\{\lambda(u_{1,1} + u_{2,2})^2 + 2\mu(u_{1,1}^2 + u_{2,2}^2) + \mu(u_{1,2} + u_{2,1})^2\right\}.$$

Then it is easy to see that $E(u,u)$ is a nonnegative quadratic form and that $E(u,u) = 0$ if and only if u is a rigid displacement

$$u = (c_1 + c_0 x_2, c_2 - c_0 x_1)^{\mathrm{T}} \tag{9}$$

with arbitrary constants c_0, c_1 and c_2. It is easily seen that

$$F_1 = (1,0)^{\mathrm{T}}, \quad F_2 = (0,1)^{\mathrm{T}}, \quad F_3 = (x_2, -x_1)^{\mathrm{T}}$$

consist of a basis of the space of such rigid displacements. For the matrix

$$F = \begin{pmatrix} F_1, & F_2, & F_3 \end{pmatrix}$$

it is clear that $AF = 0$ in $\mathbf{R}^2$, $TF = 0$ on $\partial\Omega_\pm \cup \Gamma$, and a generic vector of the form (9) can be written as Fk with an arbitrary constant vector k.

Furthermore, we introduce the class $\wp = \{u \mid u \to 0 \text{ as } \mid x \mid \to \infty\}$. One can easily verify for $u \in C^2(\Omega\setminus\Gamma) \cap C^1(\overline{\Omega\setminus\Gamma}) \cap \wp$

$$\int_{\Omega\setminus\Gamma} F^{\mathrm{T}} Au \,\mathrm{da} = \int_{\partial\Omega_\pm} F^{\mathrm{T}} Tu \,\mathrm{ds} + 2\int_\Gamma F^{\mathrm{T}} Tu \,\mathrm{ds}.$$

Also, if $u \in C^2(\Omega\setminus\Gamma) \cap C^1(\overline{\Omega\setminus\Gamma}) \cap \wp$ is a solution of (4) in $\Omega\setminus\Gamma$, then

$$2\int_{\Omega\setminus\Gamma} E(u,u) \,\mathrm{da} = \int_{\partial\Omega_\pm} u^{\mathrm{T}} Tu \,\mathrm{ds} + 2\int_\Gamma u^{\mathrm{T}} Tu \,\mathrm{ds}. \tag{10}$$

Indeed, Divergence Theorem and (4) yield that for any $u \in C^2(\Omega\setminus\Gamma) \cap C^1(\overline{\Omega\setminus\Gamma}) \cap \wp$

$$0 = \int_{\Omega\setminus\Gamma} u^{\mathrm{T}} Au \,\mathrm{da} = -2\int_{\Omega\setminus\Gamma} E(u,u) \,\mathrm{da} + \int_{\partial\Omega_\pm} u^{\mathrm{T}} Tu \,\mathrm{ds} + 2\int_\Gamma u^{\mathrm{T}} Tu \,\mathrm{ds}.$$

3 Integral equations on the boundary

It is well known that the fundamental matrix of $A(\partial_x)$ is given by

$$D(x,y) = A^*(\partial_x) t(x,y),$$

where A^* is the adjoint operator of A and $t(x,y)$ is a fundamental solution of $\mu(\lambda+2\mu)\Delta^2$,

$$t(x,y) = -\{8\pi\mu(\lambda+2\mu)\}^{-1} \mid x-y \mid^2 \ln \mid x-y \mid .$$

Hence, $D(x,y)$ is given explicitly by

$$D(x,y) = -\frac{1}{4\pi\mu(\tilde{\mu}+1)} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{11}$$

$$\begin{aligned}
D_{11} &= 2\tilde{\mu} \ln \mid x-y \mid + 2\tilde{\mu} - 1 + 2\frac{(x_2-y_2)^2}{\mid x-y \mid^2}, \\
D_{12} &= D_{21} = -2\frac{(x_1-y_1)(x_2-y_2)}{\mid x-y \mid^2}, \\
D_{22} &= 2\tilde{\mu} \ln \mid x-y \mid + 2\tilde{\mu} - 1 + 2\frac{(x_1-y_1)^2}{\mid x-y \mid^2}, \\
\tilde{\mu} &= \frac{\lambda+3\mu}{\lambda+\mu}.
\end{aligned}$$

In view of (11), $D(x,y) = D(y,x) = D(y,x)^{\mathrm{T}}$.
Along with $D(x,y)$ we consider the matrix of singular solutions

$$P(x,y) = (T(\partial_y)D(y,x))^{\mathrm{T}},$$

which is written explicitly as

$$\begin{aligned} P(x,y) &= -\frac{1}{2\pi}\left(\frac{\partial}{\partial\nu_y}\ln|x-y|\,I + \frac{\tilde{\mu}-1}{\tilde{\mu}+1}\frac{\partial}{\partial\tau_y}\ln|x-y|\,\tilde{I}\right. \\ &\qquad \left. + \frac{2}{\tilde{\mu}+1}\tilde{I}\frac{\partial}{\partial\tau_y}\frac{(x-y)^{\mathrm{T}}(x-y)}{|x-y|^2}\right) \end{aligned} \tag{12}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\tilde{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tau = (\tau_1, \tau_2)^{\mathrm{T}}$ a unit tangential vector to $\partial\Omega_\pm \cup \Gamma$.

