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Some Infinite (Singly, Doubly and Triply) Sums

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Abstract

In this article some infinite (singly, doubly and triply) sums are reported. Some of them are shown as follows for example. That is,

(i) $$\sum_{k=1}^{\infty}\frac{1}{k(k+m)(k+m+1)\cdots(k+m+n)}=\frac{(m-1)!}{(n+m)!}\sum_{k=1}^{m}\frac{1}{(n+k)},$$

(Singly infinite sum)

(ii) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{\left[(m-1)!\cdot\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)}=e-\sum_{k=0}^{m-1}\frac{1}{k!}$$

(Doubly infinite sum)

(iii) $$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left[(m-1)!\cdot(n+m)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)}=1.59063\cdots,$$

(Triply infinite sum)

where $m\in Z^{+}$ and $n\in Z_{0}^{+}$.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D=\{D_-,D_+\}$, $C=\{C_-,C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-, D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in $D(z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^{+}), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_-, $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+,

$\zeta \ne z$, $\ z \in C$, $\ \nu \in R$, $\ \Gamma$; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i} \int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin Z^-), \quad [\,\textit{Refer to} (1)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in Z^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in R), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu |\, \nu \in R\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν, for the function f such that $f \in F = \{ f ;\ 0 \ne |f_\nu| < \infty, \nu \in R \}$, where $f = f(z)$ and $z \in C$. (*vis.* $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α.)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F. (*F.O.G.* ; Fractional calculus operator group)

(III) **Lemma** [1]

(i) $$((z-c)^b)_\alpha = e^{-i\pi b} \frac{\Gamma(\alpha - b)}{\Gamma(-b)} (z-c)^{b-\alpha} \qquad \left(\left| \frac{\Gamma(\alpha - b)}{\Gamma(-b)} \right| < \infty \right).$$

(ii) $$(\log(z-c))_\alpha = -e^{-i\pi\alpha} \Gamma(\alpha) (z-c)^{-\alpha} \qquad (|\Gamma(\alpha)| < \infty),$$

(iii) $$(u \cdot v)_\alpha = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \qquad \begin{pmatrix} u = u(z) \\ v = v(z) \end{pmatrix},$$

where $z \ne c$ in (i) and (ii).

§ 1. Singly Infinite Sums

Theorem 1. *We have*

$$\sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1)\cdots(k+m+n)} = \frac{(m-1)!}{(n+m)!} \sum_{k=1}^{m} \frac{1}{n+k} \tag{1}$$

where $m \in Z^+$, $n \in Z_0^+ \ (= Z^+ \cup \{0\})$.

Proof. We have

$$(z^n \cdot \log az)_{-m} = \sum_{k=0}^{\infty} \frac{\Gamma(-m+1)}{k!\,\Gamma(-m+1-k)} (z^n)_{-m-k} (\log az)_k \quad (az \neq 0) \tag{2}$$

$$= (z^n)_{-m} \cdot \log az + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1)\cdots(m+k-1)}{k!} (z^n)_{-m-k} (\log az)_k \tag{3}$$

$$= e^{i\pi m} \frac{\Gamma(-n-m)}{\Gamma(-n)} z^{n+m} \cdot \log az + \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1)\cdots(m+k-1)}{k!}$$

$$\times e^{i\pi(k+m)} \frac{\Gamma(-n-m-k)}{\Gamma(-n)} z^{n+m+k} \cdot \left(-e^{-i\pi k} \Gamma(k) z^{-k}\right) \tag{4}$$

$$= \frac{1}{(n+1)(n+2)\cdots(n+m)} z^{n+m} \cdot \log az$$

$$- z^{n+m} \sum_{k=1}^{\infty} \frac{(-1)^k \cdot m(m+1)\cdots(m+k-1)}{k} \cdot \frac{e^{i\pi m}(-1)^{m+k}}{(n+1)(n+2)\cdots(n+m+k)} \tag{5}$$

$$= \frac{n!}{(n+m)!} z^{n+m} \log az - \frac{n!}{(m-1)!} z^{n+m} Q_{m,n} \quad , \tag{6}$$

that is,

$$(z^n \cdot \log az)_{-m} = \frac{n!}{(n+m)!} z^{n+m} \log az - \frac{n!}{(m-1)!} z^{n+m} Q_{m,n} \quad , \tag{7}$$

where

$$Q_{m,n} = \sum_{k=1}^{\infty} \frac{1}{k(k+m)(k+m+1)\cdots(k+m+n)} = \sum_{k=1}^{\infty} \frac{(m+k-1)!}{(k+m+n)!\,k} \quad . \tag{8}$$

