数理解析研究所講究録 1374 巻 2004 年 153-178

Real curves on real Hirzebruch surfaces

Sachiko SAITO (北海道教育大学・函館校　齋藤 幸子)
Hakodate Campus, Hokkaido University of Education

Most of this article is obtained by a joint work with Professor V.V.Nikulin. The author would like to thank him.

1 Hirzebruch surfaces

We call

$$\Sigma_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$$

the *n-th Hirzebruch surface* ($n \geq 0$) ([4], p.141). Σ_n is a ruled surface over $\mathbb{P}^1$. The converse assertion is also true. Every ruled surface over $\mathbb{P}^1$ is birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$, and hence, rational. For example, we have $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and Σ_1 is $\mathbb{P}^2$ blown up in one point. (The preimage of this point is C_1 defined below.) For $n \geq 1$ we set $C_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n)) \subset \Sigma_n$, and call it the *exceptional section.* We have $C_n \cdot C_n = -n$. For the details of Hirzebruch surfaces, see [4] or [9].

By a *real structure* on a complex manifold Y, we mean an anti-holomorphic involution on Y. The number of isomorphism classes of real structures on a rational surface Y is as follows ([6], p.63).

Y	number	real part of Y
$\mathbb{P}^2$	1	$\mathbb{RP}^2$
$\mathbb{P}^1 \times \mathbb{P}^1$ $(= \Sigma_0)$	4	torus if *hyperboloid*, $\emptyset$ if (*usual*, *spin*) or (*spin*, *spin*), and S^2 if *ellipsoid*.
Σ_n with $n \geq 2$ even	2	torus or $\emptyset$
Σ_n with n odd	1	Klein bottle

Here *usual* means the real structure of $\mathbb{P}^1$ given by the usual complex conjugation, and *spin* is given by $(z_0 : z_1) \mapsto (\overline{z_1} : -\overline{z_0})$ in $\mathbb{P}^1$. The real structure (*usual*, *usual*) is called *hyperboloid*.

Some Hirzebruch surface appears as the quotient space $Y := X/\tau$ of some K3 surface with a non-symplectic involution (X, τ) with their fixed point set $A := X^\tau$ non-empty (see §2). Then A is a nonsingular curve on the surface Y. This is our motive for studying curves on Hirzebruch surfaces.

2 Real K3 surfaces with non-symplectic involutions

In [22], V.V. Nikulin and the author enumerated up the connected components of moduli of real K3 surfaces with non-symplectic involution of all type (S, θ) with $\operatorname{rk} S \le 2$, where S is a fixed lattice and θ is an involution on S, and applied it to topological classifications of real curves on some real rational surfaces and topological interpretations of invariants of integral involutions.

Let us introduce some definitions. Here we say a nonsingular compact connected complex surface X is a *K3 surface* if X has a nowhere vanishing holomorphic 2-form ω_X, equivalently, $K_X = 0$, and X is simply-connected (see [4], [5], [29]). We say a smooth involution τ on a K3 surface X is *non-symplectic* (or *anti-symplectic*) if $\tau^*(\omega_X) = -\omega_X$. For an algebraic K3 surface X with a non-symplectic holomorphic involution τ, the fixed point set, denoted by $A := X^\tau$, is empty or a non-singular complex curve on X, and the quotient space $Y := X/\tau$ is a nonsingular surface. Moreover, if $A = \emptyset$, then Y is a Enriques surface, and if $A \ne \emptyset$, then Y is a rational surface and $A \in |-2K_Y|$ and $q_Y = 0$ (see Nikulin [19], [21]). Let S be the isomorphism class of the fixed part $L^\tau := \{x \in L \mid \tau_*(x) = x\}$ of τ_* in $L := H_2(X, \mathbb{Z})$. Then S is a primitive (i.e. L/S is free), 2-elementary and hyperbolic sublattice of L, where we say a lattice M is *2-elementary* if $M^*/M \cong (\mathbb{Z}/2\mathbb{Z})^{a(M)}$ for some integer $a(M) \ge 0$, and *hyperbolic* if its signature is $(1, t_{(-)}(M))$ for some integer $t_{(-)}(M) \ge 0$. We call such a pair (X, τ) a *K3 surface with non-symplectic involution of type S* (see also Yoshikawa [30], [31] for the details).

