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Reconstruction of Gaifman's characterization of Mahlo cardinals

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Introduction: This paper is an abstract of [Ta∞], in which we developed "proof-theoretically Mahlo ordinals" which are primitive recursive analogues of (weakly) Mahlo cardinals. In particular, we here concentrate our attention to explain denotation of proof-theoretically Mahlo ordinals, by constructing a primitive recursive analogue of Gaifman's characterization of Mahlo cardinals.

For detail of the content including background of this work and proofs in this paper, see [Ta∞].

We shall define primitive recursive analogues of certain regular cardinals up to the least weakly Mahlo cardinal, which were introduced in [Ta∞] (see also [Ta02]). For the definitions, we need certain Skolem hulls and collapsing functions which were essentially defined by Rathjen (cf. [Ra98] and [Ra99]).

By $+$ we denote ordinary (noncommutative) ordinal addition. An ordinal α is called an *additive principal number* if $\forall\ \xi,\zeta < \alpha\ (\xi + \zeta < \alpha)$. We also let φ denote the Veblen function, which is defined by: for any ordinals α,β, $\varphi\alpha\beta$ is the β^{th} additive principal number γ such that $\forall\xi < \alpha(\varphi\xi\gamma = \gamma)$. Note that $\varphi 0\alpha$ is often denoted by ω^α and $\varphi 1\alpha$ by ε_α. We also let ω denote the least infinite ordinal, Ω the least uncountable ordinal, and M the least weakly Mahlo cardinal.

Definition 0.1 ([Ra98]: Def.3.4) For each ordinal α and β, we define $C^M(\alpha,\ \beta)$ as well as the two functions χ^α and ψ^α by recursion on α:
(M1) $\beta\cup\{0,M\}\subset C^M(\alpha,\beta)$.
(M2) $\gamma=\gamma_1+\gamma_2\ \&\ \gamma_1,\gamma_2\in C^M(\alpha,\beta)\ \Rightarrow\ \gamma\in C^M(\alpha,\beta)$.
(M3) $\gamma=\varphi\gamma_1\gamma_2\ \&\ \gamma_1,\gamma_2\in C^M(\alpha,\beta)\ \Rightarrow\ \gamma\in C^M(\alpha,\beta)$.
(M4) $\gamma=\chi^\xi(\delta)\ \&\ \xi,\delta\in C^M(\alpha,\beta)\ \&\ \xi<\alpha\ \&\ \xi\in C^M(\xi,\gamma)\ \Rightarrow\ \gamma\in C^M(\alpha,\beta)$.
(M5) $\gamma=\psi^\xi(\kappa)\ \&\ \xi,\kappa\in C^M(\alpha,\beta)\ \&\ \xi<\alpha\ \&\ \xi\in C^M(\xi,\gamma)\ \Rightarrow\ \gamma\in C^M(\alpha,\beta)$.
$\chi^\alpha(\delta)\simeq\delta^{th}$ regular cardinal $\pi<M$ s.t. $C^M(\alpha,\pi)\cap M=\pi$.
$\psi^\alpha(\kappa)\simeq\min\{\rho<\kappa:\ C^M(\alpha,\rho)\cap\kappa=\rho\wedge\kappa\in C^M(\alpha,\rho)\}$.

Here we only mention some properties of C^M or χ, which are proved in [Ra90].

Proposition 0.2 (1) For each $\alpha<M$, $\chi^0(\alpha)$ is the α^{th} regular cardinal.
(2) If $\kappa\in C^M(\alpha,\kappa)$, then $\psi^\alpha_M(\kappa)$ is defined and $\psi^\alpha_M(\kappa)<\kappa$.
(3) Each ordinal of the form $\psi^\alpha_M(\kappa)$ is strongly critical, but it's *not* a regular cardinal.
(4) For all α, χ^α is a total function on M.

