数理解析研究所講究録 1353 巻 2004 年 72-91

L^q-L^r estimates of solution to the parabolic Maxwell equations and their application to the magnetohydrodynamic equations*

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1. Introduction and main results

Let $\mathcal{O} \subset \mathbf{R}^3$ be a simply connected and bounded domain with smooth boundary and let Ω be an exterior domain to $\mathcal{O}$, i.e. $\Omega = \mathbf{R}^3 \setminus \overline{\mathcal{O}}$. Suppose that there is some $R_0 > 0$ such that $\partial\Omega \subset B_{R_0}(0) = \{x \in \mathbf{R}^3 \mid |x| < R_0\}$. In this note we study the initial boundary value problem of the magnetohydrodynamic system (the Ohm-Navier-Stokes system) in $\Omega \times (0,\infty)$ concerning the velocity field $v = (v_1(x,t), v_2(x,t), v_3(x,t))$, the magnetic field $H = (H_1(x,t), H_2(x,t), H_3(x,t))$ and the pressure $p = p(x,t)$:

$$\begin{cases} v_t - \Delta v + (v\cdot\nabla)v + \nabla p + H \times \operatorname{curl} H = 0 & \text{in} \quad \Omega \times (0,\infty), \\ H_t + \operatorname{curl}\operatorname{curl} H + (v\cdot\nabla)H - (H\cdot\nabla)v = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{div} v = 0, \quad \operatorname{div} H = 0 & \text{in} \quad \Omega \times (0,\infty), \\ v = 0, \quad \operatorname{curl} H \times \nu = 0, \quad \nu \cdot H = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ v(x,0) = a, \quad H(x,0) = b & \text{in} \quad \Omega. \end{cases} \tag{MHD}$$

Here $a = (a_1(x), a_2(x), a_3(x))$ and $b = (b_1(x), b_2(x), b_3(x))$ are prescribed initial data and ν is the unit outward normal on $\partial\Omega$.

The magnetohydrodynamic system is known to be one of the mathematical models describing the motion of incompressible viscous and electrically conducting fluid (see Cowling [3] or Landau and Lifshitz [10]). We impose the perfectly conducting wall on the magnetic field on the boundary. The perfectly conducting wall means that the surface of the obstacle is the perfect conductor.

The main purpose of this note is to show the global existence theorem for (MHD). The initial boundary value problem of the magnetohydrodynamic system was treated mainly in a bounded domain. Sermange and Temam [14] and several authors considered the interior problem by the Galerkin method. However, in general the Galerkin

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method does not work well in unbounded domain case. So we shall take another approach. Yoshida and Giga [17] considered the interior problem by the nonlinear semigroup theory and constructed the global strong solution.

On the other hand, there are some works in exterior domain. Recently Zhao [18] considered the magnetohydrodynamic system with non-perfect conductor body case, that is the boundary conditions for the magnetic field are replaced by the homogeneous or nonhomogeneous Dirichlet conditions. However from a physical viewpoint, the case of the perfectly conducting wall is also important. The author knows only the result by Kozono [9] concerning the case of perfectly conducting wall, where the weak solution was dealt with. There has been no result on the global existence of strong solution to (MHD) in the exterior domain.

Our approach is based on the argument of T. Kato [8]. Kato showed the global solvability of the Cauchy problem of the Navier-Stokes equations in $\mathbf{R}^N$ ($N \geq 2$) with small initial velocity with respect to L^N-norm. The argument of Kato is based on the estimates of various L^q-norm of the Stokes semigroup. In particular L^q-L^r type estimates play crucial role in it. The argument of Kato was extended to the case of N-dimensional exterior domain ($N \geq 3$) by Iwashita [7]. Our aim of this note is to show the global solvability of (MHD), by use of the argument of Kato and Iwashita. In order to do this, one of the main points is to study the linearized problems corresponding to (MHD). They are consisted of two systems of equations. First is well known system of the nonstationary Stokes equations and second is the linear diffusion system with the perfectly conducting wall:

$$\begin{cases} u_t + \operatorname{curl}\operatorname{curl} u = 0, & \operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{curl} u \times \nu = 0, & \nu \cdot u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ & u(x,0) = b \quad \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is unknown vector valued function and initial data b is given. We call the above system (1.1) *the parabolic Maxwell system.* To derive good estimates to investigate (MHD), we have to study the parabolic Maxwell system.

Before stating our main results we shall introduce some notations. Throughout this note $B_R = \{x \in \mathbf{R}^3 \,|\, |x| < R\}$, $S_R = \partial B_R = \{x \in \mathbf{R}^3 \,|\, |x| = R\}$, $\Omega_R = \Omega \cap B_R$ and $B(X,Y)$ denotes the set of bounded linear operator from X to Y. $L^q(D)$ denotes the usual L^q space on D and $W^{m,q}(D)$ denotes the usual L^q-Sobolev space of order m. Furthermore we put

$$\begin{aligned} L^q_R(D) &= \{f \in L^q(D) \,|\, f(x) = 0 \text{ for } x \notin B_R\}, \\ W^{m,q}_R(D) &= \{f \in W^{m,q}(D) \,|\, f(x) = 0 \text{ for } x \notin B_R\}, \\ \dot{W}^{m,q}(D) &= \overline{C_0^\infty(D)}^{\|\cdot\|_{W^{m,q}(D)}}. \end{aligned}$$

In order to give the operator theoretic interpretation of (MHD), we shall introduce the Helmholtz decomposition. Let $1 < q < \infty$. It is well known that $L^q(\Omega)^3$ admits the Helmholtz decomposition:

$$L^q(\Omega)^3 = L^q_\sigma(\Omega) \oplus G^q(\Omega), \quad \oplus : \text{direct sum}.$$

Here

$$\begin{aligned} L^q_\sigma(\Omega) &= \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^q(\Omega)^3}}, \\ C^\infty_{0,\sigma}(\Omega) &= \{f \in C^\infty_0(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega\}, \\ G^q(\Omega) &= \{f \in L^q(\Omega)^3 \mid f = \nabla p \text{ for some } p \in L^q_{\mathrm{loc}}(\overline{\Omega})\}. \end{aligned}$$

By the assumption that $\partial\Omega$ is sufficiently smooth, the space $L^q_\sigma(\Omega)$ is characterised as (see e.g., Galdi [6, Chapter 3])

$$L^q_\sigma(\Omega) = \{f \in L^q(\Omega)^3 \mid \operatorname{div} f = 0 \text{ in } \Omega,\ \nu \cdot f = 0 \text{ on } \partial\Omega\}.$$

Let $P = P_{q,\Omega}$ be a continuous projection from $L^q(\Omega)^3$ onto $L^q_\sigma(\Omega)$ and let us define linear operators $A = A_{q,\Omega}$ and $B = B_{q,\Omega}$ as follows:

$$\begin{aligned} \mathcal{D}(A) &= W^{2,q}(\Omega)^3 \cap W^{1,q}_0(\Omega)^3 \cap L^q_\sigma(\Omega), \\ Av &= -P\Delta v \quad \text{for } v \in \mathcal{D}(A), \\ \mathcal{D}(B) &= L^q_\sigma(\Omega) \cap \{H \in W^{2,q}(\Omega)^3 \mid \operatorname{curl} H \times \nu = 0 \text{ on } \partial\Omega\}, \\ BH &= \operatorname{curl}\operatorname{curl} H \quad \text{for } H \in \mathcal{D}(B) \end{aligned}$$

From Akiyama, Kasai, Shibata and M. Tsutsumi [1], Borchers and Sohr [2] and Miyakawa [12, 13] both $-A$ and $-B$ generate the bounded analytic semigroups $\{e^{-tA}\}$ and $\{e^{-tB}\}$ in $L^q_\sigma(\Omega)$.