We can additionally fix a half-cone (the *light-cone*) $V^+(S)$ of the cone $V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}$. We can also fix a fundamental chamber $\mathcal{M} \subset V^+(S)$ for the group $W^{(-2)}(S)$ generated by reflections in all elements with square (-2) in S. This is equivalent to fixing *a fundamental subdivision* $\Delta(S) = \Delta(S)_+ \cup -\Delta(S)_+$ of all elements with square -2 in S. The $\mathcal{M}$ and $\Delta(S)_+$ define each other by the condition $(\mathcal{M}, \Delta_+) \ge 0$. These additional structures $\mathcal{M} \subset V^+(S)$ of the hyperbolic lattice S are defined uniquely up to the action of the group $\{\pm 1\} W^{(-2)}(S)$.

We can restrict considering K3 surfaces with non-symplectic involutions (X, τ) such that $V^+(S)$ contains a hyperplane section of X and the set $\Delta(S)_+$ contains only classes of effective curves with square -2 in X. Namely, $V^+(S)$ and the fundamental subdivision $\Delta(S)_+$ are prescribed by the geometry of the K3 surface X.

If a pair (X, τ) is general, then S is the Picard lattice $N(X)$ of X and $\mathcal{M}$ gives the *nef* cone (or Kählerian cone) of X. The weakest condition of degeneration (i.e., giving the most reach discriminant) is the following condition:

($\mathcal{D}$): We say that (X, τ) of type S is *degenerate* if there exists $h \in \mathcal{M}$ such that h is not nef for X. This is equivalent (see [21]) to the existence of an exceptional curve with square -2 on the quotient $Y = X/\{1, \tau\}$. This is also equivalent to having an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S^{\perp}_{N(X)}$ and $\delta_1^2 = \delta_2^2 = -4$. Remark that $(\delta_1, S) \equiv 0 \mod 2$ and $(\delta_2, S^{\perp}) \equiv 0 \mod 2$. I.e., δ_1 and δ_2 are roots with square -4 for lattices S and $S^{\perp}$ respectively.

Now we restrict ourselves to the case $\operatorname{rk} S \leq 2$. Then we have :
(1) If $\operatorname{rk} S = 1$, then $S \cong \langle 2 \rangle$, $X/\tau \cong \mathbb{P}^2$, all X are non-degenerate, and A are curves of degree 6 on $\mathbb{P}^2$.
(2) If $\operatorname{rk} S = 2$, then the lattice S is isomorphic to $U(2)$, $\langle 2 \rangle \oplus \langle -2 \rangle$ or U.
(i) If $S \cong U(2)$, then non-degenerate K3 surfaces (X, τ) give $X/\tau \cong \mathbb{P}^1 \times \mathbb{P}^1 (= \Sigma_0)$ and A are curves of bidegree $(4, 4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$; and degenerate K3 surfaces (X, τ) give $X/\tau \cong \Sigma_2$.
(ii) If $S \cong \langle 2 \rangle \oplus \langle -2 \rangle$, then $X/\tau \cong \Sigma_1$ and all X are non-degenerate. The image of A in $\mathbb{P}^2$ is a curve of degree 6 with one non-degenerate double point.
(iii) If $S \cong U$, then $X/\tau \cong \Sigma_4$ and all X are non-degenerate.

Now we endow a K3 surface with non-symplectic involution (X, τ) with a real structure. Let θ be an involution on S satisfying the following properties: $\theta(V^+(S)) = -V^+(S)$ and $\theta(\Delta(S)_+) = -\Delta(S)_+$. It follows that the lattice $S_+ := S^{\theta}$ is negative definite and it has no elements with square -2. Moreover, the linear subspace $S_- \otimes \mathbb{R}$ where $S_- := S_{\theta}$ must intersect the interior of the nef cone $\mathcal{M}$. For the fixed type (S, θ), we consider a K3 surface X with a non-symplectic involution τ of type S and an anti-holomorphic involution φ such that $\varphi(S) = S$ (This implies that $\tau \circ \varphi = \varphi \circ \tau$) and $\varphi | S = \theta$. Such triplets (X, τ, φ) are called *real K3 surfaces with non-symplectic involutions of type* (S, θ).

We consider the following real analogy for real K3 surfaces with non-symplectic involutions of type (S, θ) of the degeneration ($\mathcal{D}$) above. An element $h \in S$ is called *real* if $\theta(h) = -h$, i.e. $h \in S_-$. For a general real X we have $S = N(X)$, and all real nef elements are elements of $S_- \cap \mathcal{M}$.