It is easily verified that the columns of $D(x,y)$ and $P(x,y)$ are solutions of equation (4) for any $x \in \mathbf{R}^2$, $y \in \partial\Omega_\pm \cup \Gamma$, $x \neq y$, and that

$$D(x,y) = O(\ln|x|),\ P(x,y) = O(|x|^{-1}) \text{ as } |x| \to \infty. \tag{13}$$

Now we denote by $\tilde{D}$ and $\tilde{P}$ the reflection of $D(x,y)$ and $P(x,y)$ with respect to $\partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbf{R}\}$

$$\tilde{D}(x,y) = D\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - D\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right), \tag{14}$$

$$\tilde{P}(x,y) = P\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - P\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right). \tag{15}$$

Then it is obvious that the columns of $\tilde{D}(x,y)$ and $\tilde{P}(x,y)$ vanish on $\partial\Omega_-$.
Using a potential theory, we will find a solution of problem (4)–(7) in the form

$$u(x_1, x_2) = \tilde{V}_{\partial\Omega_+}(g) + \tilde{V}_\Gamma(f) + \tilde{W}_\Gamma(g), \tag{16}$$

where

$$\begin{aligned} \tilde{V}_{\partial\Omega_+}(g) &= \int_{\partial\Omega_+} \tilde{D}(x,y)g(y)\,\mathrm{d}y_1, \\ \tilde{V}_\Gamma(f) &= \int_\Gamma \tilde{D}(x,y)f(y)\,\mathrm{d}y_1, \\ \tilde{W}_\Gamma(g) &= \int_\Gamma \tilde{P}(x,y)g(y)\,\mathrm{d}y_1. \end{aligned}$$

Now let us introduce function spaces. By $C^{0,\alpha}(G)$ we denote a Hölder space with exponent $\alpha \in (0,1)$ of functions defined on a domain G and by $C^{1,\beta}(G)$ the subspace of functions of C^1-class whose first order derivatives belong to $C^{0,\beta}(G)$, $\beta \in (0,1)$. If $(f,g) \in C^{0,\alpha}(\Gamma) \times (C^{0,\alpha}(\partial\Omega_+) \cap C^{1,\beta}(\Gamma))$, then it is easily seen that u defined by (16) is continuous on $\partial\Omega_+ \cup \Gamma^\pm$ and satisfies (4) and (6). In order to see that u satisfies boundary conditions (5) and (7) we substitute (16) into (5) and (7) so that we deduce the integral equations for g (cf. [3], [25]). From (7) it follows

$$
\begin{aligned}
&\frac{1}{2}g\begin{pmatrix} x_1 \\ a \end{pmatrix} + \text{v.p.}\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \int_\Gamma T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) f\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \int_\Gamma T\tilde{P}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix},
\end{aligned}
\tag{17}
$$

where the integral on $\partial\Omega_+$ means a principal value. Let

$$
\begin{aligned}
Q(x,y) &= -\frac{2\mu}{\pi(\tilde{\mu}+1)}\left(\ln|x-y|I - I + \frac{(x-y)^{\mathrm{T}}(x-y)}{|x-y|^2}\right), \\
\tilde{Q}(x,y) &= Q\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) - Q\left(\begin{pmatrix} x_1 \\ -2a - x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right).
\end{aligned}
$$

Then

$$
T\tilde{P} = -\frac{\partial^2}{\partial\tau_x\partial\tau_y}\tilde{Q}.
$$

Substituting (16) with $\tilde{P}$ replaced by $\tilde{Q}$ into (5) yields

$$
\begin{aligned}
&\pm\frac{1}{2}f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) f\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad - \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1}g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}
\tag{18}
$$

where the integrals on Γ are taken as principal values. The upper and lower signs correspond to the integrals on Γ^+ and Γ^-, respectively. One can easily check that the solution u of the form (16) satisfies condition (8) (cf. [15]). Subtracting two equations in (18) implies

$$f\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \Gamma. \tag{19}$$

Therefore the integral equation (17) on $\partial\Omega_+$ becomes

$$\left(Z + \frac{1}{2}I\right) g = p \quad \text{on} \quad \partial\Omega_+ \tag{20}$$

with $Z = T(\tilde{V}_{\partial\Omega_+} + \tilde{W}_\Gamma)$. And adding two equations in (18), we obtain

$$\begin{aligned} &\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\ &\quad - \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\ &\quad + \text{v.p.} \int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{21}$$

hence

$$\begin{aligned} &\text{v.p.} \int_\Gamma \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \frac{1}{x_1 - y_1} \mathrm{d}y_1 \\ &\quad + \text{v.p.} \int_\Gamma \left\{ \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) - \frac{1}{x_1 - y_1} \right\} \frac{\partial}{\partial y_1} g\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\ &= - \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\ &\quad + \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) g\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0}. \end{aligned} \tag{22}$$

Now we introduce the new space $C_\gamma^{0,\alpha}(G)$ defined by

$$C_\gamma^{0,\alpha}(G) = \{f(x) \in C^{0,\alpha}(G) \mid f(x) = O(|\, x \,|^{-\gamma}) \text{ as } |\, x \,| \to \infty\} \quad (1 < \gamma)$$

equipped with the norm

$$\|\, g \,\|_{\gamma,\alpha} = \|\, g \,\|_{\gamma,\infty} + |\, g \,|_\alpha, \tag{23}$$

$$\|\, g \,\|_{\gamma,\infty} = \sup_{x\in G} |\, (1 + |\, x \,|^\gamma) g(x) \,|, \quad |\, g \,|_\alpha = \sup_{x,\tilde{x}\in G, x\neq\tilde{x}} \frac{|\, g(x) - g(\tilde{x}) \,|}{|\, x - \tilde{x} \,|^\alpha}.$$

Let $g \in C_{\gamma}^{0,\beta}(\Gamma)$ and vanish at the end of crack. Inverting the singular integral operator (22), we arrive at the integral equation of the second kind (cf.[20])

$$(I - Y_1)\frac{\partial}{\partial x_1} g(x) = \frac{1}{\pi^2 R(x)} \int_{-R}^{0} \frac{R(y)\,\mathrm{d}y_1}{y - x} \int_{\partial\Omega_+} T\tilde{D}(y,z) g(z)\,\mathrm{d}z_1,$$

$$\text{as} \quad R \to \infty, \quad x \in \Gamma, \tag{24}$$

where the integral on Γ is in the sense of principal value and

$$Y_1(f(x)) = \frac{1}{\pi^2 R(x)} \int_{-R}^{0} \frac{R(y)\,\mathrm{d}y_1}{y - x} \int_{\Gamma} \left(\frac{\partial}{\partial \tau_z} \tilde{Q}(z,y) - \frac{1}{z - y} \right) f(z)\,\mathrm{d}z_1,$$

$$R(x) = \sqrt{(x + R)x}.$$

4 Uniqueness and existence of solution

In this section we prove that problem (4)–(7) has a unique solution.

THEOREM 1 *Problem* (4)–(7) *has at most one solution of class* $\mathcal{K} \cap \wp$.