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Definition 0.3 (1) $\gamma =_{\text{nf}} \alpha + \beta : \Leftrightarrow \ \gamma = \alpha + \beta$ and $\gamma > \alpha \geq \beta$ and α and β are additive principal numbers.
(2) $\gamma =_{\text{nf}} \varphi\alpha\beta : \Leftrightarrow \ \gamma = \varphi\alpha\beta \ \& \ \alpha, \beta < \gamma$.
(3) $\gamma =_{\text{nf}} \Omega_\sigma : \Leftrightarrow \ \gamma = \Omega_\sigma \ \& \ \sigma < \gamma$.
(4) $\gamma =_{\text{nf}} \chi^\alpha(\beta) :\Leftrightarrow \gamma = \chi^\alpha(\beta) \ \& \ \beta < \gamma \ \& \ \alpha \in C^M(\alpha, \gamma)$.
(5) $\gamma =_{\text{nf}} \psi^\alpha_M(\kappa) : \Leftrightarrow \ \gamma = \psi^\alpha_M(\kappa) \ \& \ \alpha \in C^M(\alpha, \gamma)$.

Definition 0.4 (EORS for the proof-theoretic ordinal of **KPM**) We define the set $\mathcal{T}(M)$ of ordinals, as follows:
(i) $0, M \in \mathcal{T}(M)$.
(ii) If $\gamma =_{\text{nf}} \alpha + \beta$ and $\alpha, \beta \in \mathcal{T}(M)$, then $\gamma \in \mathcal{T}(M)$.
(iii) If $\gamma =_{\text{nf}} \varphi\alpha\beta$ and $\alpha, \beta \in \mathcal{T}(M)$ and ($\gamma < I$ or $\alpha = 0$), then $\gamma \in \mathcal{T}(M)$.
(iv) If $\gamma =_{\text{nf}} \chi^\xi(\alpha)$ and $\xi, \alpha \in \mathcal{T}(M)$, then $\gamma \in \mathcal{T}(M)$.
(v) If $\gamma =_{\text{nf}} \psi^\alpha_M(\kappa)$ and $\kappa, \alpha \in \mathcal{T}(M)$, then $\gamma \in \mathcal{T}(M)$.

Lemma 0.5 (1) Every element of $\mathcal{T}(M)$ has a unique expression, that is, every element is represented uniquely by $+, \varphi, \chi$, and ψ_M in the definition above.
(2) $\mathcal{T}(M) = C^M(\varepsilon_{M+1}, 0) \cap \varepsilon_{M+1}$ and $\mathcal{T}(M) \cap \Omega_1 = \psi^{\varepsilon_{M+1}}_M(\Omega_1)$.

Theorem 0.6 ([Ra91], [Bu92]) $|\mathbf{KPM}| = \psi^{\varepsilon_{M+1}}_M(\Omega_1)$, where $|\mathbf{KPM}|$ means the proof-theoretic ordinal of **KPM**.

Now we define primitive recursive analogues of regular cardinals up to the least (weakly) Mahlo cardinal, by using the EORS above. Let **Reg** be the class of uncountable regular cardinal.

Definition 0.7 A primitive recursive ordinal α is called a *proof-theoretically regular ordinal based on* $\mathcal{T}(M)$ if $\alpha \in \mathcal{T}(M)$ and α is of the form $\psi^\kappa_M(\Omega_1)$ for some $\kappa \in \mathbf{Reg} \cap (M+1)$.

Let $\mathbf{Reg}(\mathcal{T}(M))$ be the set of proof-theoretically regular ordinals based on $\mathcal{T}(M)$. Remark that if $\psi^\kappa_M(\Omega_1) \in \mathbf{Reg}(\mathcal{T}(M))$, then κ is of the form M or $\chi^\xi(\gamma)$.

Definition 0.8 An ordinal α is said to be *proof-theoretically Mahlo based on* $\mathcal{T}(M)$ if α satisfies the following property:

$$\textbf{(P)} \quad \alpha \in \mathbf{Reg}(\mathcal{T}(M)) \text{ and } \alpha = \sup\{\psi^{\chi^\xi(\delta)}_M(\Omega_1) \in \mathcal{T}(M) : \ \xi < \varepsilon_{M+1} \ \& \ \delta < \pi\},$$

where π is the ordinal uniquely determined by $\alpha =_{\text{nf}} \psi^\pi_M(\Omega_1)$.

The main part of [Ta∞] is devoted to develop properties of the primitive recursive ordinals above, in particular, the following.