($\mathcal{D}\mathbb{R}$): A real K3 surface (X, τ, φ) with a non-symplectic involution of type (S, θ) is called *degenerate* if there exists a real element $h \in S_- \cap \mathcal{M}$ which is not nef for X. This is equivalent to having an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S^{\perp}_{N(X)}$ and $\delta_1^2 = \delta_2^2 = -4$ (i.e., (X, θ) is degenerate in the sense of ($\mathcal{D}$) as a complex surface). Additionally, δ_1 must be orthogonal to an element $h \in S_- \cap \operatorname{int}(\mathcal{M})$ with $h^2 > 0$. Here $\operatorname{int}(\mathcal{M})$ denote the interior part of $\mathcal{M}$, i.e., the polyhedron $\mathcal{M}$ without its faces.

The condition ($\mathcal{D}\mathbb{R}$) for (X, τ, φ) implies the condition $\mathcal{D}$ for (X, τ). Thus, condition ($\mathcal{D}\mathbb{R}$) is stronger for (X, τ) than ($\mathcal{D}$). It is easy to see that ($\mathcal{D}$) implies ($\mathcal{D}\mathbb{R}$) for all lattices S of $\operatorname{rk} S \leq 2$ above and all possible θ for these lattices. But to formulate a result about connected components of moduli of non-degenerate real K3 surfaces with non-symplectic involutions (Theorem 3) in general (for arbitrary S), we have to consider the condition ($\mathcal{D}\mathbb{R}$) of degeneration.

We use the same symbol θ for the anti-holomorphic involution φ $_{\mathrm{mod}\ \tau}$ on $Y := X/\tau$. θ gives a real structure on Y, and we have $\theta(A) = A$. We have :
(1) If $S \cong \langle 2 \rangle$, then $\theta = -1$ on S, and A is a real nonsingular curve of degree 6 on $\mathbb{P}^2$. The rigid isotopic classification of such curves is known ([18]).
(2) If $S \cong U(2)$ and (X, τ) is non-degenerate, then $Y := X/\tau \cong \mathbb{P}^1 \times \mathbb{P}^1$, and (i) $\theta = -1$ on S or (ii) $S_+ \cong \langle -4 \rangle$ and $S_- \cong \langle 4 \rangle$. If (i) (called "$\mathbb{H}$ *case*"), then $Y(\mathbb{R})$ is hyperboloid or spin. If (ii), then $Y(\mathbb{R})$ is ellipsoid.
(3) If $S \cong \langle 2 \rangle \oplus \langle -2 \rangle$, then $\theta = -1$ on S.
(4) If $S \cong U$, then $\theta = -1$ on S.

In this article we mention only (2)-(i) case, i.e., $\mathbb{H}$ case (hyperboloid or spin) and (3) case (, then $X/\tau \cong \Sigma_1$). See §7 for $\mathbb{H}$ case, and §8 for Σ_1 case.

3 Integral involutions with conditions

Let L be the K3 lattice, i.e., even unimodular lattice of signature $(3, 19)$. Fix a primitive embedding of 2-elementary hyperbolic lattice S in L. It is unique up to automorphisms of L (For the details, see [31], §1). Let $\Delta(S, L)^{(-4)}$ be the set of all elements δ_1 in S such that $\delta_1^2 = -4$ and there exists $\delta_2 \in S_L^\perp$ such that $(\delta_2)^2 = -4$ and $\delta = (\delta_1 + \delta_2)/2 \in L$. Then $\delta^2 = -2$. All elements $\delta_1 \in \Delta(S, L)^{(-4)}$ are roots of S since $-4 = (\delta_1)^2$ divides $2(\delta_1, S)$ because $(\delta_1, S) \subset 2\mathbb{Z}$. Let $W^{(-4)}(S, L) \subset O(S)$ be the group generated by reflections in all roots from $\Delta(S, L)^{(-4)}$, and $W^{(-4)}(S, L)_{\mathcal{M}}$ the stabilizer subgroup of $\mathcal{M}$ in $W^{(-4)}(S, L)$. The set $\Delta(S, L)^{(-4)}$ is invariant with respect to $W^{(-2)}(S)$. It follows that the $W^{(-4)}(S, L)_{\mathcal{M}}$ is generated by reflections s_{δ_1} in $\delta_1 \in \Delta(S, L)^{(-4)}$ such that the the hyperplane $(\delta_1)_S^\perp \otimes \mathbb{R}$ in $S \otimes \mathbb{R}$ intersects the interior of $\mathcal{M}$.