By Duhamel's principle (MHD) is converted into the system of integral equations:

$$\begin{aligned} v(t) &= e^{-tA}a - \int_0^t e^{-(t-s)A} P\big[(v(s)\cdot\nabla)v(s) + H(s) \times \operatorname{curl} H(s)\big]\,ds, \\ H(t) &= e^{-tB}b - \int_0^t e^{-(t-s)B}\big[(v(s)\cdot\nabla)H(s) - (H(s)\cdot\nabla)v(s)\big]\,ds. \end{aligned} \tag{1.2}$$

To solve (1.2) by use of the successive approximation, we need some estimates for L^q-norms of the semigroups $\{e^{-tA}\}$ and $\{e^{-tB}\}$. On the Stoke semigroup $\{e^{-tA}\}$ we already have enough information by Iwashita (see Theorem 1.3). Therefore we have to study the semigroup $\{e^{-tB}\}$. The first result is concerned with the local energy decay property for $\{e^{-tB}\}$.

Theorem 1.1 (Local energy decay). *Let $1 < q < \infty$. For any $R > R_0$ and any integer $m \geq 0$, there exists a constant $C = C(q, R, m) > 0$ such that*

$$\|\partial_t^m e^{-tB} f\|_{W^{2,q}(\Omega_R)} \leq Ct^{-(3/2+m)} \|f\|_{L^q(\Omega)}, \quad t \geq 1,$$

for any $f \in L^q_\sigma(\Omega) \cap L^q_R(\Omega)$.

By use of Theorem 1.1, one can obtain the following L^q-L^r estimates for $\{e^{-tB}\}$.

Theorem 1.2 (L^q-L^r estimates).
(i) *Let $1 \leq q \leq r \leq \infty$ and $(q, r) \neq (1, 1), (\infty, \infty)$. Then there exists a constant $C = C(q, r) > 0$ such that*

$$\|e^{-tB} f\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.
(ii) *Let $1 < q \leq r \leq 3$. Then there exists a constant $C = C(q, r) > 0$ such that*

$$\|\nabla e^{-tB} f\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

The following result by Iwashita is concerning L^q-L^r estimates of the Stokes semigroup, which is refined by Maremonti and Solonnikov and Enomoto and Shibata.

Theorem 1.3 ([7, 11, 5]).
(i) *Let $1 \leq q \leq r \leq \infty$ and $(q, r) \neq (1, 1), (\infty, \infty)$. Then there exists a constant $C = C(q, r) > 0$ such that*

$$\|e^{-tA} f\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.
(ii) *Let $1 < q \leq r \leq 3$. Then there exists a constant $C = C(q, r) > 0$ such that*

$$\|\nabla e^{-tA} f\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for any $f \in L^q_\sigma(\Omega)$.

Finally, combining Theorem 1.2 and Theorem 1.3 we obtain the global solvability of (MHD) with small initial data.

Theorem 1.4. *There exists a constant* $\epsilon > 0$ *such that if* $(a,b) \in L^3_\sigma(\Omega) \times L^3_\sigma(\Omega)$ *and* $\|(a,b)\|_{L^3} < \epsilon$, *then a unique strong solution* (v,H) *to* (MHD) *exists and satisfies the following properties*:

$$t^{(1-3/q)/2}(v,H) \in BC([0,\infty); L^q_\sigma(\Omega) \times L^q_\sigma(\Omega)) \quad \text{for any } q,\ 3 \le q \le \infty, \tag{1.3}$$
$$t^{1/2}\nabla(v,H) \in BC([0,\infty); L^3(\Omega) \times L^3(\Omega)), \tag{1.4}$$

where $BC(\cdot)$ *denotes the class of bounded continuous functions. All the values in* (1.3) *and* (1.4) *vanish at* $t = 0$ *except for* $q = 3$ *in* (1.3), *and in case* $q = 3$, $(v(0), H(0)) = (a,b)$.

The basic idea to prove Theorem 1.1 and Theorem 1.2 is similar to that of Iwashita [7] deals with the nonstationary Stokes equations. However the boundary condition of (1.1), the perfectly conducting wall, is quite different from the boundary condition of the Stokes equations that is homogeneous Dirichlet condition, nonslip boundary condition. Therefore in constructing the parametrix of the resolvent problem corresponding to (1.1), we have to introduce a new idea which is based on a theorem due to von Wahl [16, Theorem 3.2]. In order to prove Theorems 1.1 and 1.2, first of all we have to study the resolvent problem corresponding to (1.1). In view of Miyakawa [12], it is suffices to study the following Laplace resolvent system with perfectly conducting wall:

$$\begin{cases} \lambda u - \Delta u = f & \text{in} \quad \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

Here $\lambda \in \mathbf{C}$ and f is given vector field.

2. Preliminaries

We consider the Laplace resolvent system in $\mathbf{R}^3$:

$$\lambda u - \Delta u = f, \quad \text{in } \mathbf{R}^3. \tag{2.1}$$

If $f \in L^q(\mathbf{R}^3)^3$, $1 < q < \infty$ and $\lambda \in \mathbf{C} \setminus (-\infty, 0]$, then a solution u to (2.1) is given by

$$u(x;\lambda) = [R_0(\lambda)f](x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{e^{-\sqrt{\lambda}|x-y|}}{|x-y|} f(y)\,dy.$$

We shall investigate $R_0(\lambda)$. Set

$$\Sigma_\epsilon = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \epsilon\}, \quad 0 < \epsilon < \frac{\pi}{2}.$$

Lemma 2.1. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and m be a non-negative integer. As $\lambda \to 0$ in Σ_ϵ, the resolvent $R_0(\lambda)$ has an expansion*

$$R_0(\lambda) = \sum_{j=0}^{\infty} \lambda^j G_j + \sum_{j=0}^{\infty} \lambda^{j+1/2} F_j$$

in $B(W_R^{m,q}(\mathbf{R}^3), W^{m+2,q}(B_R))$. Here $F_j, G_j \in B(W_R^{m,q}(\mathbf{R}^3), W^{m+2,q}(B_R))$. Especially,

$$F_0 f = \frac{1}{4\pi} \int_{\mathbf{R}^3} f(y)\, dy, \qquad G_0 f = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|}\, dy.$$

This lemma plays an important role in the study of the expansion of the resolvent $(\lambda + B)^{-1}$ in section 4.

In order to introduce the well known lemma due to Bogovskiĭ, we introduce the function space $\dot{W}_a^{m,q}(D)$ as follow

$$\dot{W}_a^{m,q}(D) = \left\{ f \in \dot{W}^{m,q}(D) \,\middle|\, \int_D f(x)\, dx = 0 \right\}.$$

Here D denotes a bounded domain in $\mathbf{R}^3$ with smooth boundary ∂D.