Therefore, operating N-fractional calculus operator N^m to the both sides of (7) (making m th order derivative of both sides) we obtain

$$((z^n \cdot \log az)_{-m})_m = \frac{n!}{(n+m)!} (z^{n+m} \log az)_m - \frac{n!}{(m-1)!} (z^{n+m})_m Q_{m,n} \tag{9}$$

hence

$$z^n \cdot \log az$$

$$= z^n \log az + \frac{n!}{(n+m)!} \left\{ (n+1)(n+2)\cdots(n+m) \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+m} \right) \right\} z^n$$

$$- \frac{(n+m)!}{(m-1)!} z^n Q_{m,n} \tag{10}$$

under the conditions.

From (10) we have

$$Q_{m,n}=\frac{(m-1)!\cdot n!}{\{(n+m)!\}^2}\left\{\prod_{k=1}^{m}(n+k)\right\}\left(\sum_{k=1}^{m}\frac{1}{n+k}\right)=\frac{(m-1)!}{(n+m)!}\sum_{k=1}^{m}\frac{1}{n+k}\ . \qquad (11)$$

Therefore, we have (1) from (11) clearly, under the conditions.

Note 1. We calculate as , for example,

$$\frac{\Gamma(-1)}{\Gamma(-2)}=\lim_{\alpha\to 0}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-2)}=\lim_{\alpha\to 0}\frac{\Gamma(\alpha-2+1)}{\Gamma(\alpha-2)}=\lim_{\alpha\to 0}\frac{(\alpha-2)\Gamma(\alpha-2)}{\Gamma(\alpha-2)}=-2. \qquad (12)$$

However we calculate this as

$$\Gamma(-1)/\Gamma(-2)=\Gamma(1-2)/\Gamma(-2)=(-2)\Gamma(-2)/\Gamma(-2)=-2\ , \qquad (13)$$

for our convenience, using the relationship $\Gamma(z+1)=z\Gamma(z)$ for Gamma function.

§ 2. Doubly Infinite Sums

Theorem 2. *Let*

$$A(k,m,n)=\frac{\left[(m-1)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)\,(k+m+1)\cdots(k+m+n)}\ . \qquad (1)$$

We have then

(i) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}A(k,m,n)=e-\sum_{k=0}^{m-1}\frac{1}{k!} \qquad (2)$$

(ii) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}A(k,m,n)=e-\sum_{k=0}^{n}\frac{1}{k!} \qquad (3)$$

and

(iii) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}A(k,p+1,n)=\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}A(k,m,p) \qquad (4)$$

where $m\in\mathbf{Z}^+$, $n\in\mathbf{Z}_0^+\ (=\mathbf{Z}^+\cup\{0\})$, $p\in\mathbf{Z}_0^+$ *and* $e=2.71828\cdots$.

Proof of (i). Now we have

$$\sum_{k=1}^{\infty}\frac{\left[(m-1)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)\,(k+m+1)\cdots(k+m+n)}=\frac{1}{(m+n)!} \qquad (5)$$

from Theorem 1.

We have then

$$\sum_{k=1}^{\infty}A(k,m,n)=\frac{1}{(m+n)!} \qquad (6)$$

from (5) and (1).

Therefore, we have

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} A(k,m,n) = \sum_{n=0}^{\infty}\frac{1}{(m+n)!} \quad . \tag{7}$$

We obtain (2) from (7) under the conditions, since

$$e = \sum_{k=0}^{m-1}\frac{1}{k!} + \sum_{k=m}^{\infty}\frac{1}{k!} \quad . \tag{8}$$

Proof of (i i). We have

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} A(k,m,n) = \sum_{m=1}^{\infty}\frac{1}{(m+n)!} \quad . \tag{9}$$

Therefore, we obtain (3) under the conditions from (9), since

$$e = \sum_{k=0}^{n}\frac{1}{k!} + \sum_{k=n+1}^{\infty}\frac{1}{k!} \quad . \tag{10}$$

Proof of (i i i). We have

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} A(k,p+1,n) = e - \sum_{k=0}^{p}\frac{1}{k!} \tag{11}$$

and

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} A(k,m,p) = e - \sum_{k=0}^{p}\frac{1}{k!} \tag{12}$$

from (2) and (3) respectively.

