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Partial difference functional equations arising from the Cauchy-Riemann equations

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1. Introduction

Let $(G,+)$ be an additive Abelian group in which it is possible to devide by 2. Let $\mathbb{C}$ be the field of complex numbers. The main aim of this note is to determine the general solution of the following new functional equation

$$(E.1) \qquad f(x+t,y)-f(x-t,y)=-i[f(x,y+t)-f(x,y-t)]$$

for all $x,y,t\in G$, where $f:G\times G\longrightarrow\mathbb{C}$ and i is the imaginary unit.

Let $\mathbb{R}$ be the field of real numbers. For a function $f:\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{C}$ we define the divided partial difference operator $\triangle_{x,t}$ and $\triangle_{y,t}$ by $(\triangle_{x,t}f)(x,y)=[f(x+t,y)-f(x,y)]/t$ and $(\triangle_{y,t}f)(x,y)=[f(x,y+t)-f(x,y)]/t$, respectively. Then the partial difference functional equation

$$\triangle_{x,t}f=-i\triangle_{y,t}f$$

may be considered as a descrete analogue of the Cauchy-Riemann equation

$$\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}.$$

The above partial difference functional equation may be rewritten in the form

$$(E.2) \qquad f(x+t,y)-f(x,y)=-i[f(x,y+t)-f(x,y)]$$

for all $x,y,t\in\mathbb{R}$, which has a simple geometric interpretation on the plane. In previous paper of J. Aczél and S. Haruki 1981 [3], and S. Haruki 1986 [6] we consider equation $(E.2)$ and show, among others, that $(E.2)$ does not lead essentially beyond a linear function in the case when $\mathbb{R}$ is replaced by an arbitrary monoid M. Further, a paper of S. Haruki and C. T. Ng 1994 [7] obtains the general solution of $(E.2)$ under more general algebraic structure than M and $\mathbb{C}$.

It is natural to ask what happens if, instead of operators $\triangle_{x,t}$ and $\triangle_{y,t}$, we impose the divided partial mean difference operators $\nabla_{x,t}$ and $\nabla_{y,t}$ defined by $(\nabla_{x,t}f)(x,y)=[f(x+t,y)-f(x-t,y)]/(2t)$ and $(\nabla_{y,t}f)(x,y)=[f(x,y+t)-f(x,y-t)]/(2t)$. In this case we have the partial difference functional equation

$$\nabla_{x,t}f=-i\,\nabla_{y,t}\,f,$$

which is also a discrete analogue of the Cauchy-Riemann equation. This leads to the above functional equation $(E.1)$ which also has a simple geometric interpretation on the plane. As a main result of this note we show that equation $(E.1)$ for $f : G \times G \longrightarrow \mathbb{C}$ does not lead essentially beyond a quadratic function.

In §2 we determine the general (§2.1) and regular (§2.2) solutions (when G is replaced by $\mathbb{R}$) of equation $(E.1)$.

We also show in §3 that similar results hold for certain related functional equations. In §3.1 we consider the functional equation

$$(E.3) \qquad f(x+t,y+t) - f(x-t,y-t) = -i[f(x-t,y+t) - f(x+t,y-t)].$$

In §3.2 we study the partial difference differential functional equations

$$\frac{\partial f(x,y)}{\partial x} = -i\left[\frac{f(x,y+t) - f(x,y-t)}{2t}\right], \quad \frac{f(x+t,y) - f(x-t,y)}{2t} = -i\frac{\partial f(x,y)}{\partial y}.$$

2. On the functional equation $(E.1)$

In this section we determine the general and regular solutions of functional equation $(E.1)$.

2.1. The general solution

A function $A^1 : G \longrightarrow \mathbb{C}$ is said to be additive if A^1 satisfies $A^1(x+y) = A^1(x) + A^1(y)$ for all $x, y \in G$. A function $A_2 : G \times G \longrightarrow \mathbb{C}$ is said to be bi-additive if A_2 satisfies both equations $A_2(x+y,z) = A_2(x,z) + A_2(y,z)$ and $A_2(x,y+z) = A_2(x,y) + A_2(x,z)$ for all $x, y, z \in G$. A function $A^2 : G \longrightarrow \mathbb{C}$ is a diagonalization of symmetric bi-additive function if $A^2(x) = A_2(x,y)|_{y=x}$, where $A_2 : G \times G \longrightarrow \mathbb{C}$ is symmetric and additive in each argument.

Our main result of this note is as follows.

