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A generalization of a non-symmetric numerical semigroup generated by three elements

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§1. Non-symmetric numerical semigroups generated by three elements.

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. Let H be a *numerical semigroup of genus* g, i.e., a subsemigroup of $\mathbb{N}$ whose complement $\mathbb{N}\backslash H$ consists of g elements. We denote by $g(H)$ the genus of H. We set

$$c(H) = \mathrm{Min}\{c \in \mathbb{N} | c + \mathbb{N} \subset H\},$$

which is called the *conductor* of H. Then $c(H) \leq 2g(H)$. A numerical semigroup H is said to be *symmetric* if $c(H) = 2g(H)$. Let $M(H) = \{a_1, a_2, \dots, a_n\}$ be the minimal set of generators for H. We set

$$\alpha_i = \mathrm{Min}\{\alpha | \alpha a_i \in < a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n >\}$$

where for any non-negative integers $b_1, \dots, b_m$ the set $< b_1, \dots, b_m >$ means the semigroup generated by $b_1, \dots, b_m$.

Example 1. i) Let $H =< 4, 5, 6 >$. Then $g(H) = 4$ and $c(H) = 8$. Hence H is symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 6$, then $\alpha_1 = 3$, $\alpha_2 = 2$ and $\alpha_3 = 2$.
ii) Let $H =< 4, 5, 7 >$. Then $g(H) = 4$ and $c(H) = 7$. Hence H is non-symmetric. If we set $a_1 = 4$, $a_2 = 5$ and $a_3 = 7$, then $\alpha_1 = 3$, $\alpha_2 = 3$ and $\alpha_3 = 2$.

Remark 2 (Herzog [1]). Let H be a non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. Then

$$\alpha_1 a_1 = \alpha_{12} a_2 + \alpha_{13} a_3,\ \alpha_2 a_2 = \alpha_{21} a_1 + \alpha_{23} a_3 \text{ and } \alpha_3 a_3 = \alpha_{31} a_1 + \alpha_{32} a_2$$

where $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{32}$, $\alpha_3 = \alpha_{13} + \alpha_{23}$ and $0 < \alpha_{ij} < \alpha_j$, all i, j. In this case α_{ij}'s are uniquely determined.

Proposition 3. *Let the notation be as in Remark 2. Then we have* $\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = a_3$.

Example 4. Let $H =< a_1 = 4, a_2 = 5, a_3 = 7 >$. Then

$$3a_1 = a_2 + a_3,\ 3a_2 = 2a_1 + a_3,\ 2a_3 = a_1 + 2a_2$$

and

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} \\ -\alpha_{21} & \alpha_2 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = 7 = a_3.$$

§2. Numerical semigroups of quasi-toric type.

In this section we introduce the notion of a numerical semigroup of quasi-toric type and give several examples.

Definition 5. i) Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. A system of relations

$$\begin{cases} \alpha_1 a_1 = \alpha_{12} a_2 + \cdots + \alpha_{1n} a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{n1} a_1 + \cdots + \alpha_{nn-1} a_{n-1} \end{cases}$$

satisfying

$$\alpha_j = \alpha_{1j} + \cdots + \alpha_{j-1j} + \alpha_{j+1j} + \cdots + \alpha_{nj}$$

for any j and $0 \le \alpha_{ij} < \alpha_j$ for all i, j is said to be *neat.*
ii) A numerical semigroup H is said to be *neat* if it has a neat system of relations.

Example 6. i) Any non-symmetric numerical semigroup H with $\sharp M(H) = 3$ is neat by Remark 2.
ii) Let $H =< a_1 = 20, a_2 = 24, a_3 = 25, a_4 = 31 >$. We have a unique neat system of relations

$$4a_1 = a_2 + a_3 + a_4,\ 4a_2 = 2a_1 + a_3 + a_4,\ 3a_3 = a_1 + a_2 + a_4 \text{ and } 3a_4 = a_1 + 2a_2 + a_3,$$

which implies that H is neat.