Theorem 0.9 ([Ta∞]) $\psi^M_M(\Omega_1)$ satisfies (**P**), that is, $\psi^M_M(\Omega_1)$ is a proof-theoretically Mahlo based on $\mathcal{T}(M)$.

In the rest of this paper, we concentrate our attention to explain the reason why the ordinals defined in Def.0.8 are primitive recursive analogues of (weakly) Mahlo cardinals. Preceding the explanation, we employ a property characterizing weakly Mahlo cardinals, which was obtained by Gaifman in [Ga67].

The definitions Def.0.10~ Def.0.12 below are essentially introduced in [Ga67].

Let **On** denote the proper class consisting of all ordinals, and $\mathcal{P}(\mathbf{On})$ the proper class consisting of all subclasses of **On**. For any $X \in \mathcal{P}(\mathbf{On})$, let c_X denote the function which assigns to an ordinal α the α^{th} element of X (we call c_X the function *enumerating elements of* X). We also let *fp* denote the function on $\mathcal{P}(\mathbf{On})$ which assigns to an subclass X the class $\{x \in X : c_X(x) = x\}$, and let *id* denote the identity function on $\mathcal{P}(\mathbf{On})$. For a function f, let $Dm(f)$ denote the domain of f and $\mathrm{Rg}(f)$ the range of f.

In what follows, we often consider sequences of functions, whose index set is an ordinal. If $\langle f_\beta \rangle_{\beta<\alpha}$ is a sequence of functions, then $\bigcap_{\beta<\alpha} f_\beta$ is a function satisfying that $Dm(\bigcap_{\beta<\alpha} f_\beta) = \bigcap_{\beta<\alpha} Dm(f_\beta)$ and that $(\bigcap_{\beta<\alpha} f_\beta)(X) = \bigcap_{\beta<\alpha} f_\beta(X)$ for any $X \in Dm(\bigcap_{\beta<\alpha} f_\beta)$.

Definition 0.10 (1) A sequence of functions $\langle f_\beta \rangle_{\beta<\alpha}$ is said to be *decreasing* if $Dm(f_\beta) = Dm(f_0)$ and if $f_\beta(X) \supset f_\gamma(X)$ for any $\beta, \gamma < \alpha$ with $\beta < \gamma$ and for any $X \in Dm(f_0)$; $\langle f_\beta \rangle_{\beta<\alpha}$ is said to be *continuously decreasing* if it is decreasing and if $f_\lambda = \bigcap_{\beta<\lambda} f_\beta$ for any limit ordinal $\lambda < \alpha$.
(2) If $F := \langle f_\beta \rangle_{\beta<\alpha}$ is a decreasing sequence, then F^D is a function whose domain is $Dm(f_0)$, which is defined by:

$$F^D(X) = \{\beta < \alpha : \ \beta \in f_\beta(X)\} \cup \bigcap_{\beta<\alpha} f_\beta(X).$$

Definition 0.11 For a limit ordinal α, we define $Q_{fp}(\alpha)$ as the smallest class E satisfying the following conditions.
(i) *fp* and *id* are in E.
(ii) $g \in E$ implies $fp \cdot g \in E$. where $fp \cdot g$ denotes the composition of *fp* and g;
(iii) If $\gamma < \alpha$ and if $\langle f_\beta \rangle_{\beta<\gamma}$ is a sequence of functions in E, then $\bigcap_{\beta<\gamma} f_\beta \in E$.
(iv) If F is a continuously decreasing sequence of functions in E whose length is α, then $F^D \in E$.

Note that $Dm(g) = \mathcal{P}(\mathbf{On})$ for any $g \in Q_{fp}(\alpha)$.

Definition 0.12 (1) For any $X \in \mathcal{P}(\mathbf{On})$, $J_{fp}(\alpha, X) := \bigcap\{g(X) : \ g \in Q_{fp}(\alpha)\}$.
(2) fp^∇ is the function with $Dm(fp^\nabla) = \mathcal{P}(\mathbf{On})$, which is defined by:

$$fp^\nabla(X) := \{\alpha : \ \alpha \in J_{fp}(\alpha, X)\}.$$

Theorem 0.13 ([Ga67]) $fp^\nabla(\mathbf{Reg})$ is the class of weakly Mahlo cardinals, where **Reg** means the class of uncountable regular cardinals.