Proof. Let $\hat{u}$ be the difference of two solutions of class $\mathcal{K} \cap \wp$ to problem (4)–(7). Then, $\hat{u}$ satisfies (4)–(7) with $p = 0$. Therefore, (10) implies

$$E(\hat{u}, \hat{u}) = 0 \quad \text{in} \quad \Omega \setminus \Gamma.$$

Hence, $\hat{u}$ is of the form (9) in $\overline{\Omega \setminus \Gamma}$. Since $\hat{u} \in \wp$, we conclude that $\hat{u}(x) = 0$, $x \in \overline{\Omega \setminus \Gamma}$. □

From (11), (12), (14), (15) and straightforward calculation one can easily obtain the following lemma. Similar result is proved in [4] in the case of a compact boundary.

LEMMA 1 *If* $f \in C_{\gamma}^{0,\alpha}(\partial\Omega_{\pm} \cup \Gamma)$, *then*
(i) $\tilde{W} f \in \wp$,

(ii) $\tilde{V} f \in \wp$.

Next we will prove the existence of the solution. As shown in the previous section, problem (4)–(7) is reduced to integral equation (20) for g on $\partial\Omega_+$. Since the kernels of Z are $1-$ *singular kernels* on $\partial\Omega_+$ defined below, it is not so easy to solve it.
Here upon, following [3], we call a matrix function $k(x,y)$ defined for all $x \in \partial\Omega_+$

and $y \in \partial\Omega_+$, $x \neq y$, and continuous there an $\omega-$ *singular kernel* on $\partial\Omega_+$, $\omega \in [0,1]$ if there exists a positive constant m such that

$$| k(x,y) | \leq m | x-y |^{-\omega} \quad \text{for all} \quad x,\ y \in \partial\Omega_+,\ x \neq y.$$

If an $\omega-$ *singular kernel* $k(x,y)$ on $\partial\Omega_+$ satisfies

$$| k(x,y) - k(\tilde{x},y) | \leq m | x-\tilde{x} || x-y |^{-\omega-1}$$

for all x, $\tilde{x} \in \partial\Omega_+$ and $y \in \partial\Omega_+$, $0 < | x-\tilde{x} | < \frac{1}{2} | x-y |$, then $k(x,y)$ is called a *proper* $\omega-$ *singular kernel* on $\partial\Omega_+$.

THEOREM 2 *If $k(x,y)$ is a proper $\omega-$singular kernel on $\partial\Omega_+$, $\omega \in [0,1)$, $k(x,y) = k(y,x)$ and $k(x,y) = O(| x |^{-1})$ as $| x | \to \infty$ for any $y \in \partial\Omega_+$, then operator K defined on $C^{0,\alpha}_{\gamma}$ by*

$$(Kg)(x) = \int_{\partial\Omega_+} k(x,y)g(y)\ \mathrm{d}y, \quad x \in \partial\Omega_+$$

is compact.

Proof. This theorem was proved in [3] in the case of a compact domain. In the case where $\partial\Omega_+$ is unbounded, however, the compactness of K is not a direct consequence of that in the compact domain. We prove here that K as a mapping from $C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ to $C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$, $\gamma > \tilde{\gamma} > 1$, with $\alpha = 1-\omega$ for $\omega \in (0,1)$ and any $\alpha \in (0,1)$ for $\omega = 0$ is compact.

Let M_1 be a bounded set in $C^{0,\alpha}_{\gamma}(\partial\Omega_+)$, that is, there exists a positive constant c such that

$$\| g \|_{\gamma,\alpha} \leq c \quad \text{for all} \quad g \in M_1, \tag{25}$$

and let $\{\theta_n\}_{n=1}^{\infty} \subset M_2 = K(M_1)$. Then there exists a sequence $\{g_n\}_{n=1}^{\infty}$ in M_1 such that $\theta_n = Kg_n$, $n = 1,2,3,\ldots$ It is obvious that $\theta_n \in C^{0,\alpha}(\partial\Omega_+)$.
(23), (25) imply that $\{g_n\}_{n=1}^{\infty}$ is uniformly bounded and equicontinuous on $C(\partial\Omega_+)$. Thus by applying *Ascoli* $-$ *Arzelà*'s theorem there exists a uniformly convergent subsequence of $\{g_n\}_{n=1}^{\infty}$, which is denoted by $\{g_n\}_{n=1}^{\infty}$ for simplicity, and a $g \in C(\partial\Omega_+)$ such that

$$\| g_n - g \|_{\gamma,\infty} \to 0 \quad \text{as} \quad n \to \infty. \tag{26}$$

Let $\theta = Kg$. Then, $\theta \in C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$ for some constant $\tilde{\gamma}$, $1 < \tilde{\gamma} < \gamma$. Really, we have

$$| \theta_n(x) - \theta(x) | \leq \int_{\partial\Omega_+} | k(x,y) || g_n(y) - g(y) |\ \mathrm{d}y_1$$

$$\leq c_1 \frac{1}{| x |^{\tilde{\gamma}}} \sup_{y \in \partial\Omega_+} \frac{| g_n(y) - g(y) |}{| 1 - \frac{y}{x} |^{\tilde{\gamma}} | x-y |^{-\gamma}} \int_{\partial\Omega_+} \frac{| k(x,y) |}{| x-y |^{\gamma-\tilde{\gamma}}}\ \mathrm{d}y_1,$$

consequently,

$$| \theta_n - \theta | (x) \leq c_2 | x |^{-\tilde{\gamma}} \| g_n - g \|_{\gamma-\tilde{\gamma},\infty}, \quad n = 1, 2, 3, .. \tag{27}$$

with some positive constants c_1, c_2. Since $k(x, y)$ is a *proper* $\omega-$*singular kernel*,

$$\begin{aligned} &| K(g_n - g)(x) - K(g_n - g)(\tilde{x}) | \\ &= \left| \int_{\partial\Omega_+} [k(x, y) - k(\tilde{x}, y)](g_n - g)(y) \, \mathrm{d}y_1 \right| \\ &\leq c_3 | x - \tilde{x} |^{\alpha} \sup_{y \in \partial\Omega_+} \| y |^{\gamma} (g_n - g)(y) | . \end{aligned}$$