The "real" analogy of the group $W^{(-4)}(S, L)_{\mathcal{M}}$ is the subgroup G defined below. Let θ be an involuion of S. We set G ($\subset W^{(-4)}(S, L)_{\mathcal{M}}$) to be the subgroup generated by all reflections s_{δ_1} in elements $\delta_1 \in \Delta(S, L)^{(-4)}$ which are contained either in S_+ or in S_- (i.e., s_{δ_1} should commute with θ) and such that $s_{\delta_1}(\mathcal{M}) = \mathcal{M}$.

We consider an *integral involution* (L, φ, S) *with condition* (S, θ) ([20]) which satisfies:

(RSK3) *L is even unimodular of signature* $(3, 19)$ *and the lattice L^{φ} is hyperbolic (of signature* $(1, t_{(-)})$*).*

Remark that a real K3 surface (X, φ) with a non-symplectic involution τ of type (S, θ) corresponds to an integral involution (L, φ, S) with condition (S, θ), where we set $L := H_2(X; \mathbb{Z})$ and φ is the action of φ on L. Then the integral involution (L, φ, S) satisfies (RSK3) (e.g. see [15] or Sect. 3.10 in [20]).

Definition 1 Two integral involutions (L, φ, S) and (L', φ', S) with condition (S, θ) to be *isomorphic with respect to the group G* if there exists an isomorphism $\xi : L \to L'$ of lattices such that $\xi\varphi = \varphi'\xi$ and $\xi|S$ belongs to the group G above.

Let $\mathrm{In}(S, \theta, G)$ denote the set of isomorphism classes (with respect to the group G) of integral involutions (L, φ, S) with condition (S, θ) satisfying (RSK3).

4 Moduli of $(\mathcal{D}\mathbb{R})$-non-degenerate real K3 surfaces with non-symplectic involutions

Definition 2 Two real K3 surfaces (X, τ, φ) and (X', τ', φ') with non-symplectic involutions of type (S, θ) are *isomorphic with respect to the group G* (see §3), if there exists an isomorphism $f : X \to X'$ such that $f\tau = \tau' f$, $f\varphi = \varphi' f$ and $f_*|S \in G$.

By monodromy consideration, two real K3 surfaces (X, φ, τ) and (X', φ', τ') with non-symplectic involutions of type (S, θ) which belong to one connected component of moduli give isomorphic integral involutions with condition (S, θ). Thus, we have *the natural map* from the set of connected components of moduli of triplets (X, φ, τ) to the set $\mathrm{In}(S, \theta, G)$.

Using Global Torelli Theorem for K3 surfaces [23] and epimorphicity of Torelli map for K3 surfaces [16], we can prove the following main theorem.

Theorem 3 ([22]) *The natural map above gives the one to one correspondence between the connected components of moduli of $(\mathcal{D}\mathbb{R})$-non-degenerate real K3 surfaces (X, τ, φ) with non-symplectic involutions of type (S, θ) and the set $In(S, \theta, G)$.*

This result is similar to Theorem 3.10.1 in [18] about moduli of real polarized K3 surfaces. Such statements reduce the problem of description of connected components of moduli of real algebraic varieties to a purely arithmetic problem.

Definition 4 (DPN pairs and DPN surfaces [19], [21], [1], [2], [6]) If Y is a non-singular surface, $A \in |-2K_Y|$ a non-singular curve, and $q_Y = 0$, then the double covering of Y ramified along A gives a K3 surface X with a non-symplectic involution (the covering transformation). A pair (Y, A) with these properties is called a *right DPN pair*, and the surface Y is called *a right DPN-surface.*

We mention that a 'general', not necessarily 'right', DPN-pair is a pair (Y, A) where Y is a non-singular surface with $q_Y = 0$, $A \in |-2K_Y|$ and A has only ADE-singularities; then the surface Y is called a DPN-surface.

Like K3 surfaces with non-symplectic involutions, we call a right DPN-pair (Y, A) *($\mathcal{D}$)-degenerate* if the corresponding K3 surface with non-symplectic involution (X, τ) is $(\mathcal{D})$-degenerate.

By a *real right DPN-pair* (Y, A, θ) we mean (Y, θ) is a non-singular projective algebraic surface with an anti-holomorphic involution θ, and $A \in |-2K_Y|$ is a non-singular curve such that $\theta(A) = (A)$. We call a real right DPN-pair (Y, A, θ) *($\mathcal{D}\mathbb{R}$)-degenerate* if the corresponding real K3 surface with non-symplectic involution (X, τ, φ) is $(\mathcal{D}\mathbb{R})$-degenerate.