We here dedicate ourselves to reconstruct (a primitive recursive analogue of) Gaifman's characterization of Mahlo cardinals mentioned in Thm.0.13. We first modify the property asserting that M is a Mahlo cardinal, as follows: [2]

$$(\mathbf{P}^*) \quad \text{For any } g \in Q_{fp}(M),\ M \in g(\mathbf{Reg}).$$

Let $Q^*_{fp}(M)$ denote the set $\{c_{g(\mathbf{Reg})}\lceil_M : g \in Q_{fp}(M)\}$, where $c_{g(\mathbf{Reg})}\lceil_M$ denotes the restriction of $c_{g(\mathbf{Reg})}$ to M, and let g^* denote $c_{g(\mathbf{Reg})}\lceil_M$.

Lemma 0.14 $(\mathbf{P}^*)$ is equivalent to the following:

$(\mathbf{P}^{**})$ M is a regular cardinal and $\forall g^* \in Q^*_{fp}(M)$ ($M = \sup \mathrm{Rg}(g^*)$).

We shall claim that a subset of $\mathcal{T}(M)$ is a primitive recursive analogue of $Q^*_{fp}(M)$. In order to do so, we first re-express the property of $Q^*_{fp}(M)$ by χ-collapsing function.

Definition 0.15 For each $\alpha < \varepsilon_{M+1}$ with cofinality M,

$$||\alpha|| := \begin{cases} M \ \text{ if } \alpha = M; \\ \{\alpha_1 + \xi : \ \xi \in ||\alpha_2||\} \ \text{ if } \alpha =_{\mathrm{nf}} \alpha_1 + \alpha_2; \\ \{\varphi 0 \xi : \ \xi \in ||\alpha_1||\} \ \text{ if } \alpha =_{\mathrm{nf}} \varphi 0 \alpha_1. \end{cases}$$

We denote the ξ^{th} element of $||\alpha||$ by $\alpha(\xi)$.

Lemma 0.16 Let α be an ordinal less than ε_{M+1} with cofinality M.
(1) χ^α enumerates the elements of $\{\gamma < M : \ \gamma \in \mathrm{Rg}(\chi^{\alpha(\gamma)})\}$.
(2) For each $\gamma < M$, $\gamma \in \mathrm{Rg}(\chi^{\alpha(\gamma)})$ if and only if $\gamma = \sup\{\chi^{\alpha(\xi)}(\zeta) : \ \xi, \zeta < \gamma\}$.

Lemma 0.17 $\{\chi^\alpha\}_{\alpha < \varepsilon_{M+1}}$ corresponds to $Q^*_{fp}(M)$ in the following sense:

(i) For each α, $\chi^{\alpha+1}$ enumerates the fixed points of χ^α, that is, for each $\delta < M$, $\chi^{\alpha+1}(\delta) \in \mathrm{Rg}(\chi^\alpha)$ and $\chi^{\alpha+1}(\delta) = \sup\{\chi^\alpha(\eta) : \eta < \chi^{\alpha+1}(\delta)\}$.

(ii) For each limit ordinal $\alpha < \varepsilon_{M+1}$ whose cofinality is less than M, χ^α enumerates $\bigcap_{\xi<\alpha} \mathrm{Rg}(\chi^\xi)$.

(iii) For each ordinal $\alpha < \varepsilon_{M+1}$ with cofinality M, χ^α enumerates the elements of $\{\gamma < M : \ \gamma \in \mathrm{Rg}(\chi^{\alpha(\gamma)})\}$, that is, for each $\delta < M$, $\chi^\alpha(\delta) = \sup\{\chi^{\alpha(\xi)}(\zeta) : \ \xi, \zeta < \chi^\alpha(\delta)\}$.

The parity between (iv) in Def.0.11 and (iii) in the lemma above becomes more clear when we consider an extension of the concept of inaccessibility.

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[2]Here we do not mention the property asserting that M is the *least* element of the set of weakly Mahlo cardinals. See [Ta∞] for the property.