Therefore, we obtain (4) from (11) and (12), under the conditions.

Theorem 3. *Let*

$$B(k,m,n) = \frac{\left[(m-1)!\cdot(m+n)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)} \quad . \tag{13}$$

We have then

(i)
$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} B(k,m,n) = \sum_{k=m}^{\infty}\frac{1}{(k!)^2} \tag{14}$$

(i i)
$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B(k,m,n) = \sum_{k=n}^{\infty}\frac{1}{\{(k+1)!\}^2} \tag{15}$$

and

(i i i)
$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} B(k,p+1,n) = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B(k,m,p) \tag{16}$$

where $m \in \mathbf{Z}^+$, $n \in \mathbf{Z}_0^+ (= \mathbf{Z}^+ \cup \{0\})$ *and* $p \in \mathbf{Z}_0^+$.

Proof of (i) and (i i). Now we have

$$\sum_{k=1}^{\infty}\frac{\left[(m-1)!\cdot(m+n)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)} = \frac{1}{\{(m+n)!\}^2} \tag{17}$$

from Theorem 1.

We have then

$$\sum_{k=1}^{\infty} B(k,m,n) = \frac{1}{\{(m+n)!\}^2} \tag{18}$$

from (13) and (17).

Therefore, we have

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} B(k,m,n) = \sum_{n=0}^{\infty} \frac{1}{\{(m+n)!\}^2} \tag{19}$$

$$= \sum_{k=m}^{\infty} \frac{1}{(k!)^2} \tag{14}$$

and

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B(k,m,n) = \sum_{m=1}^{\infty} \frac{1}{\{(m+n)!\}^2} \tag{20}$$

$$= \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \tag{15}$$

from (18) respectively.

Proof of (i i i). We have

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} B(k,p+1,n) = \sum_{k=p+1}^{\infty} \frac{1}{(k!)^2} \tag{21}$$

and

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B(k,m,p) = \sum_{k=p}^{\infty} \frac{1}{\{(k+1)!\}^2} \tag{22}$$

from (14) and (15) respectively.

Therefore, we obtain (16) from (21) and (22), under the conditions.

§ 3. A Triply Infinite Sums

Theorem 4. *We have the triple infinite sum*

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{\left[(m-1)!\cdot(n+m)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)} = 1.59063\cdots , \tag{1}$$

where $m \in Z^{+}$ *and* $n \in Z_0^{+}$.

Proof. We have

$$\sum_{k=1}^{\infty} B(k,m,n) = \frac{1}{\{(m+n)!\}^2} \tag{2}$$

then

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} B(k,m,n) = \sum_{m=1}^{\infty} \frac{1}{\{(m+n)!\}^2} \tag{3}$$

$$= \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \quad , \tag{4}$$

where $B(k,m,n)$ is the one shown by **§ 1.** (13).

Therefore, we have

$$\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} B(k,m,n) = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{1}{\{(k+1)!\}^2} \tag{5}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{\{(n+1)!\}^2} + \frac{1}{\{(n+2)!\}^2} + \frac{1}{\{(n+3)!\}^2} + \cdots \right\} \tag{6}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\{(n+1)!\}^2} + \sum_{n=0}^{\infty} \frac{1}{\{(n+2)!\}^2} + \sum_{n=0}^{\infty} \frac{1}{\{(n+3)!\}^2} + \cdots \tag{7}$$

$$= 1.279584\cdots + 0.279584\cdots + 0.029584\cdots$$

$$+0.001806\cdots + 0.000070\cdots + 0.0000011\cdots$$

$$+ 0.0000000\cdots + \cdots \tag{8}$$

$$= 1.59063\cdots \tag{1}$$

from (4).

§ 4. Illustrative Examples of Theorems 1 , 2 and 3

(I) Examples of Theorem 1.