Lemma 2.2. *Let $1 < q < \infty$ and let m be a non-negative integer. Then there exists a bounded linear operator $\mathbb{B} : \dot{W}_a^{m,q}(D) \to \dot{W}^{m+1,q}(D)^3$ such that*

$$\operatorname{div} \mathbb{B}[f] = f \textit{ in } D \textit{ and } \|\mathbb{B}[f]\|_{W^{m+1,q}(D)} \le C_{q,m,D} \|f\|_{W^{m,q}(D)}.$$

Proposition 2.3. *Let $1 < q < \infty$ and $R > R_0$.*

(i) *Let $G = \Omega$, Ω_{R+1} or $\mathbf{R}^3$, m be a non-negative integer and let $\varphi \in C_0^\infty(\mathbf{R}^3)$ be a cut-off function such that $\varphi(x) = 1$ for $|x| \le R-1$ and $\varphi(x) = 0$ for $|x| \ge R$. If $u \in W_{\mathrm{loc}}^{m,q}(G)^3$, $\operatorname{div} u = 0$ in G and $\nu \cdot u = 0$ on ∂G (when $G = \Omega$ or Ω_{R+1}), then $(\nabla\varphi) \cdot u \in \dot{W}_a^{m,q}(D_R)$, where $D_R = \{x \in \mathbf{R}^3 \mid R-1 \le |x| \le R\}$. Therefore $\mathbb{B}[(\nabla\varphi)\cdot u] \in \dot{W}^{m+1,q}(D_R)^3$, $\operatorname{div} \mathbb{B}[(\nabla\varphi)\cdot u] = (\nabla\varphi)\cdot u$ and*

$$\|\mathbb{B}[(\nabla\varphi)\cdot u]\|_{W^{m+1,q}(\mathbf{R}^3)} \le C_{q,m,\varphi,R} \|u\|_{W^{m,q}(D_R)}.$$

(ii) *If $u \in W_{\mathrm{loc}}^{m,q}(\Omega)^3$, $\operatorname{div} u = 0$ in Ω and $\nu \cdot u = 0$ on $\partial\Omega$, then there exists a $v \in W^{m,q}(\Omega)^3$ such that $v = u$ in Ω, $\operatorname{div} v = 0$ in $\mathbf{R}^3$ and*

$$\|v\|_{W^{m,q}(\mathbf{R}^3)} \le C_{q,m} \|u\|_{W^{m,q}(\Omega)}.$$

3. On the operator B_q in the exterior domain

Let B_q be an linear operator introduced in section 1. In this section we study the properties of B_q.

Theorem 3.1. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$. For any $\lambda \in \Sigma_\epsilon$ and $f \in L^q(\Omega)^3$, there is a unique solution $u \in W^{2,q}(\Omega)^3$ to (1.5). Furthermore, there is a positive constant $C = C(q, \epsilon, \Omega, \lambda_0) > 0$ such that the following estimate holds:*

$$|\lambda|\|u\|_{L^q(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{L^q(\Omega)} + \|\nabla^2 u\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}, \quad \text{for } |\lambda| \ge \lambda_0.$$

Moreover we have

$$|\lambda|\|u\|_{L^q(\Omega)} \le C\|f\|_{L^q(\Omega)}.$$

Here the constant C is independent of λ_0. If $f \in L^q_\sigma(\Omega)$, then we have $u \in L^q_\sigma(\Omega)$.

Corollary 3.2. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$. Then the resolvent set $\rho(-B) \supset \mathbf{C} \setminus (-\infty, 0]$. And there exists $C = C(p, \epsilon, \Omega) > 0$ such that*

$$\|(\lambda + B_q)^{-1}\|_{\mathscr{L}(L^q(\Omega))} \le \frac{C}{|\lambda|} \quad \text{for any } \lambda \in \Sigma_\epsilon,$$

where $\|\cdot\|_{\mathscr{L}(\cdot)}$ denotes the operator norm. Furthermore for any $\delta > 0$ there exists $C = C(\delta, q, \epsilon, \Omega) > 0$ such that the estimate

$$\|(\lambda + B_q)^{-1} f\|_{W^{2,q}(\Omega)} \le C\|f\|_{L^q(\Omega)} \quad \text{for any } \lambda \in \Sigma_\epsilon \cap \{|\lambda| > \delta\}$$

holds. And $(B_q)^* = B_{q'}$, where $1/q + 1/q' = 1$.

Proposition 3.3. *Let m be a non-negative integer and $1 < q < \infty$.*

(i) *Let $u \in \mathcal{D}(B)$ and $Bu \in W^{m,q}(\Omega)^3$. Then $u \in W^{m+2,q}(\Omega)^3$ and moreover there is $C_m > 0$ such that the following estimate holds.*

$$\|u\|_{W^{m+2,q}(\Omega)} \le C_m(\|Bu\|_{W^{m,q}(\Omega)} + \|u\|_{L^q(\Omega)}).$$

(ii) *Let $u \in \mathcal{D}(B^m)$. Then $u \in W^{2m,q}(\Omega)^3$ and moreover the following estimate holds.*

$$\|u\|_{W^{2m,q}(\Omega)} \le C_m(\|B^m u\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)}).$$

Lemma 3.4. *Let $1 < q < \infty$.*

(i) *For a non-negative integer m, there is $C_m > 0$ such that*

$$\|B^m u\|_{L^q(\Omega)} \le C_m\|u\|_{W^{2m,q}(\Omega)} \quad \text{for } u \in \mathcal{D}(B^m).$$

(ii) *Let $0 < \epsilon < \pi/2$ and m be a non-negative integer. If $f \in \mathcal{D}(B^m)$, then we obtain the estimate*

$$\|(\lambda + B)^{-1} f\|_{W^{2m+2,q}(\Omega)} \le C_m\|f\|_{W^{2m,q}(\Omega)}$$

for $\lambda \in \Sigma_\epsilon$ with $|\lambda| \ge 1$.

4. Resolvent expansion near the origin

In this section we study the following Laplace resolvent system:

$$\begin{cases} \lambda u - \Delta u = f & \text{in} \quad \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.5}$$

Here $\lambda \in \Sigma_\epsilon = \{\lambda \in \mathbf{C}\backslash\{0\} \mid |\arg \lambda| < \pi - \epsilon,\ 0 < \epsilon < \pi/2\}$ and $f = (f_1(x), f_2(x), f_3(x))$ is given function. Our aim of this section is to prove the following theorem.

Theorem 4.1. *Let $1 < q < \infty$ and m be a nonnegative integer. There exists a solution operator $R(\lambda) \in B(W_R^{m,q}(\Omega), W^{m+2,q}(\Omega_{R+2}))$ such that $R(\lambda)$ depends on $\lambda \in \Sigma_\epsilon$ meromorphically and has the following properties:*

- (i) *The set Λ of the poles is discrete.*
- (ii) *$u = R(\lambda)f$ is a solution of* (1.5) *for $\lambda \in \Sigma_\epsilon \setminus \Lambda$ and $f \in W_R^{m,q}(\Omega)$.*
- (iii) *$R(\lambda) \in B(W_R^{m,q}(\Omega), W^{m+2,q}(\Omega))$ for each $\lambda \in \Sigma_\epsilon \setminus \Lambda$.*
- (iv) *Let $\Sigma_\epsilon(\delta) = \{\lambda \in \Sigma_\epsilon \mid |\lambda| < \delta\}$. There exists $\delta_0 > 0$ such that $\Sigma_\epsilon(\delta_0) \cap \Lambda = \emptyset$ and $R(\lambda)$ has the following expansion of $\lambda \in \Sigma_\epsilon(\delta_0)$ in $B(W_R^{m,q}(\Omega), W^{m+2,q}(\Omega_R))$:*

$$R(\lambda) = \lambda^{1/2} G_1 + G_2(\lambda) + \lambda^{1/2} G_3(\lambda). \tag{4.1}$$

Here $G_1 \in B(W_R^{m,q}(\Omega), W^{m+2,q}(\Omega_R))$, $G_2(\lambda)$ is $B(W_R^{m,q}(\Omega), W^{m+2,q}(\Omega_R))$-valued holomorphic function of $\lambda \in \Sigma_\epsilon(\delta_0)$ and $G_3(\lambda)$ is bounded.