Definition 7. Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. The $\mathbb{Z}-$module

$$R = \{(r_1, r_2, \dots, r_n) \in \mathbb{Z}^n | \sum_{i=1}^{m} r_i a_i = 0\}$$

is called a *relation module for H.*

Lemma 8. *Let the notation be as in Definition 7. Then a relation module for H is a free $\mathbb{Z}-$module of rank $n-1$.*

Definition 9. Let S be a subsemigroup of $\mathbb{Z}^n$. S is said to be *saturated* if the condition $nr \in S$, where n is a positive integer and r an element of $\mathbb{Z}^n$, implies that $r \in S$.

Definition 10. Consider an order on the set

$$I = \{(i, j) | 1 \le i \le n, 1 \le j \le n, i \ne j\},$$

which is fixed. Let H be a neat numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$.

Take a neat system of relations

$$\begin{cases} \alpha_{21}a_1 + \cdots + \alpha_{n1}a_1 = \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{1n-1}a_{n-1} + \cdots + \alpha_{n-2n-1}a_{n-1} + \alpha_{nn-1}a_{n-1} = \alpha_{n-1}a_{n-1} \\ = \alpha_{n-11}a_1 + \cdots + \alpha_{n-1n-2}a_{n-2} + \alpha_{n-1n}a_n \\ \alpha_{1n}a_n + \cdots + \alpha_{n-1n}a_n = \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

Assume that

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0.$$

We set $N = \sharp\{(i,j)|\alpha_{ij} \neq 0\} - (n-1)$. We associate the vector in $\mathbb{Z}^N$ with $\alpha_{ij}a_j$ by induction on i which means the i−th relation in the neat system of relations. Let i be fixed. Let (k_i, l_i) be the maximum of the set

$$L_i = \Big(\{(j,i)|\alpha_{ji} \neq 0\} \cup \{(i,j)|\alpha_{ij} \neq 0\}\Big) \cap \Big(I \setminus \cup_{p=1}^{i-1} L_p\Big).$$

We number successively the elements (i,j) of the set L_i by $\sigma(i,j)$ in the given order if $(i,j) \neq (k_i, l_i)$. We associate the vector $b_{\sigma(i,j)} = e_{\sigma(i,j)}$ with $\alpha_{ij}a_j$ if $(i,j) \neq (k_i, l_i)$. For $\alpha_{k_i l_i}a_{l_i}$ we consider

$$\alpha_{k_i l_i}a_{l_i} = \cdots\cdots \pm \alpha_{pq}a_p \cdots\cdots$$

from the i− th relation in the neat system. Using the relation we can associate the vector b_{N+i} with $\alpha_{k_i l_i}a_{l_i}$, because we already have associated some vector with $\alpha_{pq}a_p$. Thus, we can construct the subsemigroup $S = < b_1, \ldots, b_{N+n-1} >$ of $\mathbb{Z}^N$. The neat numerical semigroup H is said to be of *quasi-toric type* if the semigroup S is saturated.

The reason why H is said to be of quasi-toric type is that if the associated semigroup S is saturated then the affine scheme Spec $k[S]$ becomes an affine toric variety where k is an algebraically closed field.

Remark 11. Let a neat system of relations be fixed. Then the property of "quasi-toric type" does not depend on the choices of the numbering of the elements of $M(H)$ and the order on the set

$$I = \{(i,j)|1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}.$$

Example 12. Let H be a non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$. We have the neat system of relations

$$\begin{cases} \alpha_1 a_1 = (\alpha_{21} + \alpha_{31})a_1 = \alpha_{12}a_2 + \alpha_{13}a_3 \\ \alpha_2 a_2 = (\alpha_{12} + \alpha_{32})a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 \\ \alpha_3 a_3 = (\alpha_{13} + \alpha_{23})a_3 = \alpha_{31}a_1 + \alpha_{32}a_2 \end{cases}$$

We define the order on the set

$$\{(i,j)|i,j=1,2,3 \text{ and } i\neq j\}$$

as follows: $(i,j)\leq(i',j')$ if "$j<j'$" or "$j=j'$, $i\leq i'$". The associated subsemigroup S of $\mathbb{Z}^4$ is generated by $b_1=e_1$, $b_2=e_2$, $b_3=e_3$, $b_4=e_4$, $b_5=(1,1,-1,0)$ and $b_6=(-1,0,1,1)$. Then S is saturated, which implies that H is of quasi-toric type.

