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# On the Stationary Solution of the Mathematical Model for Grain Boundary Grooving

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1. Introduction

In this talk, we will present some stationary solution for nonlinear partial differential equation called Mullins Equation which is occered in the theory of grain boundary grooving.

$$u_t = -C_1^E(u)(1+u_x^2)^{1/2}exp(-C_2^E(u)\frac{u_{xx}}{(1+u_x^2)^{3/2}}) + C_1^C(u)(1+u_x^2)^{1/2}. \quad (1)$$

The main tool, which we can use, is the admissibility property between weighted continuous function spaces for the integral operator,as follows.

$$T_\xi x(t) = -\int_t^\infty e^{\zeta_1(t-s)}F(x(s),y(s))ds,$$

$$T_\xi y(t) = \xi e^{\zeta_2 t} + \int_0^t e^{\zeta_2(t-s)}F(x(s),y(s))ds. \quad (2)$$

From this admissibility we can prove the existence theorem for the special simultaneous differential equation. This existence theorem can be applied for the second order differential equation,

$$u'' = f(u,u') = \frac{kT(u)(1+u'^2)^{3/2}}{v\gamma} ln(\frac{P_0(u)}{P_c}). \quad (3)$$

The solution of this equation is one of the stationary solution for Mullins Equation.

2. Theorems

On the equation (1), we are interested in the stational solution. So we shall consider the equation (3) which we can make by putting $u_t = 0$ for the equation (1). To prove the existence theorem for the stational solution, we use the next two theorems.

Theorem1

For the second oreder differential equation,

$$u'' = f(u, u'), \quad (4)$$

suppose that the following hypotheses.

$$f(u,p) \in C^1(R^2), \quad x > 0, \quad {}^{\exists}\lambda \in R^1 \quad s.t \quad .f(\lambda, 0) = 0, \quad f_u(\lambda, 0) > 0$$

.

Then there exits the solution on $(0, \infty)$ and it satisfies that

$${}^{\exists}D > 0 \quad s.t. \quad |u(x) - \lambda| \leq Dexp(-\tau x),$$

where

$$0 < \tau < \left| \frac{f_p(\lambda, 0) - \sqrt{f_p(\lambda, 0)^2 + 4f_u(\lambda, 0)}}{2} \right|.$$

Theorem2

On the differential equation,

$$\omega_1' = \zeta_1 \omega_1 + F(\omega_1, \omega_2), \quad \omega_2' = \zeta_2 \omega_2 + F(\omega_1, \omega_2), \quad x > 0,$$

where,

$$f(\eta_1, \eta_2) \in C^1(R^2), \quad F(0,0) = 0, \quad F_{\eta_1}(0,0) = 0, \quad \zeta_1 > 0, \zeta_2 < 0,$$

there exists some global nontrivial solution

$$\omega(x) = (\omega_1(x), \omega\ _2(x)), \quad x > 0,$$

for every $\tau, 0 < \tau < |\zeta_2|$, and the next inequality is satisfied.

$$|e^{\tau x}\omega_1(x)| + |e^{\tau x}\omega_2(x)| < \infty, \quad x > 0.$$

At first we consider Theorem2. By using the addmissibility of the integral operator(2), we can establish the proof of Theorem2. Let consider the integral operator on the following function set B,

$$B = \omega(x) = (\omega_1(x), \omega_2(x)) \in C^0([0,\infty)); ||\omega|| \leq 2|\xi|,$$

$$||\omega|| = \sup_{x \geq 0}(e^{\tau x}\omega_1(x) + e^{\tau x}\omega_2(x)).$$

On this set the integral operator(2) satisfies the contraction princeple. Then the operator $T_\xi : B \longrightarrow B$ has the unique fixes point $\omega(x) = (\omega_1(x), \omega_2(x))$.
Theorem3
On the set B the integral operator(2) satisfies the contraction princeple. Then the integral equation which is made by the integral operator(2) has unique solution in the set B.
The proof of this threorem3 is essentially depended the following inequalities.

$$
\begin{aligned}
|e^{\tau x}T_\xi\omega_1(x)| &= |e^{\tau x}\int_x^\infty e^{\zeta_1(x-y)}F(\omega_1(y),\omega_2(y))dy| \\
&\leq \int_x^\infty e^{\zeta_1(x-y)}|e^{\tau x}(F(\omega_1(y),\omega_2(y)) - F(0,0))|dy \\
&= \int_x^\infty e^{\zeta_1(x-y)}|(F_{\eta 1}(\theta\omega_1(y),\theta\omega_2(y))\omega_1(y)e^{\tau x} \\
&\quad +F_{\eta 2}(\theta\omega_1(y),\theta\omega_2(y))\omega_2(y)e^{\tau x}|dy \\
&\leq M\int_x^\infty e^{\zeta_1(x-y)}(|e^{\tau x}\omega_1(y)| + |e^{\tau x}\omega_2(y)|)dy,
\end{aligned}
$$

$$
\begin{aligned}
|e^{\tau x}T_\xi\omega_1(x)| &= |e^{\tau x}(\xi e + \int_0^x e^{\zeta_2(x-y)}F(\omega_1(y),\omega_2(y))dy) \\
&\leq |\xi|e^{(\tau+\zeta_2)x} + \int_0^x e^{\tau x+\zeta_2 x-\zeta_2 y-\tau y} \\
&\quad \times|e^{\tau y}|F(\omega_1(y),\omega_2(y))|dy \\
&= |\xi|e^{(\tau+\zeta_2)x} + \int_0^x e^{(\tau+\zeta_2)(x-y)} \\
&\quad \times|e^{\tau y}|F(\omega_1(y),\omega_2(y))|dy \\
&\leq |\xi| + \frac{M}{\tau_2}||\omega||,
\end{aligned}
$$

where

$$0 < \theta < 1, \tau + \zeta_2 < 0, \tau + \zeta_2 = -\tau_2.$$

Hence we can prove Theorem2. Next we treat Theorem1, by using the results of Theorem2. Let define the function $F(\omega_1, \omega_2)$ in Theorem2 by the next equation,

$$F(\eta_1,\eta_2) = f(\frac{\eta_1-\eta_2}{\zeta_1-\zeta_2}+\lambda, \frac{\zeta_1\eta_1-\zeta_2\eta_2}{\zeta_1-\zeta_2}) - \frac{\eta_1-\eta_2}{\zeta_1-\zeta_2}f_u(\lambda,0) - \frac{\zeta_1\eta_1-\zeta_2\eta_2}{\zeta_1-\zeta_2}f_p(\lambda,0),$$

where

$$\zeta_1 = \frac{f_p(\lambda,0)+\sqrt{f_p(\lambda,0)^2+4f_u(\lambda,0)}}{2} > 0,$$

$$\zeta_2 = \frac{f_p(\lambda,0) - \sqrt{f_p(\lambda,0)^2 + 4f_u(\lambda,0)}}{2} < 0,$$

where the function $f$ as in Theorem1. By the result of Theorem2 there exists the solution $\omega(x) = (\omega_1(x), \omega_2(x))$. Define

$$u(x) = \frac{\omega_1(x) - \omega_2(x)}{\zeta_1 - \zeta_2} + \lambda, \quad x > 0.$$

This function $u$ is the solution in Theorem1. At last, we can apply Theorem1 for the equation (3), we get the stational solution of (1).

References

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