Theorem 2.1. *A function $f : G \times G \longrightarrow \mathbb{C}$ satisfies equation $(E.1)$ for all $x, y, z \in G$ if and only if there exist*

(i) *an arbitrary complex constant A^0,*

(ii) *an arbitrary additive function $A^1 : G \longrightarrow \mathbb{C}$,*

(iii) *an arbitrary symmetric bi-additive function $A_2 : G \times G \longrightarrow \mathbb{C}$, and*

(iv) *a diagonalization $A^2 : G \longrightarrow \mathbb{C}$ of the above function A_2*

such that

$$(S.1) \qquad f(x,y) = A^0 + A^1(x) + iA^1(y) + A^2(x) - A^2(y) + 2iA_2(x,y)$$

for all $x, y \in G$.

We impose the following notations. Define the shift operators X^t and Y^t by $(X^t f)(x,y) = f(x+t,y)$ and $(Y^t f)(x,y) = f(x,y+t)$ for all $x, y, t \in G$. In particular $1 = X^0 = Y^0$ denote the identity operator. Further, define the partial mean difference operators $\delta_{x,t}$ and $\delta_{y,t}$ by

$\delta_{x,t} = X^t - X^{-t}$ and $\delta_{y,t} = Y^t - Y^{-t}$ for all $x, y, t \in G$. Notice that the ring of operators generated by this family of operators is commutative and distributive.

In order to prove Theorem 2.1 we need the following two lemmas. One of them is:

Lemma 2.1. *If a function $f : G \times G \longrightarrow \mathbb{C}$ satisfies equation (E.1) for all $x, y, t \in G$, then f also satisfies each one of the following three functional equations*

$$(\delta^3_{x,t} f)(x,y) = 0 \quad \textit{and} \quad (\delta^3_{y,t} f)(x,y) = 0 \tag{2.1}$$

$$((\delta^2_{x,t} + \delta^2_{y,t})f)(x,y) = 0 \tag{2.2}$$

or as the expanded form (2t replaced by t)

$$f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) = 4f(x,y) \tag{2.3}$$

for all $x, y \in G$.

The above Lemma 2.1 shows that equation (E.1) yields equation (2.3). On the other hand, J. Aczél, H. Haruki, M. A. McKiernan and G. N. Sakovič 1968 [2, p.43, Lemma 3] proved that equation (2.3) is equivalent to the Haruki functional equation (M. A. McKiernan [11], H. Światak [13], among others)

$$f(x+t,y+t) + f(x+t,y-t) + f(x-t,y+t) + f(x-t,y-t) = 4f(x,y). \tag{2.4}$$

Hence, if we directly apply a general theorem of M. A. McKiernan 1970 [12, p.32, Theorem 2] to the equation (2.4), then we obtain

$$(\delta^k_{x,t} f)(x,y) = 0 \quad \textit{and} \quad (\delta^k_{y,t} f)(x,y) = 0 \tag{2.5}$$

with $k = 11$. On the other hand, it is also known in [2, p.43, Lemma 3] that if an arbitrary f satisfies (2.4), then f also satisfies difference equations (2.5) for $k = 4$. However, in order to obtain the general solution of equation (E.1) our result (2.5) with $k = 3$, that is, (2.1), is better result and easier to prove the *'only if'* part of Theorem 2.1, since if $k > 3$ in (2.5), then a solution of (2.1) contains more symmetric multiadditive functions in higher order (cf. S. Mazur and W. Orlicz 1934 [9], and M. A. McKiernan 1967 [10], among others). We emphasis that we do not apply the above mensioned result for the case $k = 4$.

The other is the following lemma which is a particular case of Lemma 6 in [2, p.49-50]. We note that if we replace $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ by $f : G \times G \longrightarrow \mathbb{C}$ in Lemma 6 of [2], then it follows from a general theorem of S. Mazur and W. Orlicz [9] that the result of Lemma 6 in [2] still holds for the case $\delta_{x,t} = X^t - X^{-t}$ and $\delta_{y,t} = Y^t - Y^{-t}$ instead of $\Delta_{x,t} = X^{t/2} - X^{-t/2}$ and $\Delta_{y,t} = Y^{t/2} - Y^{-t/2}$ defined in [2, p.43].