Proposition 13. *Let H be a neat numerical semigroup with $M(H)=\{a_1,\dots,a_n\}$ such that it has a neat system of relations*

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2+\cdots+\alpha_{1n}a_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{n1}a_1+\cdots+\alpha_{nn-1}a_{n-1} \end{cases}$$

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0.$$

i) *If there is some j_0 with $1\leq j_0\leq n$ such that $\alpha_{ij_0}>0$ for all i, then H is of quasi-toric type.*
ii) *If there is some i_0 with $1\leq i_0\leq n$ such that $\alpha_{i_0j}>0$ for all j, then H is of quasi-toric type.*

Example 14. Let $H=<a_1=20, a_2=24, a_3=25, a_4=31>$. We have a neat system of relations

$$4a_1=a_2+a_3+a_4,\ 4a_2=2a_1+a_3+a_4,$$

$$3a_3=a_1+a_2+a_4 \text{ and } 3a_4=a_1+2a_2+a_3.$$

Since we have

$$\begin{vmatrix} 4 & -1 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 31 \neq 0,$$

by Proposition 13 H is of quasi-toric type (Cf. Example 6 ii)).

Proposition 15. *Let H be a neat numerical semigroup with $M(H)=\{a_1,\dots,a_n\}$ such that it has a neat system of relations*

$$\begin{cases} \alpha_1 a_1 = \alpha_{1n}a_n+\alpha_{12}a_2 \\ \alpha_2 a_2 = \alpha_{21}a_1+\alpha_{23}a_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_i a_i = \alpha_{ii-1}a_{i-1}+\alpha_{ii+1}a_{i+1} \quad (2\leq i\leq n-1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_n a_n = \alpha_{nn-1}a_{n-1}+\alpha_{n1}a_1 \end{cases}$$

with

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} \neq 0$$

where we set $\alpha_{ij} = 0$ if $\alpha_{ij}a_j$ does not appear in the system of relations. Then H is of quasi-toric type.

Example 16 (Komeda [3]). For any $n \geq 5$, let H_n be the numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \dots, a_{n-1} = 2n+n-1\}.$$

Then we have a neat system of relations

$$\alpha_1 a_1 = 4a_1 = a_2 + a_{n-1},\ \alpha_2 a_2 = 3a_2 = a_1 + a_3,\ \alpha_3 a_3 = 2a_3 = 2a_2 + a_4,$$

$$\alpha_i a_i = 2a_i = a_{i-1} + a_{i+1}\ (4 \leq i \leq n-2),\ \alpha_{n-1}a_{n-1} = 2a_{n-1} = 3a_1 + a_{n-2}.$$

By Proposition 15, H_n is a neat numerical semigroup of quasi-toric type.

Theorem 17. *Let H be a neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$. Then H is of quasi-toric type.*

Proof. Let

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2 + \alpha_{13}a_3 + \alpha_{14}a_4 \\ \alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 + \alpha_{24}a_4 \\ \alpha_3 a_3 = \alpha_{31}a_1 + \alpha_{32}a_2 + \alpha_{34}a_4 \\ \alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{42}a_2 + \alpha_{43}a_3 \end{cases}$$

be a unique neat system of relations for H. We note that

$$D = \begin{vmatrix} \alpha_1 & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_2 & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_3 \end{vmatrix} > 0.$$

By Proposition 13 first we may assume that $\alpha_1 a_1 = \alpha_{12}a_2 + \alpha_{14}a_4$, which implies that $\alpha_3 = \alpha_{23} + \alpha_{43}$. Moreover, we have

$$\text{"}\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3 \text{ or } \alpha_{23}a_3 + \alpha_{24}a_4\text{"}$$

and

$$\text{"}\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3 \text{ or } \alpha_{42}a_2 + \alpha_{43}a_3\text{"}.$$

i) $\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3$, $\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3$. Then $\alpha_3 a_3 = \alpha_{32}a_2 + \alpha_{34}a_4$. This case is reduced to Proposition 15.