There exist two real double coverings of Y ramified along A, which are two real K3 surfaces with non-symplectic involutions (X, τ, φ) and $(X, \tau, \widetilde{\varphi})$ where $\widetilde{\varphi} := \tau\varphi = \varphi\tau$. They both define the same real right DPN-pair $(Y := X/\tau,\ A := X^\tau,\ \theta := \varphi \bmod \tau)$.

Definition 5 (related real K3 surfaces, positive real right DPN-pair) We say these two real K3 surfaces with non-symplectic involutions (X, τ, φ) and $(X, \tau, \widetilde{\varphi})$ are *related.* A real right DPN-pair (Y, A, θ) together with a choice of one (between two) real K3 surface with non-symplectic involution (X, τ, φ) such that its quotient by τ gives (Y, A, θ) is called a *positive real right DPN-pair.*

We denote a positive real right DPN-pair by $(Y, A, \theta)^+$. Then the *related positive real right DPN-pair* will be denoted by $(Y, \theta, A)^-$. If $(Y, A, \theta)^+$ is given by (X, τ, φ), then the related positive DPN-pair is given by $(X, \tau, \widetilde{\varphi})$. We can define *positive real right DPN-pairs of type* (S, θ) and an *isomorphism of positive real right DPN-pairs with respect to the group G defined by* (S, θ). Obviously, an isomorphism of real K3 surfaces with non-symplectic involutions (X, τ, φ) and (X', τ', φ') defines the corresponding isomorphism of the related real K3 surfaces with non-symplectic involutions $(X, \tau, \widetilde{\varphi})$ and $(X', \tau', \widetilde{\varphi'})$. Moreover, the type (S, θ) and the group G don't change for related real K3 surfaces with non-symplectic involutions. Moreover, we can see related positive real right DPN-pairs are $(\mathcal{D}\mathbb{R})$-degenerate simultaneously.

We formulate an equivalent theorem from Theorem 3:

Theorem 6 ([22]) *The natural map gives the one to one correspondence between connected components of moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate positive real right DPN-pairs* $(Y, \theta, A)^+$ *of type* (S, θ) *and the set* $In(S, \theta, G)$.

The integral involutions (L, φ, S) and $(L, \widetilde{\varphi}, S)$ corresponding to related positive real right DPN-pairs $(Y, A, \theta)^+$ and $(Y, A, \theta)^-$ are related as $\widetilde{\varphi} = \tau\varphi$. Recall that τ acts as $+1$ on S and as -1 on $S_L^\perp$. Thus, we say naturally integral involutions (L, φ, S) and $(L, \widetilde{\varphi}, S)$ are *related.* From Theorem 6 we get

Theorem 7 (moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate real right DPN-pairs, [22]) *The natural map gives the one to one correspondence between connected components of moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate real right DPN-pairs* (Y, A, θ) *of type* (S, θ) *and the set* $In(S, \theta, G)/\{1, \tau\}$ *of pairs* $\{(L, \varphi, S), (L, \widetilde{\varphi}, S)\}$ *of isomorphism classes (with respect to G) of related integral involutions.*

5 Invariants of integral involutions with conditions

Let S be a 2-elementary and hyperbolic (of signature $(1, t_{(-)}(S))$), even lattice having a primitive embedding $S \subset L$ to the K3 lattice L and such that there exists an involution τ of L with $L^\tau = S$. (The last property is equivalent for S to be 2-elementary.) By [18],Th.3.6.2, the isomorphism class of S is determined by the triplet of invariants $(r(S), a(S), \delta(S))$, where $r(S) = 1 + t_{(-)}(S)$ is the rank of S, $a(S)$ is defined by $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}$, and $\delta(S)$ $(= 0, 1)$ is the *parity* of the discriminant form of S, namely, $\delta(S) = 0$ if and only if $(x^*)^2 \in \mathbb{Z}$ for any $x^* \in S^*$. All possible triplets $(r, a, \delta) = (r(S), a(S), \delta(S))$ are presented in Figure 2 in [21].