In order to prove Theorem 4.1, first of all we construct the parametrix to (1.5). Choose a positive number $R > 0$ such that $R > R_0 + 3$. Here R_0 is introduced in the previous section. Let Φ be a mapping of $f \in L^q(\Omega_{R+3})$ to unique solution $u \in W^{2,q}(\Omega_{R+3})$ of the following problem:

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega_{R+3}, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega_{R+3}, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega_{R+3}. \end{cases}$$

Then $\Phi \in B(L^q(\Omega_{R+3}), W^{2,q}(\Omega_{R+3}))$. Put $\varphi \in C_0^\infty(\mathbf{R}^3)$ such that $\varphi = 1$ for $|x| \leq R+1$ and $= 0$ for $|x| \geq R+2$. For $f \in L^q(\Omega)$, let f_{R+3} be the restriction of f to Ω_{R+3} and let f_0 be the zero extension of f to $\mathbf{R}^3$, that is, $f_0 = f$ in Ω and $f_0 = 0$ in $\mathbf{R}^3 \setminus \Omega$. Let us define an operator $A(\lambda)$ by

$$A(\lambda)f = (1 - \varphi)R_0(\lambda)f_0 + \varphi\Phi f_{R+3}. \tag{4.2}$$

For $A(\lambda)f$ we have

$$\begin{cases} (\lambda-\Delta)A(\lambda)f = f + S(\lambda)f & \text{in} \quad \Omega, \\ \operatorname{curl}(A(\lambda)f) \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot (A(\lambda)f) = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$S(\lambda)f = 2\nabla\varphi\cdot\nabla R_0(\lambda)f_0 + (\Delta\varphi)R_0(\lambda)f_0 + \lambda\varphi\Phi f_{R+3} - 2\nabla\varphi\cdot\nabla\Phi f_{R+3} - (\Delta\varphi)\Phi f_{R+3}$$

From the definition of the cut-off function φ, $\operatorname{supp} S(\lambda)f \subset \Omega_{R+3}$. If $f \in L^q_{R+2}(\Omega)$, then by the Fourier multiplier theorem and the property of $R_0(\lambda)$, we obtain

$$\|R_0(\lambda)f_0\|_{W^{2,q}(B_{R+3})} \leq C\|f\|_{L^q(\Omega)}.$$

Lemma 4.2. *The inverse $(I+S(\lambda))^{-1}$ of $I+S(\lambda)$ exists as a $B(L^q_{R+2}(\Omega), L^q_{R+2}(\Omega))$-valued meromorphic function of $\lambda \in \Sigma_\epsilon$. The set Λ of poles is discrete and has no intersection with $\Sigma_\epsilon(\delta_0)$ for some $\delta_0 > 0$. Furthermore, $(I+S(\lambda))^{-1}$ has the same type of expansion as* (4.1).

This lemma will follow from the following lemma.

Lemma 4.3. *$I + S(0)$ has the bounded inverse $(I+S(0))^{-1}$.*

Before stating the proof of Lemma 4.3, we introduce the following uniqueness result which will be required in the proof of Lemma 4.3.

Proposition 4.4. *Let $1 < q < \infty$. Suppose that $u \in W^{2,q}_{\text{loc}}(\Omega)$ satisfies*

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u \times \nu = 0 & \text{on} \quad \partial\Omega, \\ \nu \cdot u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{4.3}$$

and $u(x) = O(|x|^{-1})$, $\nabla u(x) = O(|x|^{-2})$. Then $u = 0$ in Ω.

Proof of Proposition 4.4. By virtue of the local regularity theory for the elliptic equations, one can take $u \in W^{2,r}_{\text{loc}}(\Omega)$ for any $r \in (1,\infty)$. In particular, now we take $u \in W^{2,2}_{\text{loc}}(\Omega)$. We consider a function $\psi \in C_0^\infty(\mathbf{R}^3)$ with the properties $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ for $|x| \leq 1/2$ and $= 0$ for $|x| \geq 1$ and define $\psi_R(x) := \psi(x/R)$. According to the well known formula $\Delta u = \nabla \operatorname{div} u - \operatorname{curl}\operatorname{curl} u$, the divergence theorem and the assumption we get

$$\begin{aligned} 0 = \int_\Omega -\Delta u\cdot\psi_R u\,dx = &\int_{\Omega_R} \operatorname{curl} u\cdot(\nabla\psi_R\times u)\,dx + \int_{\Omega_R}(\operatorname{div} u)(\nabla\psi_R\cdot u)\,dx \\ &+ \int_{\Omega_R}\psi_R[(\operatorname{div} u)^2 + \operatorname{curl} u\cdot\operatorname{curl} u]\,dx. \end{aligned}$$

Since $\operatorname{supp} \nabla \psi_R \subset \{x \in \mathbf{R}^3 \mid R/2 < |x| < R\}$, we have

$$\left| \int_{\Omega_R} \operatorname{curl} u \cdot (\nabla \psi_R \times u)\, dx + \int_{\Omega_R} (\operatorname{div} u)(\nabla \psi_R \cdot u)\, dx \right| \leq \frac{C}{R}.$$

Therefore letting $R \to \infty$, we have $\| \operatorname{curl} u \|^2_{L^2(\Omega)} + \| \operatorname{div} u \|^2_{L^2(\Omega)} = 0$. This implies that $\operatorname{curl} u = 0$ and $\operatorname{div} u = 0$ in Ω and moreover by virtue of theorem due to von Wahl [16], we obtain $\nabla u = 0$ in Ω. Hence $u = \text{const}$ in Ω. From the assumption that u satisfies $\nu \cdot u = 0$ on $\partial \Omega$, we have $u = 0$ in Ω. This completes the proof. □

Now we shall show Lemma 4.3.

Proof of Lemma 4.3. Since the operator $S(0)$ is compact, by the Fredholm alternative theorem it suffices to show injectivity of $I + S(0)$. Let us pick up $f \in L^q_{R+2}(\Omega)$ so that $(I + S(0))f = 0$. Then it follows from (4.2), $A(0)f$ satisfies (4.3) and moreover $A(0)f$ has the properties that $A(0)f = O(|x|^{-1})$ and $\nabla(A(0)f) = O(|x|^{-2})$. Therefore from Proposition 4.4, $A(0)f = 0$. Namely we have

$$(1 - \varphi) R_0(0) f_0 + \varphi \Phi f_{R+3} = 0 \quad \text{in } \Omega.$$

By the definition of the cut-off function φ we have $\Phi f_{R+3} = 0$ for $|x| \leq R + 1$ and $R_0(0) f_0 = 0$ for $|x| \geq R + 2$. Put $w = \Phi f_{R+3}$ for $x \in \Omega_{R+3}$ and $= 0$ for $x \notin \Omega$. Then w satisfies

$$\begin{cases} -\Delta w = f_0 & \text{in} \quad B_{R+3}, \\ \operatorname{curl} w \times \nu = 0 & \text{on} \quad S_{R+3}, \\ \nu \cdot w = 0 & \text{on} \quad S_{R+3}. \end{cases}$$

On the other hand, from $R(0) f_0 = 0$ for $|x| \geq R + 2$, we also have

$$\begin{cases} -\Delta R_0(0) f_0 = f_0 & \text{in} \quad B_{R+3}, \\ \operatorname{curl} (R_0(0) f_0) \times \nu = 0 & \text{on} \quad S_{R+3}, \\ \nu \cdot (R_0(0) f_0) = 0 & \text{on} \quad S_{R+3}. \end{cases}$$