Lemma 2.2. *A function $f : G \times G \longrightarrow \mathbb{C}$ satisfies both equations* (2.1) *for all $x, y, t \in G$ if and only if f is given by*

$$\begin{aligned} f(x,y) = {} & A^0 + A^1(x) + A^2(x) + B^1(y) + B^2(y) \\ & + A^{1,1}(x;y) + A^{2,1}(x;y) + A^{1,2}(x;y) + A^{2,2}(x;y) \end{aligned} \tag{2.6}$$

for all $x, y \in G$, where A^0, and A^1, $A^2 : G \longrightarrow \mathbb{C}$ are defined in Theorem 2.1, $B^1 : G \longrightarrow \mathbb{C}$ is an arbitrary additive function, and $B^2 : G \longrightarrow \mathbb{C}$ is a diagonalization of an arbitrary symmetric bi-additive function. Further, functions $A^{1,1}$, $A^{2,1}$, $A^{1,2}$, and $A^{2,2} : G \times G \longrightarrow \mathbb{C}$ are defined as follows. $A^{1,1}(x; y)$ is additive in x and y, and not symmetric. $A^{2,1}(x; y) := A_{2,1}(x, x; y)$ is additive in all variables, symmetric in the first two. $A^{1,2}(x; y) := A_{1,2}(x; y, y)$ is additive in all variables, symmetric in the last two places. $A^{2,2}(x; y) := A_{2,2}(x, x; y, y)$ is additive in all variables, symmetric in the first two and in the last two places.

2.2. Regular solutions

In addition, as soon as some suitable regularity assumptions are imposed on f for the case $G = \mathbb{R}$ in the above Theorem 2.1, then it can be readily shown by the following lemma that f is an ordinary complex polynomial of degree at most two. The following lemma is a consequence of Theorem 2.1.

Lemma 2.3. *Let $(F, +)$ be an additive group. If $f : F \times F \longrightarrow \mathbb{C}$ is given by $(S.1)$ for all $x, y \in F$, then all functions A^1, $A^2 : F \longrightarrow \mathbb{C}$ and $A_2 : F \times F \longrightarrow \mathbb{C}$ can be represented in terms of f and an arbitrary constant A^0 by*

$$A^1(x) = [f(x, y) - f(-x, -y) - f(-x, y) + f(x, -y)]/4, \tag{2.26}$$

$$A^1(y) = [f(x, y) - f(-x, -y) + f(-x, y) - f(x, -y)]/(4i), \tag{2.27}$$

$$A_2(x, y) = [f(x, y) + f(-x, -y) - f(-x, y) - f(x, -y)]/(8i), \tag{2.28}$$

$$A^2(x) = f(x, 0) - A^0 - [f(x, y) - f(-x, -y) - f(-x, y) + f(x, -y)]/4, \tag{2.29}$$

$$A^2(y) = -f(0, y) + A^0 + [f(x, y) - f(-x, -y) + f(-x, y) - f(x, -y)]/4 \tag{2.30}$$

for all $x, y \in F$.

If we assume that, for example, $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ is continuous everywhere, then by applying the above Lemma 2.3 we have the following result.

Theorem 2.2. *A continuous function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies $(E.1)$ for all $x, y, t \in \mathbb{R}$ if and only if f is given by*

$$f(x, y) = a(x^2 - y^2 + 2ixy) + b(x + iy) + c \tag{2.38}$$

for all $x, y \in \mathbb{R}$, where a, b, and c are arbitrary complex constants.

Equation $(E.1)$ can also be rewritten in the complex form

$$f(z + t) - f(z - t) = -i[f(z + it) - f(z - it)] \tag{2.39}$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$, where $f(z) := f(x, y)$ for all $x, y \in \mathbb{R}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$. In this case the continuous solution (2.38) is given by a complex polynomial of degree at most two such that

$$f(z) = az^2 + bz + c$$

for all $z \in \mathbb{C}$.

3. Certain related functional equations

3.1. Related equations I.

In §3.1 we mainly consider the functional equation

$$(E.3) \qquad f(x+t, y+t) - f(x-t, y-t) = -i[f(x-t, y+t) - f(x+t, y-t)]$$

for all $x, y, t \in G$, where $f : G \times G \longrightarrow \mathbb{C}$, and determine the general and regular solution of $(E.3)$.

There are various applications of functional equations. One of them is an application of functional equations to a geometric characterization of complex polynomials from the standpoint of conformal-mapping properties. From this point of view H. Haruki 1971 [4] obtains the functional equation

$$(3.1) \qquad f(z + te^{\pi i/4}) - f(z - te^{\pi i/4}) = i[f(z + te^{-\pi i/4}) - f(z + te^{-\pi i/4})]$$