We obtain

$$Q_{1,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \qquad \text{([17] p. 656, [18] p. 42.)} \tag{1}$$

$$Q_{1,1} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} = \frac{1}{2!2} \qquad \text{([17] p. 666, [18] p. 43.)} \tag{2}$$

$$Q_{1,2} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{3!3} \qquad \text{([17] p. 675, [18] p. 43.)} \tag{3}$$

$$Q_{1,3} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4!4} \tag{4}$$

$$Q_{1,4} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)(k+4)(k+5)} = \frac{1}{5!5} = \frac{1}{600} \tag{5}$$

for $m = 1$,

$$Q_{2,0}=\sum_{k=1}^{\infty}\frac{1}{k(k+2)}=\frac{3}{4} \qquad \text{([17] p. 656, [18] p. 45.)} \qquad (6)$$

$$Q_{2,1}=\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+3)}=\frac{5}{(3!)^2}=\frac{5}{36} \qquad \text{([17] p. 666.)} \qquad (7)$$

$$Q_{2,2}=\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+3)(k+4)}=\frac{2!7}{(4!)^2}=\frac{7}{288} \qquad (8)$$

$$Q_{2,3}=\sum_{k=1}^{\infty}\frac{1}{k(k+2)(k+3)(k+4)(k+5)}=\frac{3!9}{(5!)^2}=\frac{3}{800} \qquad (9)$$

for $m=2$,

$$Q_{3,0}=\sum_{k=1}^{\infty}\frac{1}{k(k+3)}=\frac{11}{18} \qquad \text{([17] p. 656.)} \qquad (10)$$

$$Q_{3,1}=\sum_{k=1}^{\infty}\frac{1}{k(k+3)(k+4)}=\frac{2\cdot 26}{(4!)^2}=\frac{13}{144} \qquad (11)$$

$$Q_{3,2}=\sum_{k=1}^{\infty}\frac{1}{k(k+3)(k+4)(k+5)}=\frac{2!2\cdot 47}{(5!)^2}=\frac{47}{3600} \qquad (12)$$

$$Q_{3,3}=\sum_{k=1}^{\infty}\frac{1}{k(k+3)(k+4)(k+5)(k+6)}=\frac{3!2\cdot 74}{(6!)^2}=\frac{37}{21600} \qquad (13)$$

for $m=3$,

$$Q_{4,0}=\sum_{k=1}^{\infty}\frac{1}{k(k+4)}=\frac{25}{48} \qquad \text{([17] p. 656.)} \qquad (14)$$

$$Q_{4,1}=\sum_{k=1}^{\infty}\frac{1}{k(k+4)(k+5)}=\frac{3!\cdot 154}{(5!)^2}=\frac{77}{1200} \qquad (15)$$

$$Q_{4,2}=\sum_{k=1}^{\infty}\frac{1}{k(k+4)(k+5)(k+6)}=\frac{2!\cdot 3!\cdot 342}{(6!)^2}=\frac{19}{2400} \qquad (16)$$

$$Q_{4,3}=\sum_{k=1}^{\infty}\frac{1}{k(k+4)(k+5)(k+6)(k+7)}=\frac{(3!)^2 638}{(7!)^2}=\frac{319}{352800} \qquad (17)$$

for $m=4$,

$$Q_{5,0}=\sum_{k=1}^{\infty}\frac{1}{k(k+5)}=\frac{137}{300} \qquad (18)$$

$$Q_{5,1}=\sum_{k=1}^{\infty}\frac{1}{k(k+5)(k+6)}=\frac{4!\cdot 1044}{(6!)^2}=\frac{29}{600} \qquad (19)$$

for $m = 5$, and

$$Q_{m,0} = \sum_{k=1}^{\infty} \frac{1}{k(k+m)} = \frac{1}{m}\sum_{k=1}^{m}\frac{1}{k} \qquad (\,[17]\ \text{p.}\,656\,). \qquad (20)$$

(II) Examples of Theorem 2. (i);

1)
$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)} = 1.71828\cdots . \qquad (21)$$

(Set $m = 1$ in § 2. (2).)

2)
$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(n+1)(n+2)\cdot(2n+3)^{-1}}{k(k+2)(k+3)\cdots(k+2+n)} = 0.71828\cdots . \qquad (22)$$

(Set $m = 2$ in § 2. (2).)

3)
$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{[2\sum_{k=1}^{3}(n+k)^{-1}]^{-1}}{k(k+3)(k+4)\cdots(k+3+n)} = 0.21828\cdots . \qquad (23)$$

(Set $m = 3$ in § 2. (2).)