Let (X, τ) be a K3 surface with a non-symplectic involution of type S, and X^τ be the fixed point set of τ. Then, we have the following interesting results ([19] and [21]).

$$X^\tau = \begin{cases} \emptyset \ \text{(and } Y \text{ is Enriques)} & \text{if } (r(S), a(S), \delta(S)) = (10, 10, 0); \\ C_1 + C_1{}' & \text{if } (r(S), a(S), \delta(S)) = (10, 8, 0); \\ C_{g(S)} + E_1 + \cdots + E_{k(S)} & \text{otherwise;} \end{cases} \tag{5.1}$$

where $g(S) = (22 - r(S) - a(S))/2$, $k(S) = (r(S) - a(S))/2$ and C_g denote a curve of genus g and $E_i \cong \mathbb{P}^1$. We have

$$X^\tau \sim 0 \pmod 2 \text{ in } H_2(X, \mathbb{Z}) \quad \text{if and only if} \quad \delta(S) = 0. \tag{5.2}$$

The dimension of moduli of pairs (X, τ) and the corresponding DPN-pairs (Y, A) is equal to $20 - r(S)$.

We now consider the type (S, θ) of a real K3 surface with a non-symplectic involution (X, τ, φ). Any θ for which S^θ is negative definite and does not contain elements $x \in S^\theta$ with $x^2 = -2$ can be taken as a type. See [6] about some results in this direction. In this article we shall consider lattices S with $r(S) \leq 2$. Then the problem of finding possible types (S, θ) is very simple.

Assume that the type (S, θ) is fixed. We denote $S_+ = S^\theta$ and $S_- = S_\theta$. We shall use invariants

$$s = \mathrm{rk}\, S,\ p = \mathrm{rk}\, S_+ \tag{5.3}$$

Then S has the signature $(s_{(+)}, s_{(-)}) = (1, s-1)$ and S_+ has the signature $(p_{(+)}, p_{(-)}) = (0, p)$.

We say two integral involutions (L, φ, S) and (L', φ', S) of type (S, θ) *have the same genus with respect to the group* G if there exists an automorphism $\xi : S \to S$ from G which can be continued to an isomorphism $(L, \varphi, S) \otimes \mathbb{R} \to (L', \varphi', S) \otimes \mathbb{R}$ over $\mathbb{R}$, and an isomorphism $(L, \varphi, S) \otimes \mathbb{Z}_p \to (L', \varphi', S) \otimes \mathbb{Z}_p$ over the ring $\mathbb{Z}_p$ of p-adic integers for any prime p. All the genus invariants (for an arbitrary even lattice S with an involution θ) were found in [20] together with necessary and sufficient conditions of existence. In many cases a genus has only one isomorphism class. Then the genus invariants give isomorphism invariants.

We assume that the integral involution (L, φ, S) satisfies the condition (RSK3) above. Then the only real invariant of (L, φ, S) is

$$r = \mathrm{rk}\, L^\varphi = 1 + t_{(-)}. \tag{5.4}$$

Below we describe the genus invariants of the integral involution (L, φ, S) of type (S, θ). To simplify notations, we temporarily denote $L_+ = L^\varphi$ and $L_- = L_\varphi$.

Since L is unimodular, we have

$$A_{L_\pm} = {L_\pm}^*/L_\pm \cong L/(L_+ \oplus L_-) \cong (\mathbb{Z}/2\mathbb{Z})^a \tag{5.5}$$

where $a \geq 0$ is an integer. It is one of the most important genus invariants.

We also have another genus invariants

$$\delta_\varphi = \begin{cases} 0 & \text{if } (x, \varphi(x)) \equiv 0 \mod 2\ \forall x \in L \\ 1 & \text{otherwise} \end{cases}, \tag{5.6}$$

and

$$\delta_{\varphi S} = \begin{cases} 0 & \text{if } (x, \varphi(x)) \equiv (x, s_\varphi) \mod 2\ \forall x \in L \\ & \text{for some element } s_\varphi \text{ in } S \\ 1 & \text{otherwise} \end{cases} . \tag{5.7}$$

If $\delta_{\varphi S} = 0$, then the element s_φ occurring in the definition of $\delta_{\varphi S}$ is uniquely defined modulo $2S$. It is called *the characteristic element of the involution* φ.

For convenience, we often divide (L, φ, S) into the following 3 *types*.

Type 0: $\delta_{\varphi S} = 0$ and $\delta_\varphi = 0$;
Type Ia: $\delta_{\varphi S} = 0$ and $\delta_\varphi = 1$;
Type Ib: $\delta_{\varphi S} = 1$.

For $x_\pm \in S_\pm$ we put

$$\delta_{x_\pm} = \begin{cases} 0 & \text{if } (x_\pm, L_\pm) \equiv 0 \mod 2, \\ 1 & \text{otherwise.} \end{cases} \tag{5.8}$$

Equivalently, $\delta_{x_\pm} = 0$ iff $\frac{1}{2}x_\pm \in 2L_\pm^*$.