Definition 0.18 If $\alpha = M$ or there exist several ordinals $\beta_1, \ldots, \beta_n$ satisfying (a) $\alpha = \beta_1 + \cdots + \beta_n + M$; (b) each β_i is an additive principal number; and (c) $\beta_1 \geq \cdots \geq \beta_n \geq M$, then α is called an *M-type ordinal.* We also call $\beta_1 + \cdots + \beta_n$ above by the *base segment* of α. We regard 0 as the base segment of M.

Definition 0.19 Let $\mathbf{b}$ be the base segment of an M-type ordinal. Then, for each ordinal γ, we define γ-$\mathbf{b}$-inaccessibility, as follows:

(i) An ordinal ξ is said to be 0-$\mathbf{b}$-inaccessible if $\xi \in \mathrm{Rg}(\chi^{\mathbf{b}})$.

(ii) ξ is $(\gamma+1)$-$\mathbf{b}$-inaccessible if

$$\xi \in \mathrm{Rg}(\chi^{\gamma}) \text{ and } \xi = \sup\{\zeta < \xi : \zeta \text{ is } \gamma\text{-}\mathbf{b}\text{-inaccessible}\};$$

(iii) Let λ is a limit ordinal. Then, ξ is λ-$\mathbf{b}$-inaccessible if for every $\eta < \lambda$ ξ is η-$\mathbf{b}$-inaccessible.

Since $\mathrm{Rg}(\chi^0)$ is the set of regular cardinals less than M, α-0-inaccessible cardinals are exactly α-weakly inaccessible cardinals usually defined in Set theory (cf. [Ra98] or [Dr74]).

From Le.0.17, we can obtain the following corollary, which is a natural extension of Prop.3.6 in [R98].

Corollary 0.20 Let $\mathbf{b}$ be the base segment of an M-type ordinal. Then, the diagonal set $\{\kappa < M : \kappa \text{ is } \kappa\text{-}\mathbf{b}\text{-inaccessible}\}$ is enumerated by the function $\chi^{\mathbf{b}+M}$.

The result (ii) above is the special case of (iii) in Le.0.17, which corresponds to the property (iv) in Def.0.11. In fact, $\{\kappa < M : \kappa \text{ is } \kappa\text{-}\mathbf{b}\text{-inaccessible}\}$ is the set obtained from $\{I^{\mathbf{b}}_{\alpha}\}_{\alpha<M}$ in a way similar to Def.0.10.(2), where $I^{\mathbf{b}}_{\gamma}$ denotes the function which enumerates all γ-$\mathbf{b}$-inaccessible ordinals.

We can also obtain the following corollary (Cor.0.22) from Le.0.17.

Definition 0.21 For each ordinal α, an ordinal γ is said to be *proof-theoretically α-inaccessible* if $\gamma \in \mathcal{T}(M)$ of the form $\gamma = \psi^{\kappa}_{M}(\Omega_1)$ for some $\kappa \in Rg(\chi^{\alpha})$.

Let *α-PTIO* be the set of proof-theoretically α-inaccessible ordinals.

For the κ in Def.0.21, it is true that $\kappa \in \mathcal{T}(M)$, but it is possible that κ is of the form $\chi^{\delta}(\eta)$ for some $\delta > \alpha$ even though $\kappa \in Rg(\chi^{\alpha})$.

Corollary 0.22 (i) If $\psi^{\chi^{\alpha+1}(\beta)}_{M}(\Omega_1) \in \mathcal{T}(M)$, then

$$\begin{aligned}\psi^{\chi^{\alpha+1}(\beta)}_{M}(\Omega_1) &= \sup\{\ \xi\ :\ \xi \in \alpha\text{-PTIO} \cap \psi^{\chi^{\alpha+1}(\beta)}_{M}(\Omega_1)\ \}\\ &= \sup\{\ \psi^{\chi^{\alpha}(\delta)}_{M}(\Omega_1) \in \mathcal{T}(M)\ :\ \delta < \chi^{\alpha+1}(\beta)\ \}\end{aligned}$$

(Note that $\psi^{\chi^{\alpha+1}(\beta)}_{M}(\Omega_1)$ is an element of α-PTIO itself.)