Hence,

$$| \theta_n - \theta |_{\alpha} \leq c_3 \| g_n - g \|_{\gamma,\infty}, \quad n = 1, 2, 3, ... \tag{28}$$

The assertion now follows from the fact that the constants c_1, c_2, c_3 are independent of x and $\tilde{x}$. (27), (28), (23) and (26) yield

$$\| \theta_n - \theta \|_{\alpha} \to 0 \quad \text{as} \quad n \to \infty,$$

which proves that $K : C^{0,\alpha}_{\gamma}(\partial\Omega_+) \to C^{0,\alpha}_{\tilde{\gamma}}(\partial\Omega_+)$ is compact. □

THEOREM 3 *Problem* (4)–(7) *has a unique solution* $u \in \mathcal{K} \cap \wp$ *for any* $p \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$ *with any* $\alpha \in (0, 1)$ *and any* $\gamma > 1$.

Proof. In (20) Z is represented as $Zg = K_1 g + K_2 g$, where

$$\begin{aligned} K_1 g &= \text{v.p.} \int_{\partial\Omega_+} \frac{1}{x_1 - y_1} g \begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1, \\ K_2 g &= (Z - K_1) g. \end{aligned} \tag{29}$$

Then K_1 has a 1-singular kernel and K_2 is a non-singular operator. Applying the operator $(K_1 - \frac{1}{2}I)$ to both sides of (20) yields

$$\left((K_1)^2 + K_1 K_2 - \frac{1}{2} K_2 - \frac{1}{4} I \right) g = \left(K_1 - \frac{1}{2} I \right) p. \tag{30}$$

Here we claim that

$$\begin{aligned} &((K_1)^2 g)(x) \\ &= \int_{\partial\Omega_+} \frac{1}{x - y} \left[\int_{\partial\Omega_+} \frac{g(z)}{y - z} \mathrm{d}z_1 \right] \mathrm{d}y_1 \\ &= -\pi^2 g(x) + \int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{g(z)}{(x - y)(y - z)} \mathrm{d}y_1 \right] \mathrm{d}z_1. \end{aligned} \tag{31}$$

LEMMA 2 *If $g \in C^{0,\alpha}_{\gamma}(\partial\Omega_+)$, then* (31) *holds.*

Proof. In the case of a compact boundary (31) is well-known as a Poincaré-Bertrand formula ([20], §23). For convenience we consider the functions of a real variable $x = (x_1, x_2)$ as the functions of a complex variable $x = x_1 + ix_2$. Let $x = x_1 + ix_2$, $y = y_1 + iy_2$ and $z = z_1 + iz_2$. In the present case where $\partial\Omega_+$ is unbounded first we prove the formula

$$\int_{\partial\Omega_+} \frac{1}{x-y}\left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z}\,\mathrm{d}z_1\right]\mathrm{d}y_1$$
$$= -\pi^2\phi(x,x) + \int_{\partial\Omega_+}\left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)}\,\mathrm{d}y_1\right]\mathrm{d}z_1$$

for $\phi \in C^{0,\alpha}_{\gamma}(\partial\Omega_+ \times \partial\Omega_+)$. Let

$$\Phi(t) = \int_{\partial\Omega_+} \frac{1}{t-y}\left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z}\,\mathrm{d}z_1\right]\mathrm{d}y_1,$$
$$\Psi(t) = \int_{\partial\Omega_+}\left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(t-y)(y-z)}\,\mathrm{d}y_1\right]\mathrm{d}z_1,$$

where $t = t_1 + it_2$ is a point on the plane, not on $\partial\Omega_+$. Then,

$$\Phi(t) = \Psi(t) \tag{32}$$

holds. Indeed, it is sufficient to prove

$$I_1 = \int_{-\infty}^{\infty} \frac{1}{t-y}\left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\phi(y,z)}{y-z}\,\mathrm{d}z_1\right]\mathrm{d}y_1 \to 0,$$
$$I_2 = \int_{-\infty}^{\infty}\left[\int_{z_1-\varepsilon}^{z_1+\varepsilon} \frac{\phi(y,z)}{(t-y)(y-z)}\,\mathrm{d}y_1\right]\mathrm{d}z_1 \to 0$$
$$\text{as} \quad \varepsilon \to 0^+.$$

For I_1, we divide the integral over $(-\infty, \infty)$ three

$$\int_{-\infty}^{\infty} = \int_{R}^{\infty} + \int_{-\infty}^{-R} + \int_{-R}^{R}.$$

Since the above assertion for the third integral was proved in [20], we consider them for the first and second integrals. Since $\phi(y,z) \in C^{0,\alpha}_{\gamma}(\partial\Omega_+ \times \partial\Omega_+)$, when R is sufficiently large, the first integral can be estimated as follows.

$$\left|\int_{R}^{\infty} \frac{1}{t-y}\left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\phi(y,z)}{y-z}\,\mathrm{d}z_1\right]\mathrm{d}y_1\right|$$

$$
\begin{aligned}
&= \left| \int_R^\infty \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \left(\frac{\phi(y,z)-\phi(y,y)}{y-z} + \frac{\phi(y,y)}{y-z} \right) \mathrm{d}z_1 \right] \mathrm{d}y_1 \right| \\
&\leq \left| \int_R^\infty \frac{1}{t-y} \left[\int_{y_1-\varepsilon}^{y_1+\varepsilon} \left(\frac{C(\mid \phi(y,z) \mid + \mid \phi(y,y) \mid)^{\tilde{\alpha}}}{\mid y-z \mid^{1-(1-\tilde{\alpha})\alpha}} + \frac{\phi(y,y)}{y-z} \right) \mathrm{d}z_1 \right] \mathrm{d}y_1 \right| \\
&\leq \varepsilon^{(1-\tilde{\alpha})\alpha} \sup_{R-\varepsilon<y_1<\infty} \| y_1 \mid \phi(y,y)|^{\tilde{\alpha}} \\
&\to 0 \quad \text{as} \quad \varepsilon \to 0^+,
\end{aligned}
$$