ii) $\alpha_2 a_2 = \alpha_{21}a_1 + \alpha_{23}a_3$, $\alpha_4 a_4 = \alpha_{42}a_2 + \alpha_{43}a_3$. Then $\alpha_3 a_3 = \alpha_{31}a_1 + \alpha_{34}a_4$. Hence we get $\alpha_1 = \alpha_{21} + \alpha_{31}$, $\alpha_2 = \alpha_{12} + \alpha_{42}$, $\alpha_3 = \alpha_{23} + \alpha_{43}$ and $\alpha_4 = \alpha_{14} + \alpha_{34}$. We introduce the order on the set

$$I = \{(i,j)|1 \le i \le 4, 1 \le j \le 4, i \ne j\}$$

as follows:
$(i,j) \le (i',j')$ if "$j < j'$" or "$j = j'$, $i \le i'$". Then we get the associated subsemigroup

$$S =< b_1 = e_1, \dots, b_5 = e_5, b_6, b_7, b_8 >$$

of $\mathbb{Z}^5$ through the method in Definition 10 where $b_6 = e_1 + e_2 - e_3$, $b_7 = e_1 + e_4 - e_3$ and $b_8 = e_2 + e_5 - e_4$. We can show that S is saturated.

iii) $\alpha_2 a_2 = \alpha_{23}a_3 + \alpha_{24}a_4$, $\alpha_4 a_4 = \alpha_{41}a_1 + \alpha_{43}a_3$. In the similar way to ii) we can show that H is of quasi-toric type.

iv) $\alpha_2 a_2 = \alpha_{23}a_3 + \alpha_{24}a_4$, $\alpha_4 a_4 = \alpha_{42}a_2 + \alpha_{43}a_3$. In this case at most one α_{i1} appears. This contradicts the neatness of H. *Q.E.D.*

Problem 18. Let $n \ge 5$. If H is a neat numerical semigroup with $\sharp M(H) = n$, then is it of quasi-toric type ?

§3. Numerical semigroups of toric type.

We study the relation between a 1-neat numerical semigroup and a numerical semigroup of toric type whose definitions are given in this section.

Definition 19. Let H be a neat numerical with $M(H) = \{a_1, \dots, a_n\}$ such that it has a neat system of relations

$$\begin{cases} \alpha_1 a_1 = \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n. \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \alpha_n a_n = \alpha_{n1}a_1 + \cdots + \alpha_{nn-1}a_{n-1} \end{cases}$$

It is said to be *1-neat* if

$$\begin{vmatrix} \alpha_1 & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ -\alpha_{21} & \alpha_2 & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & & \vdots \\ -\alpha_{n-11} & -\alpha_{n-12} & \cdots & \alpha_{n-1} \end{vmatrix} = a_n.$$

Example 20. By Proposition 3 a non-symmetric numerical semigroup H with $M(H) = \{a_1, a_2, a_3\}$ is 1-neat.

Proposition 21. *Let H be a numerical semigroup with $M(H) = \{a_1, \dots, a_n\}$. Let $\mathbf{r}_i = (r_{i1}, r_{i2}, \dots, r_{in})$ be an element of the relation module R for H with*

$i = 1, \ldots, n-1$. *Assume that*

$$\begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n-1} \\ r_{21} & r_{22} & \cdots & r_{2n-1} \\ \vdots & \vdots & & \vdots \\ r_{n-11} & r_{n-12} & \cdots & r_{n-1n-1} \end{vmatrix} = \pm a_n.$$

Then $\mathbf{r}_1, \ldots, \mathbf{r}_{n-1}$ *form a basis for the* $\mathbb{Z}$*-module* R.