Hence we obtain $w = R_0(0) f_0$ in Ω_{R+3}. Therefore

$$0 = A(0) f = R_0(0) f_0 + \varphi (\Phi f_{R+3} - R_0(0) f_0) = R_0(0) f_0.$$

This implies $f_0 = 0$ in Ω. □

Proof of Lemma 4.2. Let $M = \|(I + S(\lambda))^{-1}\|$, where $\| \cdot \|$ denotes the operator norm. From the fact that $S(\lambda)$ is continuous in $\Sigma_\epsilon \cup \{0\}$, there is some $\delta_0 > 0$ such that $\|S(\lambda) - S(0)\| < 1/2M$ for any $\lambda \in \Sigma_\epsilon(\delta_0)$. Hence, for $\lambda \in \Sigma_\epsilon(\delta_0)$,

$$(I + S(\lambda))^{-1} = \sum_{j=1}^{\infty} [(I + S(0))^{-1}(S(0) - S(\lambda))]^j (I + S(0))^{-1}. \tag{4.4}$$

Since $S(\lambda)$ is holomorphic in $\lambda \in \Sigma_\epsilon$, by analytic Fredholm's alternative theorem (see e.g., Dunford and Schwartz [4, p. 592, Lemma 13]) we obtain $(I + S(\lambda))^{-1}$ for any $\lambda \in \Sigma_\epsilon$ as a meromorphic function and we see that the set Λ of poles is discrete in Σ_ϵ. The expansion of $(I + S(\lambda))^{-1}$ follows from Lemma 2.1, Lemma 4.3 and (4.4). □

With help of Lemma 4.2, we prove Theorem 4.1.

Proof of Theorem 4.1. Define $R(\lambda)$ by

$$R(\lambda) = A(\lambda)(I + S(\lambda))^{-1}. \tag{4.5}$$

Then the assertions are immediately derived from the expansion of $R_0(\lambda)$, Lemma 4.2 and (4.5). □

5. Proof of Theorem 1.1

In this section we will prove Theorem 1.1 with aid of Theorem 4.1. Let $0 < \epsilon < \epsilon_1 < \pi/2$ and let γ be a contour as follows: $\gamma = \gamma_1 \cup \gamma_2$, where

$$\begin{aligned} \gamma_1 &= \{\lambda \in \mathbf{C} \,|\, 0 < |\lambda| \le \delta_0/2, |\arg \lambda| = \pi - \epsilon_1\}, \\ \gamma_2 &= \{\lambda \in \mathbf{C} \,|\, |\lambda| > \delta_0/2, |\arg \lambda| = \pi - \epsilon_1\}. \end{aligned}$$

According to Theorem 4.1, the semigroup e^{-tB} is represented as

$$e^{-tB} = \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t} R(\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{\gamma_2} e^{\lambda t}(\lambda + B)^{-1}\, d\lambda,$$

in $L^q_\sigma(\Omega) \cap L^q_{R+2}(\Omega)$. Hence we have

$$\begin{aligned} \partial_t^m e^{-tB} &= \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^m R(\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{\gamma_2} e^{\lambda t}\lambda^m(\lambda + B)^{-1}\, d\lambda \\ &=: I_1(t) + I_2(t). \end{aligned}$$

By Corollary 3.2 we easily see that

$$\|I_2(t)\|_{B(L^q_{R+2}(\Omega), W^{2,q}(\Omega_{R+2}))} \le Ce^{-ct}, \tag{5.1}$$

for $t \ge 1$. Therefore it suffices to estimate $I_1(t)$. In order to do this, we introduce the following well known lemma concerning the gamma function $\Gamma(\sigma)$.

Lemma 5.1. *For $\sigma > 0$ and $t > 0$, it holds that*

$$\left|\frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^{\sigma-1}\, d\lambda - \frac{\sin \sigma\pi}{\pi}\Gamma(\sigma)t^{-\sigma}\right| \le Ce^{-ct}.$$

From Theorem 4.1 we see that

$$I_1(t) = \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^{m+1/2}G_1\,d\lambda + \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^{m}G_2(\lambda)\,d\lambda + \frac{1}{2\pi i}\int_{\gamma_1} e^{\lambda t}\lambda^{m+1/2}G_3(\lambda)\,d\lambda$$
$$=: J_1(t) + J_2(t) + J_3(t).$$

Here $G_1, G_2(\lambda)$ and $G_3(\lambda)$ are given in Theorem 4.1. By use of Lemma 5.1, we have

$$\|J_1(t)\|_{B(L^q_{R+2}(\Omega),W^{2,q}(\Omega_{R+2}))} \le Ct^{-m-3/2}. \tag{5.2}$$

Since $G_2(\lambda)$ is holomorphic, by use of Cauchy's integral theorem we see that

$$\|J_2(t)\|_{B(L^q_{R+2}(\Omega),W^{2,q}(\Omega_{R+2}))} \le Ce^{-ct}. \tag{5.3}$$

Finally, since $\|G_3(\lambda)\|_{B(L^q_{R+2}(\Omega),W^{2,q}(\Omega_R+2))} \le C$, we have

$$\|J_3(t)\|_{B(L^q_{R+2}(\Omega),W^{2,q}(\Omega_{R+2}))} \le Ct^{-m-3/2}. \tag{5.4}$$

Combining (5.1), (5.2), (5.3) and (5.4), we obtain Theorem 1.1.

6. Proof of Theorem 1.2

In this section we will give the proof of Theorem 1.2. Here and hereafter $T(t)$ denotes the analytic semigroup generated by $-B_q$ ($T(t) = e^{-tB}$). Let us define an operator $E(t)$ by

$$E(t)u(x) = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbf{R}^3} \exp\left(-\frac{|x-y|^2}{4t}\right)u(y)\,dy.$$

If $b \in L^q_\sigma(\mathbf{R}^3)$, then $u = E(t)b$ solves the following equations:

$$\begin{cases} u_t + \operatorname{curl}\operatorname{curl} u = 0, \quad \operatorname{div} u = 0 & \text{in } \mathbf{R}^3 \times (0,\infty), \\ u(x,0) = b & \text{in } \mathbf{R}^3. \end{cases} \tag{6.1}$$

By Young's inequality and Sobolev's embedding theorem we have

Lemma 6.1. *Let $1 \le q \le r \le \infty$ and put $\sigma = 3(1/q - 1/r)/2$. Then for any integer j we have*

$$\begin{aligned} &\|\partial_t^j\partial_x^\alpha E(t)b\|_{L^r(\mathbf{R}^3)} \le Ct^{-\sigma-j-|\alpha|/2}\|b\|_{L^q(\mathbf{R}^3)}, \quad t \ge 1, \\ &\|\partial_t^j\partial_x^\alpha E(t)b\|_{L^r(\mathbf{R}^3)} \le C(1+t)^{-\sigma-j-|\alpha|/2}\|b\|_{W^{[2\sigma]+1+|\alpha|+2j,q}(\mathbf{R}^3)}, \quad t \ge 0, \end{aligned} \tag{6.2}$$

where $[\cdot]$ denotes the Gauss symbol.

Next we prove the local regularity property of $T(t)$.

Lemma 6.2. *Let $1 < q < \infty$ and $R \geq R_0$. Assume that $b \in L^q_\sigma(\Omega) \cap L^q_R(\Omega) \cap \mathcal{D}(B^N_q)$ for some integer $N \geq 1$. Then the estimate*

$$\|\partial_t^j T(t)b\|_{W^{2(N-j),q}(\Omega_R)} \leq C_{q,N,R}(1+t)^{-3/2-j}\|b\|_{W^{2N,q}(\Omega)},$$

holds for any $t > 0$ and $j = 0, 1, 2$.