where C is a constant and $1 > \alpha > \tilde{\alpha} > 0$. In the same way the second integral tends to 0 as $\varepsilon \to 0^+$. Similarly one can show that $I_2 \to 0$ as $\varepsilon \to 0^+$.
We denote by $\Phi^+(x)$ and $\Phi^-(x)$ the limits of $\Phi(t)$ as $t \to x$ from the upper and from the lower of $\partial\Omega_+$, respectively. By the Plemelj's formula, the relation

$$
\Phi^+(x) + \Phi^-(x) = 2 \int_{\partial\Omega_+} \frac{1}{x-y} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{y-z} \mathrm{d}z_1 \right] \mathrm{d}y_1 \tag{33}
$$

holds. Furthermore, $\Psi(t)$ is represented as

$$
\Psi(t) = \int_{\partial\Omega_+} \frac{\psi(z;t)}{z-t} \mathrm{d}z_1, \tag{34}
$$

$$
\psi(z;t) = \int_{\partial\Omega_+} \left(\frac{1}{y-t} - \frac{1}{y-z} \right) \phi(y,z) \, \mathrm{d}y_1.
$$

Denoting by $\psi^+(z;x)$ and $\psi^-(z;x)$ the limits of $\psi(z;t)$ as $t \to x$ from the upper and from the lower of $\partial\Omega_+$, respectively. Again by the Plemelj's formula we obtain

$$
\psi^+(z;x) - \psi^-(z;x) = 2\pi i \phi(x,z), \tag{35}
$$

$$
\begin{aligned}
\psi^+(z;x) + \psi^-(z;x) &= 2 \int_{\partial\Omega_+} \left(\frac{1}{y-x} - \frac{1}{y-z} \right) \phi(y,z) \, \mathrm{d}y_1 \\
&= 2(z-x) \int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)} \mathrm{d}y_1.
\end{aligned}
$$

Put

$$
\begin{aligned}
\psi(z;t) &= \psi^+(z;x) + \varepsilon^+ \quad (\text{if } t \text{ is in the upper of } \partial\Omega_+), \\
\psi(z;t) &= \psi^-(z;x) + \varepsilon^- \quad (\text{if } t \text{ is in the lower of } \partial\Omega_+).
\end{aligned} \tag{36}
$$

Then it is obvious that $\varepsilon^+ \to 0$, $\varepsilon^- \to 0$ as $t \to x$. Moreover, one can prove

$$\int_{\partial\Omega_+} \frac{\varepsilon^+}{z-t}\,\mathrm{d}z_1 \to 0, \quad \int_{\partial\Omega_+} \frac{\varepsilon^-}{z-t}\,\mathrm{d}z_1 \to 0 \tag{37}$$

as $t \to x$ along $\pm\nu_x$. In fact,

$$\mid \varepsilon^+ \mid = \mid \psi(z;t) - \psi^+(z;x) \mid \leq C\delta^{\alpha(1-\tilde{\alpha})} \mid \psi(z;t) - \psi^+(z;x) \mid^{\tilde{\alpha}},$$

where C is a constant, $\delta = \mid t - x \mid$, and α, $\tilde{\alpha}$ are the same as above. Therefore

$$\left| \int_{\partial\Omega_+} \frac{\varepsilon^+}{z-t}\,\mathrm{d}z_1 \right| \leq C\delta^{\alpha(1-\tilde{\alpha})} \int_{\partial\Omega_+} \frac{\mid \psi(z;t) - \psi^+(z;x) \mid^{\tilde{\alpha}}}{\mid z-t \mid}\,\mathrm{d}z_1 \to 0 \quad \text{as} \quad \delta \to 0.$$

The case of ε^- can be treated in exactly the same manner. Replacing $\psi(z;t)$ in (34) by expression (36) and using (37), we obtain

$$\begin{aligned} \Psi^+(x) &= \pi i \psi^+(x;x) + \int_{\partial\Omega_+} \frac{\psi^+(z;x)}{z-x}\,\mathrm{d}z_1, \\ \Psi^-(x) &= -\pi i \psi^-(x;x) + \int_{\partial\Omega_+} \frac{\psi^-(z;x)}{z-x}\,\mathrm{d}z_1, \end{aligned}$$

hence by (35)

$$\begin{aligned} &\Psi^+(x) + \Psi^-(x) \\ &\quad = -2\pi^2\phi(x,x) + 2\int_{\partial\Omega_+} \left[\int_{\partial\Omega_+} \frac{\phi(y,z)}{(x-y)(y-z)}\,\mathrm{d}y_1 \right] \mathrm{d}z_1. \end{aligned} \tag{38}$$

Since from (32) the left sides of (33) and (38) are equal, the formula is proved. Hence, for any $g \in C_\gamma^{0,\alpha}(\partial\Omega_+)$ and $x \in \partial\Omega_+$ (31) holds. □

Now we return to the proof of THEOREM 3. Using Cauchy's integral theorem to the integral in the right-hand side of (31) yields

$$\begin{aligned} ((K_1)^2 g)(x) &= -\pi^2 g(x) \\ &\quad + \int_{\partial\Omega_+} \left[\frac{1}{x-z} \left(\int_{\partial\Omega_+} \frac{\mathrm{d}y_1}{x-y} - \int_{\partial\Omega_+} \frac{\mathrm{d}y_1}{z-y} \right) g(z) \right] \mathrm{d}z_1 \\ &= -\pi^2 g(x). \end{aligned}$$

Hence, equation (30) can be written as

$$\left(K_1K_2 - \frac{1}{2}K_2 - (\frac{1}{4} + \pi^2)I \right) g = \left(K_1 - \frac{1}{2}I \right) p. \tag{39}$$