Note. Indeed, having $m = 1$ we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1 \quad (\text{from } Q_{1,0}) \qquad ([17]\,\text{p.}\,656,\ [18]\,\text{p.}\,42.) \qquad (24)$$

$$\sum_{k=1}^{\infty} \frac{2}{k(k+1)(k+2)} = \frac{1}{2!} \quad (\text{from } Q_{1,1}) \qquad ([17]\,\text{p.}\,666,\ [18]\,\text{p.}\,43.) \qquad (25)$$

$$\sum_{k=1}^{\infty} \frac{3}{k(k+1)(k+2)(k+3)} = \frac{1}{3!} \quad (\text{from } Q_{1,2}) \qquad ([17]\,\text{p.}\,675,\ [18]\,\text{p.}\,43.) \qquad (26)$$

$$\sum_{k=1}^{\infty} \frac{4}{k(k+1)(k+2)(k+3)(k+4)} = \frac{1}{4!} \quad (\text{from } Q_{1,3}) \qquad (27)$$

.... (28)

$$\sum_{k=1}^{\infty} \frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)} = \frac{1}{(n+1)!} \quad (\text{from } Q_{1,n}) \qquad (29)$$

.... (30)

from Theorem 1 in § 1.

Therefore, by the summing up from (24) to (30) we obtain

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{n+1}{k(k+1)(k+2)\cdots(k+1+n)}=\sum_{n=0}^{\infty}\frac{1}{(n+1)!} \tag{31}$$

$$=e-1=1.71828\cdots, \tag{32}$$

clearly, for example.

(III) Examples of Theorem 2. (ii);

1) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot(\sum_{k=1}^{m}k^{-1})]^{-1}}{k(k+m)}=1.71828\cdots. \tag{33}$$

(Set $n=0$ in **§ 2.** (3).)

2) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot(\sum_{k=1}^{m}(1+k)^{-1})]^{-1}}{k(k+m)(k+m+1)}=0.71828\cdots \tag{34}$$

(Set $n=1$ in **§ 2.** (3).)

3) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot(\sum_{k=1}^{m}k^{-1})]^{-1}}{k(k+m)}=0.21828\cdots \tag{35}$$

(Set $n=2$ in **§ 2.** (3).)

(IV) Examples of Theorem 3 (i);

1) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{[n!]^{-1}}{k(k+1)(k+2)\cdots(k+1+n)}=1.279584\cdots. \tag{36}$$

(Set $m=1$ in **§ 2.** (14).)

2) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{[n!\cdot(2n+3)]^{-1}}{k(k+2)(k+3)\cdots(k+2+n)}=0.279584\cdots. \tag{37}$$

(Set $m=2$ in **§ 2.** (14).)

3) $$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{[n!\cdot 2(3n^2+12n+11)]^{-1}}{k(k+3)(k+4)\cdots(k+3+n)}=0.029584\cdots. \tag{38}$$

(Set $m=3$ in **§ 2.** (14).)

(V) Examples of Theorem 3. (ii);

1) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot m!(\sum_{k=1}^{m}k^{-1})]^{-1}}{k(k+m)}=1.279584\cdots. \tag{39}$$

(Set $n=0$ in **§ 2.** (15).)

2) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot(1+m)!\,(\sum_{k=1}^{m}(1+k)^{-1})]^{-1}}{k(k+m)(k+m+1)}=0.279584\cdots . \qquad (40)$$

(Set $n=1$ in § **2.** (15).)

3) $$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{[(m-1)!\cdot(2+m)!\,(\sum_{k=1}^{m}(2+k)^{-1})]^{-1}}{k(k+m)(k+m+1)(k+m+2)}=0.21828\cdots . \qquad (41)$$

(Set $n=2$ in § **2.** (15).)

References

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$$Q_{m,n}=\sum_{k=1}^{\infty}\frac{1}{k(k+m)(k+m+1)\cdots(k+m+n)} \quad (n\in Z^{+}\cup\{0\},\ m\in Z^{+})$$

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$$\sum_{k=1}^{\infty}\frac{(n-1)!\cdot 2^{n-1}}{\prod_{k=0}^{n}(2k+3)} \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{(n-1)!\cdot(n+1)\cdot 2^{n-1}}{\prod_{k=0}^{n+1}(2k+3)}$$

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$$R_{m,\beta}=(-1)^{n}\sum_{k=1}^{\infty}\frac{(m+k-1)!\cdot(-1)^{k}}{(m-1)!k}\cdot\frac{\Gamma(-m-k-\beta)}{\Gamma(-\beta)}$$

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