Proof. By virtue of the local energy decay theorem we have

$$\|\partial_t^m T(t)b\|_{W^{2,q}(\Omega_{R+1})} \leq Ct^{-3/2-m}\|b\|_{L^q(\Omega)} \quad \text{for any } t \geq 1,\ m \geq 0. \tag{6.3}$$

Let $0 < r < 1$ and let us choose $\varphi \in C_0^\infty(\mathbf{R}^3)$ in such a way that $\varphi(x) = 1$ for $|x| \leq R + r$ and $\varphi(x) = 0$ for $|x| \geq R + 1$. Since $b \in L^q_\sigma(\Omega)$ and $u(t) = T(t)b$ satisfies the equations:

$$\begin{cases} u_t - \Delta u = 0, & \operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{curl} u \times \nu = 0, & \nu \cdot u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty). \end{cases} \tag{6.4}$$

If we put $v = \varphi u - \mathbb{B}[(\nabla\varphi)\cdot u]$ and $w = \partial_t^m v$, then by Proposition 2.3 and (6.4), w satisfies the equations

$$\begin{cases} -\Delta w = h, & \operatorname{div} w = 0 \quad \text{in} \quad \Omega_{R+1}, \\ \operatorname{curl} w \times \nu = 0, & \nu \cdot w = 0 \quad \text{on} \quad \partial\Omega_{R+1} = \partial\Omega \cup S_{R+1}, \end{cases}$$

for any $t > 0$ and

$$h = -2\nabla\varphi \cdot \nabla\partial_t^m u - (\Delta\varphi)\partial_t^m u - (\partial_t - \Delta)\partial_t^m \mathbb{B}[(\nabla\varphi)\cdot u] - \varphi\partial_t^{m+1} u.$$

From (6.3) we have $\partial_t^m u, \partial_t^{m+1} u \in W^{2,q}(\Omega_{R+1})$, therefore $h \in W^{1,q}(\Omega_{R+1})$. So by the local regularity theorem, we obtain $w \in W^{3,q}(\Omega_{R+1})$ and

$$\|w\|_{W^{3,q}(\Omega_{R+1})} \leq Ct^{-\frac{3}{2}-m}\|b\|_{L^q(\Omega)}, \quad \text{for } t \geq 1.$$

Therefore we get

$$\|\partial_t^m u\|_{W^{3,q}(\Omega_{R+r})} \leq Ct^{-\frac{3}{2}-m}\|b\|_{L^q(\Omega)}, \quad \text{for } t \geq 1.$$

Repeated use of the above argument implies that

$$\|\partial_t^m T(t)b\|_{W^{2N,q}(\Omega_R)} \leq Ct^{-3/2-m}\|b\|_{L^q(\Omega)} \quad \text{for } t \geq 1.$$

for any integers $m \geq 0$ and $N \geq 1$. When $0 < t \leq 1$, by using the analytic semigroup theory and Proposition 3.3, we obtain

$$\|\partial_t^m T(t)b\|_{W^{2(N-m),q}(\Omega)} \leq C\|b\|_{W^{2N,q}(\Omega)}.$$

This completes the proof. □

Put $\tilde{b} = e^{-B}b$ for $b \in L^q_\sigma(\Omega)$. Then $\tilde{b} \in \mathcal{D}(B^N_q)$ for any integer $N \geq 0$ and

$$\|\tilde{b}\|_{W^{2N,q}(\Omega)} \leq C_{q,N}\|b\|_{L^q(\Omega)}. \tag{6.5}$$

Put $u(t) = T(t)\tilde{b} = T(t+1)b$. Then $u(t)$ is smooth in t and x and satisfies the following system:

$$\begin{cases} u_t + \operatorname{curl}\operatorname{curl} u = 0, & \operatorname{div} u = 0 \quad \text{in} \quad \Omega \times (0,\infty), \\ \operatorname{curl} u \times \nu = 0, & \nu \cdot u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \\ & u(0) = \tilde{b} \quad \text{in} \quad \Omega. \end{cases}$$

Since the asymptotic behavior of $T(t)b$ for large $t > 0$ follows from that of $u(t)$, so we shall start with the following step.

1st step. For any $m \geq 0$ and $t \geq 0$ we have the relations:

$$\|\partial_t^j u(t)\|_{W^{2m,q}(\Omega_R)} \leq C(1+t)^{-3/2-j}\|b\|_{L^q(\Omega)} \quad \text{for } t \geq 0. \tag{6.6}$$

In fact, let N be an integer so that $N \geq (3/q + 2m + 4)/2$. Since $\tilde{b} \in \mathcal{D}(B^N_q) \subset L^q_\sigma(\Omega)$, we have $\operatorname{div}\tilde{b} = 0$ in Ω, $\nu \cdot \tilde{b} = 0$ on $\partial\Omega$. Therefore by virtue of Proposition 2.3, there is $c \in W^{2N,q}(\mathbf{R}^3)$ such that $c = \tilde{b}$ in Ω, $\operatorname{div} c = 0$ in $\mathbf{R}^3$ and

$$\|c\|_{W^{2N,q}(\mathbf{R}^3)} \leq C\|\tilde{b}\|_{W^{2N,q}(\Omega)} \leq C\|b\|_{L^q(\Omega)}.$$

Put $v(t) = E(t)c$. Then v satisfies (6.1) and by Lemma 6.1 and Sobolev's embedding theorem, we have

$$\|\partial_t^j v(t)\|_{W^{2m,q}(\Omega_R)} \leq C(1+t)^{-\frac{3}{2q}-j}\|b\|_{L^q(\Omega)}, \tag{6.7}$$

for any $t \geq 0$ and $j = 0, 1$. Take $\varphi \in C_0^\infty(\mathbf{R}^3)$ such a way that $\varphi(x) = 1$ for $|x| \leq R$ and $= 0$ for $|x| \geq R + 1$. In view of Proposition 2.3, put

$$w(t) = u(t) - (1-\varphi)\operatorname{div} v(t) - \mathbb{B}[(\nabla\varphi)\cdot v(t)].$$

We see that $\operatorname{div} w(t) = 0$ in Ω and $\nu \cdot w = 0$ on $\partial\Omega$ for any $t \geq 0$. Moreover from Proposition 2.3 and (6.7) we have

$$\|\partial_t^j \mathbb{B}[(\nabla\varphi)\cdot v(t)]\|_{W^{2m+2,q}(\mathbf{R}^3)} \leq C(1+t)^{-\frac{3}{2q}-j}\|b\|_{L^q(\Omega)} \tag{6.8}$$

for any $t \geq 0$ and $j = 0, 1$. Since $\operatorname{supp}\mathbb{B}[(\nabla\varphi)\cdot v(t)] \subset \{x \in \mathbf{R}^3 \mid R-1 \leq |x| \leq R\}$ and $1 - \varphi(x) = 0$ for $|x| \leq R$, we see that $w = u$ n Ω_R. Therefore, if we obtain

$$\|\partial_t^j w(t)\|_{W^{2m,q}(\Omega_R)} \leq C(1+t)^{-\frac{3}{2q}-j}\|b\|_{L^q(\Omega)} \tag{6.9}$$

for $j = 0, 1$, then we get (6.6). To obtain (6.9) we set

$$d = \varphi\tilde{b} - \mathbb{B}[(\nabla\varphi)\cdot\tilde{b}],$$
$$g(t) = 2\nabla\varphi\cdot\nabla v(t) + (\Delta\varphi)v(t) - (\partial_t - \Delta)\mathbb{B}[(\nabla\varphi)\cdot v(t)],$$

and then w satisfies the following equations:

$$\begin{cases} \partial_t w + \operatorname{curl}\operatorname{curl} w = g, \quad \operatorname{div} w = 0 & \text{in} \quad \Omega\times(0,\infty), \\ \operatorname{curl} w\times\nu = 0, \quad \nu\cdot w = 0 & \text{in} \quad \partial\Omega\times(0,\infty), \\ w(x,0) = d & \text{in} \quad \Omega. \end{cases}$$