It is easily seen that $K_2 g$ satisfies the Lipschitz condition if $g \in C_\gamma^{0,\alpha}(\partial\Omega_+)$ and the right-hand side of (39) also belongs to $C_{\tilde\gamma}^{0,\alpha}(\partial\Omega_+)$ if $p \in C_\gamma^{0,\alpha}(\partial\Omega_+)$. Since K_1K_2 and K_2 have *proper* $0-$ *singular kernels*, by THEOREM 2, we can apply Fredholm's theorem to problem (39) in the dual system

$$\langle \bigcup_{\gamma<\gamma_0} C_\gamma^{0,\alpha}(\partial\Omega_+), \bigcup_{\tilde\gamma<\gamma<\gamma_0} C_{\tilde\gamma}^{0,\alpha}(\partial\Omega_+)\rangle$$

with a fixed $\gamma_0 > 1$ (cf. [5]).
We can apply the same argument to (24). The operator Y_1 can be decomposed into

$$Y_1 = Y_{11} + Y_{10},$$

where Y_{11} has a 1-singular kernel and Y_{10} is a non-singular operator. Similarly, if $\frac{\partial}{\partial x_1} g \in C_\gamma^{0,\beta}(\Gamma)$ which vanish at the crack tip, then we can apply Fredholm's theorem in the dual system

$$\langle \bigcup_{\gamma<\gamma_0} C_\gamma^{0,\beta}(\Gamma), \bigcup_{\tilde\gamma<\gamma<\gamma_0} C_{\tilde\gamma}^{0,\beta}(\Gamma)\rangle.$$

It is not difficult to prove that u defined by (16) with g given above is a desired solution to problem (4)–(7). □

Moreover, we require stronger regularity of g.

THEOREM 4 *If $p \in C_\gamma^{1,\alpha}(\partial\Omega_+)$, then $g \in C_\gamma^{1,\alpha}(\partial\Omega_+) \cap C_\gamma^{2,\beta}(\Gamma)$ whose first order derivative vanishes at the crack tip.*

This THEOREM 4 can be proved in a similar way as in the proof of THEOREM 2 in [12].

5 The model of crack propagation

In this section we consider a quasi-stationary model of crack propagation. To obtain an explicit formula we adopt the energy criterion given by Griffith [11]. According to his theory, when a crack is extended, there is a flow of energy from the stress field in the body to the crack tip. This energy is stored on both faces of the newly enlarged crack. In the case of linear elasticity, we call the released potential energy G as the crack increases a unit area the energy release rate. Following [22], we represent G in the form

$$G = -\lim_{\varepsilon\to 0} \frac{\Pi(u_\varepsilon) - \Pi(u)}{\varepsilon}, \tag{40}$$

where Π is the potential energy functional defined by

$$\Pi(u) = \int_{\Omega\setminus\Gamma} E(u,u)\,\mathrm{d}x - \int_{\partial\Omega_\pm} s\cdot u\,\mathrm{d}x_1 \tag{41}$$

and $s = (s_i) = (\sigma_{ij}\nu_j) = Tu$.

Now let us consider the virtual kinked crack extension

$$\Gamma_\varepsilon = \{x_\varepsilon \mid x_\varepsilon = x_0 + \tilde{x}, x_0 \in \Gamma, \tilde{x} \in \tilde{\Gamma}\} \tag{42}$$

with $\tilde{\Gamma} = \{\kappa X = \kappa(\cos\theta_0,\ \sin\theta_0) \mid 0 < \kappa < \varepsilon\}$. This means that the virtual crack extension Γ_ε propagates with an angle θ_0. Then we deduce the boundary value problem with respect to the displacement u_ε

$$(*)\begin{cases} Au_\varepsilon = 0 \quad \text{in} \quad \Omega\setminus\Gamma_\varepsilon, \\ Tu_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon^\pm, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_-, \\ Tu_\varepsilon = p \quad \text{on} \quad \partial\Omega_+, \end{cases}$$

where $\Gamma_\varepsilon^\pm$ mean both sides of Γ_ε. We seek a solution u_ε of problem $(*)$ in the form

$$u_\varepsilon = u + \varepsilon\hat{u}, \tag{43}$$

where u is a solution of problem (4)–(7). Differentiation of Tu_ε on $\Gamma_\varepsilon^\pm$ with respect to ε yields

$$0 = T\left(\frac{\partial u_\varepsilon}{\partial\varepsilon} + \frac{\partial u_\varepsilon}{\partial x_1}\frac{\partial}{\partial\varepsilon}\varepsilon\cos\theta_0 + \frac{\partial u_\varepsilon}{\partial x_2}\frac{\partial}{\partial\varepsilon}\varepsilon\sin\theta_0\right)\Bigg|_{\Gamma_\varepsilon^\pm}.$$

Letting $\varepsilon \to 0$, we get

$$T\left(\hat{u} + \frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right)\Bigg|_{\Gamma^\pm} = 0.$$

In view of (4)–(7), (43) and $(*)$ we obtain the boundary value problem of $\hat{u}$:

$$(**)\begin{cases} A\hat{u} = 0 \quad \text{in} \quad \Omega\setminus\Gamma, \\ T\hat{u} = -T\left(\frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm, \\ \hat{u} = 0 \quad \text{on} \quad \partial\Omega_-, \\ T\hat{u} = 0 \quad \text{on} \quad \partial\Omega_+. \end{cases}$$

Similarly for u we can apply the potential theory to problem $(**)$, so that the solution of $(**)$ is described in the form

$$\hat{u}(x_1, x_2) = \tilde{V}_{\partial\Omega_+}(h_1) + \tilde{V}_\Gamma(h_2) + \tilde{W}_\Gamma(h_1), \tag{44}$$

where $(h_2, h_1) \in C^{0,\alpha}_{\gamma}(\Gamma) \times (C^{0,\alpha}_{\gamma}(\partial\Omega_+) \cap C^{1,\beta}_{\gamma}(\Gamma))$, $\gamma > 1$, have the similar properties as (f, g). In order for $\hat{u}$ in (44) to satisfy the boundary condition in (**) we substitute (44) into (**) and derive the integral equations on $\partial\Omega_+$ and Γ.