Then we have the function $\delta_\pm : S_\pm \to \mathbb{Z}/2\mathbb{Z}$ where $x_\pm \mapsto \delta_{x_\pm}$. We set

$$H_\pm := \delta_\pm^{-1}(0)/2S_\pm \subset S_\pm/2S_\pm.$$

These subgroups are equivalent to the invariants $\delta_{x_\pm}$. We have a more exact range for the subgroups $H_\pm$:

$$\Gamma_\pm = (2S)_\pm/2S_\pm \subset H_\pm \subset 2(S_\pm{}^* \cap (\frac{1}{2}S_\pm))/2S_\pm \cong (S_\pm{}^* \cap (\frac{1}{2}S_\pm))/S_\pm = A_{S_\pm}^{(2)} \subset A_{S_\pm}.$$

Here $(2S)_\pm$ are the orthogonal projections of $2S \subset S_+ \oplus S_-$ to $S_\pm$ respectively. This projections also give the graph Γ of the isomorphism γ of the groups Γ_+ and Γ_-. The $A_{S_\pm}^{(2)}$ denote the subgroup of $A_{S_\pm}$ generated by all elements of order two. Let

$$H := H_+ \oplus_\gamma H_- := (H_+ \oplus H_-)/\Gamma.$$

For simplicity we identify $H_\pm = H_\pm \mod \Gamma \subset H$.

Since L is unimodular,

$$\delta_{x_\pm} = 0 \Leftrightarrow \exists x'_\pm \in L_\mp : \frac{1}{2}(x_\pm + x'_\pm) \in L. \tag{5.9}$$

The elements $x'_\pm$ are defined by the elements $x_\pm$ uniquely modulo $2L_\mp$; this enables us, for elements $x_+ \in S_+$ and $x_- \in S_-$ for which $\delta_{x_+} = \delta_{x_-} = 0$, to define the invariant

$$\frac{1}{2}(x_+, x'_-) \mod 2 = -\frac{1}{2}(x'_+, x_-) \mod 2 \ (\in \mathbb{Z}/2\mathbb{Z}).$$

We can define a finite quadratic form $q_\rho : H \to \frac{1}{2}\mathbb{Z}/2\mathbb{Z}$ by

$$q_\rho[(x_+, x_-)] := q_{S_+}(\frac{1}{2}x_+) + \frac{1}{2}(x_+, x'_-) - q_{S_-}(\frac{1}{2}x_-) \quad \text{for } x_\pm \in H_\pm.$$

Moreover (see [20]), when $\delta_{\varphi S} = 0$, we define *the characteristic element* v by

$$v := s_\varphi \in H = H_+ \oplus_\gamma H_- \subset (S_+ \oplus S_-)/2S.$$

The element v should be *characteristic for the quadratic form* q_ρ, namely, $q_\rho(x, v) \equiv q_\rho(x, x) \pmod 1$ for any $x \in H$. The element v is zero if $\delta_\varphi = 0$, and v is not zero if $\delta_\varphi = 1$ and $\delta_{\varphi S} = 0$.

Finally, we have the following very useful theorems:

Theorem 8 ([20]) *Two integral involutions with condition (S, θ) have the same genus with respect to G if and only if the corresponding two lists of genus invariants*

$$(r, a; H_+, H_-, q_\rho; \delta_\varphi, \delta_{\varphi S}, v). \tag{5.10}$$

are conjugate by the group G of the condition (S, θ).

Theorem 9 ([20]) *The invariants* (5.10) *give complete genus invariants of integral involutions (L, φ, S) of type (S, θ) in the set $In(S, \theta, G)$.* **Conditions 1.8.1 and 1.8.2** *in* [20] *are necessary and sufficient for the existence of an integral involution in $In(S, \theta, G)$ with the genus invariants.*

Now we discuss related involutions. We denote by

$$\big(r(\varphi), a(\varphi); H(\varphi)_+, H(\varphi)_-, q_{\rho(\varphi)}; \delta_\varphi, \delta_{\varphi S}, v(\varphi)\big) \tag{5.11}$$

the genus invariants (5.10) for φ.