In order to represent $w(t)$ by Duhamel's principle and to estimate the resulting formula by using Lemma 6.2, we require the following facts:

$$d \in \mathcal{D}(B_q^N)\cap L^q_\sigma(\Omega)\cap L^q_{R+1}\Omega, \tag{6.10}$$
$$\partial_t^j g(t) \in \mathcal{D}(B_q^{N-1-j})\cap L^q_\sigma(\Omega)\cap L^q_{R+1}(\Omega), \quad \text{for any } t\ge 0, \tag{6.11}$$
$$\|d\|_{W^{2N,q}(\Omega)} \le C_{q,N}\|b\|_{L^q(\Omega)}, \tag{6.12}$$
$$\|\partial_t^j g(t)\|_{W^{2(N-1-j),q}(\Omega)} \le C_{q,m,R}(1+t)^{-3/2p-j}\|a\|_{L^q(\Omega)}, \tag{6.13}$$

where $j = 0, 1$. From Proposition 2.3 and (6.5), (6.12) holds and from (6.7) and (6.8), (6.13) holds. The following lemma tells us that (6.10) and (6.11) hold.

Lemma 6.3. *Let $1 < q < \infty$, U be a neighborhood of $\overline{\mathcal{O}}$ in $\mathbf{R}^3$ and $N \ge 1$ be an integer. If $b \in W^{2N,q}(\Omega)$ satisfies $\operatorname{div} b = 0$ in Ω and $b = 0$ in $\Omega\cap U$, then $b \in \mathcal{D}(B_q^N)$. As a consequence, if $b \in W^{2N,q}(\Omega)\cap L^q_\sigma(\Omega)$ coincides with some $c \in \mathcal{D}(B_q^N)$ in $\Omega\cap U$, then $b \in \mathcal{D}(B_q^N)$.*

By (6.10), (6.11) and Duhamel's principle

$$w(t) = T(t)d + \int_0^t T(t-s)g(s)\,ds,$$

and from Lemma 6.2, (6.12) and (6.13) we obtain

$$\begin{aligned}\|w(t)\|_{W^{2m,q}(\Omega_R)} &\le C\left\{(1+t)^{-\frac{3}{2}}\|d\|_{W^{2m,q}(\Omega)} + \int_0^t (1+t-s)^{-\frac{3}{2}}\|g(s)\|_{W^{2m,q}(\Omega)}\,ds\right\} \\ &\le C(1+t)^{-3/2q}\|b\|_{L^q(\Omega)}, \quad \text{for any } t>0,\end{aligned}$$

because $3/2q < 3/2$. And we see that

$$\partial_t w(t) = \partial_t T(t)d + T\left(\frac{t}{2}\right)g\left(\frac{t}{2}\right) + \int_0^{t/2}\partial_t T(t-s)g(s)\,ds + \int_0^{t/2} T(s)(\partial_t g)(t-s)\,ds,$$

therefore by Lemma 6.2, (6.12) and (6.13) we obtain

$$\|\partial_t w(t)\|_{W^{2m,q}(\Omega_R)} \le C(1+t)^{-3/2q-1}\|b\|_{L^q(\Omega)}.$$

This completes the proof of (6.9), and therefore we obtain (6.6). By Sobolev's embedding theorem and (6.6), we obtain

$$\|\partial_t^j u(t)\|_{W^{2m,\infty}(\Omega_R)} \le C(1+t)^{-\frac{3}{2q}-j}\|b\|_{L^q(\Omega)}, \quad j=0,1, \tag{6.14}$$

for any $t>0$, $1<q<\infty$, $m\ge 0$.

2nd step. Take $\psi(x)\in C_0^\infty(\mathbf{R}^3)$ such that $\psi(x)=1$ for $|x|\le R-1$ and $=0$ for $|x|\ge R$. Let us put

$$z(t)=(1-\psi)u(t)+\mathbb{B}[(\nabla\psi)\cdot u(t)].$$

Then we see that $\operatorname{div} z(t)=0$ in $\mathbf{R}^3$ and z satisfies

$$\begin{cases} z_t-\Delta z=h, \quad \operatorname{div} z=0 & \text{in } \mathbf{R}^3\times(0,\infty) \\ \qquad\qquad z(0)=e, & \text{in } \mathbf{R}^3, \end{cases} \tag{6.15}$$

where

$$h=2\nabla\psi\cdot\nabla u+(\Delta\psi)u+(\partial_t-\Delta)\mathbb{B}[(\nabla\psi)\cdot u],$$
$$e=(1-\psi)\tilde{b}+\mathbb{B}[(\nabla\psi)\cdot\tilde{b}].$$

From Proposition 2.3, (6.5) and (6.14), we obtain $\operatorname{div} e=0$ in $\mathbf{R}^3$ and

$$\|e\|_{W^{2m,q}(\mathbf{R}^3)} \le C\|b\|_{L^q(\Omega)}, \tag{6.16}$$
$$\|h(t)\|_{W^{2m,q}(\mathbf{R}^3)} \le C(1+t)^{-3/2q}\|b\|_{L^q(\Omega)}. \tag{6.17}$$

Especially from (6.15) we see that $\operatorname{div} h=0$. Therefore by Duhamel's principle

$$z(t)=E(t)e+Z(t), \quad Z(t)=\int_0^t E(t-s)h(s)\,ds.$$

Note that $z=u$ if $|x|\ge R$ because $1-\psi(x)=1$ and $\mathbb{B}[(\nabla\psi)\cdot u]=0$ for $|x|\ge R$, so that we shall estimate $z(t)$. From (6.2) and (6.16) we have

$$\|E(t)e\|_{L^r(\mathbf{R}^3)} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|b\|_{L^q(\Omega)}, \tag{6.18}$$
$$\|\nabla E(t)e\|_{L^r(\mathbf{R}^3)} \le C(1+t)^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)-\frac{1}{2}}\|b\|_{L^q(\Omega)}. \tag{6.19}$$

Next we shall estimate $Z(t)$ for $1 < q \le r \le \infty$. In order to do this choose ρ in such a way that $1 < \rho < \min\{q, 3/2\}$. Since $\operatorname{supp} h(x,t) \subset \{x \in \mathbf{R}^3 \mid R-1 \le |x| \le R\}$ by the Hölder inequality and (6.16) we have

$$\|h(s)\|_{L^\rho(\mathbf{R}^3)} \le C(1+s)^{-\frac{3}{2q}}\|b\|_{L^q(\Omega)}. \tag{6.20}$$

Let κ be an integer such that $\kappa > 3(1/q - 1/r) + 1$. Then by Sobolev's embedding theorem and (6.17) we have

$$\|h(s)\|_{W^{1,r}(\mathbf{R}^3)} \le C\|h(s)\|_{W^{\kappa,q}(\mathbf{R}^3)} \le C(1+s)^{-\frac{3}{2q}}\|b\|_{L^q(\Omega)}. \tag{6.21}$$

From (6.2), (6.20), (6.21) we have

$$\|Z(t)\|_{L^r(\mathbf{R}^3)} \le CI_\rho(t)\|b\|_{L^q(\Omega)}, \tag{6.22}$$
$$\|\nabla Z(t)\|_{L^r(\mathbf{R}^3)} \le CJ_\rho(t)\|b\|_{L^q(\Omega)}, \tag{6.23}$$

where

$$I_\rho(t) = \int_0^t (1+t-s)^{-\frac{3}{2}(\frac{1}{\rho}-\frac{1}{r})}(1+s)^{-\frac{3}{2q}}\, ds,$$
$$J_\rho(t) = \int_0^t (1+t-s)^{-\frac{3}{2}(\frac{1}{\rho}-\frac{1}{r}-\frac{1}{2})}(1+s)^{-\frac{3}{2q}}\, ds.$$