It is easily obtained

$$\begin{aligned}
&\frac{1}{2}h_1\begin{pmatrix} x_1 \\ a \end{pmatrix} + \text{v.p.}\int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \int_{\Gamma} T\tilde{D}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \int_{\Gamma} T\tilde{P}\left(\begin{pmatrix} x_1 \\ a \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on} \quad \partial\Omega_+. \qquad (45)
\end{aligned}$$

It yields

$$\begin{aligned}
&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&\quad - \frac{\partial^2}{\partial\tau_x\partial\tau_y}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix}\Bigg|_{y_1=-\infty}^{0} \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} \frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) \frac{\partial}{\partial y_1}h_1\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1 \\
&= -T\left(\frac{\partial u}{\partial x_1}\cos\theta_0 + \frac{\partial u}{\partial x_2}\sin\theta_0\right) \quad \text{on} \quad \Gamma^\pm, \qquad (46)
\end{aligned}$$

since h_1 vanishes at the crack tip. Note that

$$\begin{aligned}
\frac{\partial^2}{\partial x_2\partial\tau_x}\ln|x-y| &= \frac{\partial^2}{\partial x_1\partial\nu_x}\ln|x-y|, \\
\frac{\partial^2}{\partial x_2\partial\nu_x}\ln|x-y| &= -\frac{\partial^2}{\partial x_1\partial\tau_x}\ln|x-y|.
\end{aligned} \qquad (47)$$

Then using integration by parts and THEOREM 4, we can rewrite (46) to

$$\begin{aligned}
&\pm\frac{1}{2}h_2\begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \int_{\partial\Omega_+} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ a \end{pmatrix}\right) h_1\begin{pmatrix} y_1 \\ a \end{pmatrix} \mathrm{d}y_1 \\
&\quad + \text{v.p.}\int_{\Gamma^\pm} T\tilde{D}\left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix}\right) h_2\begin{pmatrix} y_1 \\ 0 \end{pmatrix} \mathrm{d}y_1
\end{aligned}$$

$$
\begin{aligned}
&+\text{v.p.}\int_{\Gamma^\pm}\frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial}{\partial y_1}h_1\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\\
=\;&-\Bigg\{\Bigg(\int_{\partial\Omega_+}\frac{\partial}{\partial x_1}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\a\end{pmatrix}\right)g\begin{pmatrix}y_1\\a\end{pmatrix}\,\mathrm{d}y_1\\
&+\text{v.p.}\int_{\Gamma^\pm}\frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial^2}{\partial y_1^2}g\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\Bigg)\cos\theta_0\\
&+\Bigg(\int_{\partial\Omega_+}\frac{\partial}{\partial x_2}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\a\end{pmatrix}\right)g\begin{pmatrix}y_1\\a\end{pmatrix}\,\mathrm{d}y_1\pm\frac{1}{2}\frac{\partial^2}{\partial x_1^2}g(x)\\
&+\text{v.p.}\int_{\Gamma^\pm}\frac{\partial}{\partial\nu_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial^2}{\partial y_1^2}g\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\Bigg)\sin\theta_0\Bigg\}. \qquad (48)
\end{aligned}
$$

Subtracting and adding two equations in (48) yield

$$
h_2(x) = -\frac{\partial^2}{\partial x_1^2}g(x), \qquad (49)
$$

$$
\begin{aligned}
&\int_{\partial\Omega_+}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\a\end{pmatrix}\right)h_1\begin{pmatrix}y_1\\a\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\text{v.p.}\int_{\Gamma}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)h_2\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\text{v.p.}\int_{\Gamma}\frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial}{\partial y_1}h_1\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\\
&=\;-\Bigg\{\Bigg(\int_{\partial\Omega_+}\frac{\partial}{\partial x_1}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\a\end{pmatrix}\right)g\begin{pmatrix}y_1\\a\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\text{v.p.}\int_{\Gamma}\frac{\partial}{\partial\tau_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial^2}{\partial y_1^2}g\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\Bigg)\cos\theta_0\\
&\quad+\Bigg(\int_{\partial\Omega_+}\frac{\partial}{\partial x_2}T\tilde{D}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\a\end{pmatrix}\right)g\begin{pmatrix}y_1\\a\end{pmatrix}\,\mathrm{d}y_1\\
&\quad+\text{v.p.}\int_{\Gamma}\frac{\partial}{\partial\nu_x}\tilde{Q}\left(\begin{pmatrix}x_1\\0\end{pmatrix},\begin{pmatrix}y_1\\0\end{pmatrix}\right)\frac{\partial^2}{\partial y_1^2}g\begin{pmatrix}y_1\\0\end{pmatrix}\,\mathrm{d}y_1\Bigg)\sin\theta_0\Bigg\}. \qquad (50)
\end{aligned}
$$

Substituting (49) into (50) leads to the similar formula as (24)

$$
\begin{aligned}
(I-Y_1)\frac{\partial}{\partial x_1}h_1(x) &= \frac{1}{\pi^2R(x)}\int_{-R}^{0}\frac{R(y)}{y-x}\Bigg\{T\tilde{V}_{\partial\Omega_+}h_1 - T\tilde{V}_{\Gamma}\frac{\partial^2}{\partial x_1^2}g\\
&\quad+\cos\theta_0\left(\frac{\partial}{\partial x_1}T\tilde{V}_{\partial\Omega_+}g+\frac{\partial}{\partial\tau_x}Y_2g\right)
\end{aligned}
$$

$$+ \sin\theta_0 \left(\frac{\partial}{\partial x_2} T\tilde{V}_{\partial\Omega_+} g + \frac{\partial}{\partial \nu_x} Y_2 g \right) \Bigg\} \, dy_1$$

$$\text{as} \quad R \to \infty, \quad x \in \Gamma, \tag{51}$$

where

$$Y_2(f) = \text{v.p.} \int_\Gamma \tilde{Q} \left(\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \right) \frac{\partial^2}{\partial y_1^2} f \begin{pmatrix} y_1 \\ 0 \end{pmatrix} \, dy_1.$$

Applying THEOREM 3 and 4 for problem (**), we can get a unique solution $\hat{u}$.