for all $z \in \mathbb{C}$ and $t \in \mathbb{R}$, where $f : \mathbb{C} \longrightarrow \mathbb{C}$, under two geometric properties on f. Equation (3.1) yields the above equation $(E.3)$ for all $x, y, t \in \mathbb{R}$, where $f(x, y) := f(z)$ for $z = x + iy$ and $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$. It is obtained in [4] that the continuous solution of equation (3.1) for $f : \mathbb{C} \longrightarrow \mathbb{C}$ by applying the regularity of Haruki's functional equation (2.4). We note that the continuity assumption in order to consider equation (3.1) is natural and meaningful with two geometric properties in [4]. Further, it is possible to obtain the general solution of equation $(E.3)$ and (3.1) for $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ when no regularity assumptions are imposed on f, since it is shown in [4] that (3.1) implies (2.4), and the general solution of (2.4) is obtained in [2, p.50-51, Theorem 5]. However, in this section we first show that equation $(E.3)$ is equivalent to $(E.1)$ when no regularity assumptions are imposed on f so that by Theorem 2.1 in §2.1 we immediately obtain the general solution of equation $(E.3)$ under no regularity assumptions on f. We emphasis that we do not apply the general solution of equation (2.4). We will be able to generalize our results from $\mathbb{R}$ to G.

Theorem 3.1. *If $f : G \times G \longrightarrow \mathbb{C}$ satisfies equation $(E.1)$ for all $x, y, t \in G$, then also equation $(E.3)$ for all $x, y, t \in G$ and conversely so that $(E.1)$ and $(E.3)$ are equivalent.*

The following result is an immediate consequence of Theorem 2.1 and Theorem 3.1.

Corollary 3.1. *A function $f : G \times G \longrightarrow \mathbb{C}$ satisfies equation $(E.3)$ for all $x, y, t \in G$ if and only if f is given by expression $(S.1)$ in §2.1 for all $x, y \in G$.*

We also readily obtain from Theorem 3.1 and Theorem 2.2 in the case of $G = \mathbb{R}$ that a continuous function $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ satisfies $(E.3)$ for all $x, y, t \in \mathbb{R}$ if and only if f is given by (2.38) for all $x, y \in \mathbb{R}$.

H. Haruki proved the following theorem in [4, p.37, Theorem] by applying a regularity of functional equation (2.4).

Theorem 3.2. *A continuous function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *satisfies equation* (3.1) *for all* $z \in \mathbb{C}$ *and* $t \in \mathbb{R}$ *if and only if* f *is given by a quadratic polynomial of* z.

Theorem 3.3. *If* $f : G \times G \longrightarrow \mathbb{C}$ *satisfies each one of the following three functional equations*

$$(E.1) \qquad f(x+t,y) - f(x-t,y) = -i[f(x,y+t) - f(x,y-t)]$$

$$(E.2) \qquad f(x+t,y) - f(x,y) = -i[f(x,y+t) - f(x,y)]$$

$$(E.3) \qquad f(x+t,y+t) - f(x-t,y-t) = -i[f(x-t,y+t) - f(x+t,y-t)]$$

for all $x, y, t \in G$, *then each one of* $(E.1)$, $(E.2)$, *or* $(E.3)$ *also satisfies either the Haruki functional equation*

$$(2.4) \qquad f(x+t,y+t) + f(x-t,y+t) + f(x+t,y-t) + f(x-t,y-t) = 4f(x,y)$$

or

$$(2.3) \qquad f(x+t,y) + f(x-t,y) + f(x,y+t) + f(x,y-t) = 4f(x,y)$$

for all $x, y, t \in G$.

3.2. Related equations II.

If one side of the Cauchy-Riemann equation $\partial f/\partial x = -i\partial f/\partial y$ is replaced by the operators $\nabla_{x,t}f$ or $\nabla_{y,t}f$ defined in §1, then we have the following two partial difference differential functional equations

$$(3.7) \qquad \frac{\partial f(x,y)}{\partial x} = -i\left[\frac{f(x,y+t) - f(x,y-t)}{2t}\right]$$

$$(3.8) \qquad \frac{f(x+t,y) - f(x-t,y)}{2t} = -i\frac{\partial f(x,y)}{\partial y}.$$

In §3.2 we determine the general solutions of each one of the above two equations.

Theorem 3.4. *A function* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.7) *for all* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by*

$$(3.9) \qquad f(x,y) = \frac{1}{2}a(y^2 - x^2 - 2ixy) + b(y - ix) + c,$$

where $a, b,$ *and* c *are arbitrary complex constants.*

In view of the forms (3.7) and (3.8) the following theorem readily follows from a proof similar to the above proof of Theorem 3.4.

Theorem 3.5. *A function* $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ *satisfies equation* (3.8) *for all* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R} \setminus \{0\}$ *if and only if* f *is given by*

$$f(x,y) = \frac{1}{2}a(x^2 - y^2 + 2ixy) + b(x + iy) + c,$$

where $a, b,$ *and* c *are arbitrary complex constants.*

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