For notational simplicity, we put $\sigma = 3(1/q - 1/r)/2$. We shall estimate $I_\rho(t)$ and $J_\rho(t)$.

$$I_\rho(t) \le \left(1+\frac{t}{2}\right)^{-\sigma} \int_0^{t/2} (1+t-s)^{-\frac{3}{2}(\frac{1}{\rho}-\frac{1}{q})}(1+s)^{-\frac{3}{2q}}\, ds$$
$$+ \left(1+\frac{t}{2}\right)^{-\sigma} \int_0^{t/2} (1+s)^{-\frac{3}{2}(\frac{1}{\rho}-\frac{1}{r})}(1+t-s)^{-\frac{3}{2r}}\, ds.$$

Since $0 \le s \le t/2$, we see that $1+t-s \ge 1+s$. Therefore

$$I_\rho(t) \le C\left(1+\frac{t}{2}\right)^{-\sigma} \int_0^{t/2} (1+s)^{-\frac{3}{2\rho}}\, ds.$$

Next we consider $J_\rho(t)$. Note that $1/2 \le 3/2r$ and $r \le 3$, below. By the similar calculation we obtain the following.

$$J_\rho(t) \le C\left(1+\frac{t}{2}\right)^{-\sigma-\frac{1}{2}} \int_0^{t/2} (1+s)^{-\frac{3}{2\rho}}\, ds.$$

Therefore combining (6.18), (6.19), (6.22) and (6.23) we obtain

$$\|Z(t)\|_{L^r(\mathbf{R}^3)} \le C(1+t)^{-\sigma}\|b\|_{L^q(\Omega)},$$

for $1 < q \le r \le \infty$ and $t > 0$ and

$$\|\nabla Z(t)\|_{L^r(\mathbf{R}^3)} \le C(1+t)^{-\sigma-\frac{1}{2}}\|b\|_{L^q(\Omega)},$$

for $1 < q \le r \le 3$ and $t > 0$. Recalling that $u = z$ when $|x| \ge R$, we obtain

$$\|u(t)\|_{L^r(\Omega)} \le C(1+t)^{-\sigma}\|b\|_{L^q(\Omega)}, \quad t > 0 \tag{6.24}$$

for $1 < q \le r \le \infty$ and

$$\|\nabla u(t)\|_{L^r(\Omega)} \le C(1+t)^{-\sigma-\frac{1}{2}}\|b\|_{L^q(\Omega)}, \quad t > 0 \tag{6.25}$$

for $1 < q \le r \le 3$. Since $u(t) = T(1+t)b$, from (6.24) and (6.25) we have Theorem 1.2 in the case of $t > 1$, except for the case when $q = 1$.

3rd step. We consider the case when $0 < t \le 1$. For any real number $s \in (0, 2m)$, by the complex interpolation theorem we have $W^{s,q}(\Omega) = (L^q(\Omega), W^{2m,q}(\Omega))_\theta$ with $s = 2m\theta$ (see e.g., Triebel [15]). By Proposition 3.3 (ii) and the theory of analytic semigroup, we have

$$\|\nabla^j T(t)b\|_{L^q(\Omega)} \le Ct^{-\frac{j}{2}}\|b\|_{L^q(\Omega)}, \tag{6.26}$$

$$\|\nabla^j T(t)b\|_{W^{2m,q}(\Omega)} \le C^{-\frac{j}{2}}\{\|B_q^m T(t)b\|_{L^q(\Omega)} + \|T(t)b\|_{L^q(\Omega)}\} \le Ct^{-m}\|b\|_{L^q(\Omega)}, \tag{6.27}$$

for $j = 0, 1$. Therefore interpolating (6.26) and (6.27) for $s = 2m\theta$ we have

$$\|\nabla^j T(t)b\|_{W^{s,q}(\Omega)} \le Ct^{-\frac{s}{2}-\frac{j}{2}}\|b\|_{L^q(\Omega)},$$

for $j = 0, 1$. In particular by virtue of Sobolev's embedding theorem, for $s = 3(1/q - 1/r)$ we have

$$\|\nabla^j T(t)b\|_{L^r(\Omega)} \le C\|\nabla^j T(t)b\|_{W^{s,q}(\Omega)} \le Ct^{-\sigma-\frac{j}{2}}\|b\|_{L^q(\Omega)}. \tag{6.28}$$

Next we consider the case when $1 < q < \infty$ and $r = \infty$. Let $B^s_{q,r}(\Omega)$ be the Besov space and we shall use the fact that $B^{3/p}_{q,1} \subset L^\infty(\Omega)$ to estimate the L^∞-norm of $T(t)b$. If we choose an integer m such that $0 < 3/q < 2m$, then we have $B^{3/q}_{q,1}(\Omega) = [L^q(\Omega), W^{2m,q}(\Omega)]_{\theta,1}$ with $3/p = 2m\theta$. Here $[\cdot,\cdot]_{\theta,1}$ denotes the real interpolation. Hence, interpolating (6.26) and (6.27) we obtain

$$\|\nabla^j T(t)b|B^{3/q}_{q,1}(\Omega)\| \le Ct^{\frac{3}{2q}-\frac{j}{2}}\|b\|_{L^q(\Omega)},$$

for $j = 0, 1$. Hence we have

$$\|\nabla^j T(t)b\|_{L^\infty(\Omega)} \le C\|\nabla^j T(t)b|B^{3/q}_{q,1}(\Omega)\| \le Ct^{\frac{3}{2q}-\frac{j}{2}}\|b\|_{L^q(\Omega)}, \tag{6.29}$$

for $j = 0, 1$. Therefore from (6.28) and (6.29) we have Theorem 1.2 for $0 < t \le 1$, except for the case when $q = 1$.

4th step. Finally we consider the case when $q = 1$. By the duality we have

$$|(T(t)b, \varphi)| = |(b, T(t)\varphi)| \leq \|b\|_{L^1(\Omega)} \|T(t)\varphi\|_{L^\infty(\Omega)} \leq C\|b\|_{L^q(\Omega)} t^{-3/2q'} \|\varphi\|_{L^{q'}(\Omega)}$$

for any $\varphi \in C^\infty_{0,\sigma}(\Omega)$ which is dense in $L^{q'}_\sigma(\Omega)$, where $q' = q/(q-1)$. Therefore we have

$$\|T(t)b\|_{L^q(\Omega)} \leq Ct^{-\frac{3}{2}(1-\frac{1}{q})} \|b\|_{L^1(\Omega)} \tag{6.30}$$

for $1 < q < \infty$ and $t > 0$. Furthermore (6.30) also holds for $q = \infty$. In fact we have

$$\begin{aligned} \|T(t)b\|_{L^\infty(\Omega)} &= \left\| T\left(\frac{t}{2}\right) T\left(\frac{t}{2}\right) b \right\|_{L^\infty(\Omega)} \leq Ct^{-\frac{3}{2q}} \left\| T\left(\frac{t}{2}\right) b \right\|_{L^q(\Omega)} \\ &\leq Ct^{-\frac{3}{2q}} t^{-\frac{3}{2}(1-\frac{1}{q})} \|b\|_{L^1(\Omega)} = Ct^{-\frac{3}{2}} \|b\|_{L^1(\Omega)}. \end{aligned}$$

This completes the proof of Theorem 1.2.

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