6 The direction of crack extension

In this section we calculate G defined by (40). Taking into account (10), if u is a solution of problem (*), then $\Pi(u)$ vanishes except on $\partial\Omega_+$. Then from (41), (43) $\Pi(u_\varepsilon)$ is written by

$$\Pi(u_\varepsilon) = -\frac{1}{2} \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot u_\varepsilon \, dx_1 = \Pi(u) + \varepsilon \Pi(\hat{u}). \tag{52}$$

In order to determine the crack direction θ_0 we apply maximum energy release rate criterion in 2-dimensional plane (cf. Wu [26]). Thus by virtue of (40), (52) we seek the angle θ_0 such that

$$\max_{-\pi<\theta_0<\pi} G = \max_{-\pi<\theta_0<\pi} (-\Pi(\hat{u})). \tag{53}$$

From (24) it implies that

$$\frac{\partial}{\partial x_1} g(x) = Y_3(T\tilde{V}_{\partial\Omega_+} g) \quad \text{on} \quad \Gamma, \tag{54}$$

where

$$\begin{aligned} Y_3 g &= \lim_{R\to\infty} \left((1+\pi^2) I - Y_{11} Y_{10} - Y_{10} \right)^{-1} \\ &\quad \left\{ (I + Y_{11}) \left(\lim_{R\to\infty} \frac{1}{\pi^2 R(z)} \int_{-R}^{0} \frac{R(y) g}{y - z} \, dy_1 \right) \right\}. \end{aligned}$$

Substituting (54) into (39) yields that

$$g(x) = \left(K_1 K_2 - \frac{1}{2} K_2 - (\frac{1}{4} + \pi^2) I \right)^{-1} \left\{ \left(K_1 - \frac{1}{2} I \right) p \right\} \quad \text{on} \quad \partial\Omega_+. \tag{55}$$

Similarly, h_1 is described by g and θ_0. Indeed, from (51) it follows that

$$\frac{\partial}{\partial x_1} h_1(x) = Y_3 \left(T\tilde{V}_{\partial\Omega_+} h_1 - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) + A_1 \cos\theta_0 + B_1 \sin\theta_0 \quad \text{on} \quad \Gamma, \tag{56}$$

where A_1, B_1 are functions defined by

$$\begin{aligned} A_1 &= Y_3\left(\frac{\partial}{\partial x_1}T\tilde{V}_{\partial\Omega_+}g + \frac{\partial}{\partial\tau_x}Y_2 g\right), \\ B_1 &= Y_3\left(\frac{\partial}{\partial x_2}T\tilde{V}_{\partial\Omega_+}g + \frac{\partial}{\partial\nu_x}Y_2 g\right). \end{aligned}$$

Substituting (49), (56) into (45), we have

$$h_1(x) = C + A_2\cos\theta_0 + B_2\sin\theta_0 \quad \text{on} \quad \partial\Omega_+, \tag{57}$$

where

$$\begin{aligned} C &= \left(K_1K_2 - \frac{1}{2}K_2 - (\frac{1}{4}+\pi^2)I\right)^{-1} \\ &\qquad \left\{\left(K_1 - \frac{1}{2}I\right)\left(I + \int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}(x,y)Y_3\right)\left(T\tilde{V}_\Gamma\frac{\partial^2}{\partial x_1^2}g\right)\right\}, \\ A_2 &= \left(K_1K_2 - \frac{1}{2}K_2 - (\frac{1}{4}+\pi^2)I\right)^{-1} \\ &\qquad \left\{\left(K_1 - \frac{1}{2}I\right)\left(\int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}(x,y)\right)(-A_1)\right\}, \\ B_2 &= \left(K_1K_2 - \frac{1}{2}K_2 - (\frac{1}{4}+\pi^2)I\right)^{-1} \\ &\qquad \left\{\left(K_1 - \frac{1}{2}I\right)\left(\int_\Gamma \frac{\partial}{\partial\tau_x}\tilde{Q}(x,y)\right)(-B_1)\right\}. \end{aligned}$$

Since A_i, B_i and C are functions depending on g, h_i depends only on surface force p for $i = 1, 2$. Hence, substituting (49), (56), (57) into (44), we have

$$\begin{aligned} \hat{u} &= \tilde{V}_{\partial\Omega_+}(C + A_2\cos\theta_0 + B_2\sin\theta_0) + \tilde{V}_\Gamma\left(-\frac{\partial^2}{\partial x_1^2}g\right) \\ &\quad +\tilde{V}_\Gamma^*\left(Y_3\left(T\tilde{V}_{\partial\Omega_+}(C + A_2\cos\theta_0 + B_2\sin\theta_0) - T\tilde{V}_\Gamma\frac{\partial^2}{\partial x_1^2}g\right)\right. \\ &\quad \left. +A_1\cos\theta_0 + B_1\sin\theta_0\right), \end{aligned}$$

since (47) leads to

$$\tilde{W}_\Gamma = \frac{\partial}{\partial x_1}\tilde{V}_\Gamma^*.$$

Thus, from (52) $\Pi(\hat{u})$ is written as

$$-2\Pi(\hat{u}) = D + A_3\cos\theta_0 + B_3\sin\theta_0,$$

where

$$
\begin{aligned}
D &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} C + \tilde{V}_\Gamma \left(-\frac{\partial^2}{\partial x_1^2} g \right) \right. \\
&\qquad \left. + \tilde{V}_\Gamma^* \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} C - T\tilde{V}_\Gamma \frac{\partial^2}{\partial x_1^2} g \right) \right) \right) \mathrm{d}x_1, \\
A_3 &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} A_2 + \tilde{V}_\Gamma^* \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} A_2 \right) + A_1 \right) \right) \mathrm{d}x_1, \\
B_3 &= \int_{\partial\Omega_+} p^{\mathrm{T}} \cdot \left(\tilde{V}_{\partial\Omega_+} B_2 + \tilde{V}_\Gamma^* \left(Y_3 \left(T\tilde{V}_{\partial\Omega_+} B_2 \right) + B_1 \right) \right) \mathrm{d}x_1.
\end{aligned}
$$

(40) is equivalent to

$$G = \frac{1}{2}(D + A_3 \cos\theta_0 + B_3 \sin\theta_0).$$

From this it is easy to see that G attains the maximum value in $(-\pi, \pi)$ at

$$\theta_0 = \mathrm{Tan}^{-1}\left(\frac{B_3}{A_3}\right). \tag{58}$$

Hence, summing up the above

THEOREM 5 *Suppose a homogeneous elastic body Ω with a crack Γ is loaded a surface force p. Then according to maximum energy release rate criterion Γ propagates along the direction θ_0 given by* (58) *dependent only on surface force p.*

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