(ii) If $\psi_M^{\chi^\alpha(\beta)}(\Omega_1) \in \mathcal{T}(M)$ and α is a limit ordinal whose cofinality is less than M, then

$$\begin{aligned}\psi_M^{\chi^\alpha(\beta)}(\Omega_1) &= \sup\{\ \xi\ :\ \exists\delta < \alpha\ (\xi \in \delta\text{-PTIO} \cap \psi_M^{\chi^\alpha(\beta)}(\Omega_1))\ \}\\ &= \sup\{\ \psi_M^{\chi^\eta(\delta)}(\Omega_1) \in \mathcal{T}(M)\ :\ \eta < \alpha\ \&\ \delta < \chi^\alpha(\beta)\ \}.\end{aligned}$$

(iii) If $\psi_M^{\chi^\alpha(\beta)}(\Omega_1) \in \mathcal{T}(M)$ and α is an ordinal with cofinality M, then

$$\begin{aligned}\psi_M^{\chi^\alpha(\beta)}(\Omega_1) &= \sup\{\ \xi\ :\ \exists\zeta < \chi^\alpha(\beta)\ (\xi \in \alpha(\zeta)\text{-PTIO} \cap \psi_M^{\chi^\alpha(\beta)}(\Omega_1))\ \}\\ &= \sup\{\ \psi_M^{\chi^{\alpha(\zeta)}(\eta)}(\Omega_1) \in \mathcal{T}(M)\ :\ \zeta, \eta < \chi^\alpha(\beta)\ \}.\end{aligned}$$

We conjecture the propeties in Cor.0.22 can be expressed more explicitly. For example, we conjecture that, if $\delta = \psi_M^{\chi^\alpha(\beta)}(\Omega_1) \in \mathcal{T}(M)$ and if α is an ordinal with cofinality M, then δ is the η^{th} element $\gamma \in \mathcal{T}(M)$ satsfying

$$\gamma = \sup\{\Phi(\gamma', \gamma'') : \gamma', \gamma'' < \gamma\ \&\ \Phi(\gamma', \gamma'')\text{ is defined}\},$$

where

$$\eta = otyp(\{\psi_M^{\chi^\alpha(\beta')}(\Omega_1) \in \mathcal{T}(M) : \beta' < \beta\})$$

and $\Phi(\xi, \zeta)$ denotes the ζ^{th} element of

$$\{\psi_M^{\chi^{\theta_\xi}(\rho)}(\Omega_1) \in \mathcal{T}(M) : \rho \in \mathcal{T}(M)\}$$

and θ_ξ denotes the ξ^{th} element of

$$\{\theta \in \mathcal{T}(M) : \exists\rho\ (\ \psi_M^{\chi^\theta(\rho)}(\Omega_1) \in \mathcal{T}(M)\ )\ \wedge\ \theta \in [\![\alpha]\!]\ \}.$$

Here,

$$[\![\alpha]\!] := \begin{cases} \mathcal{T}(M) \cap M & \text{if } \alpha = M; \\ \{\alpha_1 + \xi \in \mathcal{T}(M) : \xi \in [\![\alpha_2]\!]\} & \text{if } \alpha =_{\text{nf}} \alpha_1 + \alpha_2\ \wedge\ t(\alpha_2) = \hat{m}; \\ \{\varphi 0\xi \in \mathcal{T}(M) : \xi \in [\![\alpha_1]\!]\} & \text{if } \alpha =_{\text{nf}} \varphi 0\alpha_1\ \wedge\ t(\alpha_1) = \hat{m}. \end{cases}$$

By Cor.0.22, one can regard each $c_{\alpha\text{-}PTIO}$ which is the function enumerating elements of α-*PTIO* is a primitive recursive analogue of an element of $Q^*_{fp}(M)$. Therefore, the property (**P**) is a primitive recursive analogue of (**P****), which is obtained from (**P****) by replacing $Q^*_{fp}(M)$ by $\{c_{\alpha\text{-}PTIO}\}_{\alpha < \varepsilon_{M+1}}$. So, we can consider that proof-theoretically Mahlo ordinals defined in Def.0.8 are primitive recursive analogues of Mahlo cardinals.

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