Let H be a neat numerical semigroup with $M(H) = \{a_1, \ldots, a_n\}$ with its fixed neat system of relations. We set $N = \sharp\{(i,j)|\alpha_{ij} \neq 0\} - (n-1)$. Let $S = < b_1, \ldots, b_{N+n-1} >$ of $\mathbb{Z}^N$ be the associated subsemigroup. Let k be a field. Let $\varphi_H : k[X] = k[X_1, \ldots, X_n] \longrightarrow k[H] = k[t^h]_{h \in H}$ be a k-algebra homomorphism sending X_i to t^{a_i}, $\pi : k[Y] = k[Y_1, \ldots, Y_{N+n-1}] \longrightarrow k[S] = k[T^b]_{b \in S}$ a k-algebra homomorphism sending Y_l to T^{b_l}, $\eta : k[Y] \longrightarrow k[X]$ a k-algebra homomorphism sending Y_l to $g_l = X_j^{\alpha_{ij}}$ if b_l corresponds to $\alpha_{ij} a_j$ and $\zeta : k[\mathbb{N}^N] = k[t_1, \ldots, k_N] \longrightarrow k[H]$ a k-algebra homomorphism sending t_i to $t^{w(g_i)}$ where the weight w on $k[X]$ is defined by $w(X_i) = a_i$ and $w(c) = 0$ for $c \in k^\times$. By the definition of b_l's, ζ extends to $\zeta' : k[S] \longrightarrow k[H]$. Then we get $\varphi_H \circ \eta = \zeta' \circ \pi$, which implies that Ker $\varphi_H \supseteq \eta(\text{Ker } \pi)$.

Definition 22. A neat numerical semigroup H is said to be *of toric type* if it is of quasi-toric type and we have an isomorphism $k[H] \cong k[S] \otimes_{k[Y]} k[X]$, that is to say, Ker $\varphi_H = (\eta(\text{Ker } \pi))$.

Remark 23 (Komeda [2]). A numerical semigroup of toric type is Weierstrass, where a numerical semigroup H is said to be *Weierstrass* if there is a pointed non-singular complete curve (C, P) over an algebraically closed field such that

$$H = \{n \in \mathbb{N} | \text{ there is a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

Example 24. Any non-symmetric numerical semigroup with $M(H) = \{a_1, a_2, a_3\}$ is of toric type, because we know that the ideal Ker φ_H is generated by

$$X_1^{\alpha_1} - X_2^{\alpha_{12}} X_3^{\alpha_{13}},\ X_2^{\alpha_2} - X_1^{\alpha_{21}} X_3^{\alpha_{23}} \text{ and } X_3^{\alpha_3} - X_1^{\alpha_{31}} X_2^{\alpha_{32}} \text{ (Herzog [1]).}$$

We note that H is 1-neat (Cf. Example 20).

Example 25. For any integer $n \geq 5$, let H_n be a numerical semigroup with

$$M(H_n) = \{a_1 = n, a_2 = n+1, a_3 = 2n+3, a_4 = 2n+4, \ldots, a_{n-1} = 2n+n-1\}$$

(Cf. Example 16). Then the ideal Ker φ_{H_n} is generated by

$$X_2^3 - X_1 X_3,\ X_2 X_j - X_1 X_{j+1} (3 \leq j \leq n-2),\ X_2 X_{n-1} - X_1^4,$$

$$X_3X_j - X_2^2X_{j+1}(3 \leq j \leq n-2),\ X_3X_{n-1} - X_2^2X_1^3,$$

$$X_iX_j - X_{i-1}X_{j+1}(4 \leq i \leq n-2,\ i \leq j \leq n-2),\ X_iX_{n-1} - X_{i-1}X_1^3(4 \leq i \leq n-1).$$

It is proved that H_n is of toric type. In this case H_n is also 1-neat.

Theorem 26. *A 1-neat numerical semigroup with $M(H) = \{a_1, a_2, a_3, a_4\}$ is of toric type.*

Problem 27. Let $n \geq 5$. If H is a 1-neat numerical semigroup with $\sharp M(H) = n$, is it of toric type ?

References

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