Theorem 10 ([22]) *We have the following relations between genus invariants of the related involutions (L, φ, S) and $(L, \widetilde{\varphi} = \tau\varphi, S)$ (for undefined symbols below, see* [22]*).*

$$r(\varphi) + r(\tau\varphi) = 22 - s + 2p,$$

$$a(\tau\varphi) - a(\varphi) = a(S) - 2a_{H(\varphi)} + 2\,\mathrm{rk}\,\rho(\varphi),$$

$$s_\varphi + s_{\tau\varphi} \equiv s_\theta \in S \mod 2L,$$

where $\mathrm{rk}\,\rho$ *means the rank of the matrix which gives ρ in some bases of H_+ and H_- over the field $\mathbb{Z}/2\mathbb{Z}$, and $s_\theta \in S$ is the characteristic element of (S, θ) defined by the* *property* $2(x, \theta(x)) \equiv (x, s_\theta) \mod 2$ *for any* $x \in S^*$ *(the* s_θ *is defined* $\mod 2S$*). In particular,*

$$\delta_{\varphi S} = \delta_{(\tau\varphi)S}.$$

The $H(\varphi)_\pm$ *and* $H(\tau\varphi)_\pm$ *are orthogonal with respect to the discriminant bilinear form* $b_{S_\pm}$ *on* $A_{S_\pm}$ *respectively.*

Moreover, we have

$$l(A_{S_+}) + a(\varphi) - 2a_{H(\varphi)_+} = l(A_{S_-}) + a(\tau\varphi) - 2a_{H(\tau\varphi)_-}.$$

Theorem 10 permits to choose one from two related involutions by some conditions on their invariants. It helps the classification of the pairs of related involutions, which is important for the classification of the corresponding DPN-pairs.

Possible types of (S, θ) with $\operatorname{rk} S \le 2$. By classification in [18] and also in [19], [21], there are the following and only the following possibilities for S with $r(S) = \operatorname{rk} S \le 2$. We have

$$(r(S), a(S), \delta(S)) = (1,1,1),\ (2,2,0),\ (2,2,1),\ (2,0,0).$$

We consider all possible θ for these cases.

The case: $(r(S), a(S), \delta(S)) = (1,1,1)$. Then $S \cong \langle 2\rangle$. Since S_- should be hyperbolic, $S_- = S$ and $\theta = -1$ on S. Then $S_+ = \{0\}$. For this case $Y = X/\{1, \tau\} = \mathbb{P}^2$.

The case $(r(S), a(S), \delta(S)) = (2,2,0)$. Then $S \cong U(2)$. We have $Y = \mathbb{P}^1 \times \mathbb{P}^1$ in the non-degenerate case. Let us consider possible θ. Dividing form of S by 2, we get the unimodular lattice U. It follows that if $\operatorname{rk} S_- = 1$, then $S_- \cong \langle 4\rangle$ and $S_+ \cong \langle -4\rangle$. If $\operatorname{rk} S_- = 2$, then $\theta = -1$ on S. Thus, we get two cases

The case $S \cong U(2)$, the involution θ is -1 on S. We consider this case in §7. Then $Y = X/\{1, \tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ in the non-degenerate case. Moreover, $Y = X/\{1, \tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{R}$ is a hyperboloid, if $Y(\mathbb{R}) \ne \emptyset$.

The case $S \cong U(2)$, $S_- \cong \langle 4\rangle$, $S_+ \cong \langle -4\rangle$. For this case, $Y = X/\{1, \tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ and Y over $\mathbb{R}$ is an ellipsoid. Really, if $Y = \Sigma_2$, then any anti-holomorphic involution of Y acts as -1 in $H^2(Y, \mathbb{R}) = \mathbb{R}^2$, and then $\theta = -1$ in $S \otimes \mathbb{R}$.

The case $(r(S), a(S), \delta(S)) = (2,2,1)$. Then $S \cong \langle 2\rangle \oplus \langle -2\rangle$. Assume that $\operatorname{rk} S_- = 1$. Since the lattice $S(2^{-1}) \cong \langle 1\rangle \oplus \langle -1\rangle$ is unimodular and odd, it follows that $S_- = \langle 2\rangle$ and $S_+ \cong \langle -2\rangle$. Again the lattice S_+ has elements with square -2 which is impossible for θ. Thus, $\theta = -1$ on S and $S_+ = \{0\}$. We consider this case in §8. For this case $Y = X/\{1, \tau\} = \Sigma_1$.

The case $(r(S), a(S), \delta(S)) = (2,0,0)$. Then $S \cong U$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since S_- is hyperbolic, $\operatorname{rk} S_- = 1$ or 2. Let $\operatorname{rk} S_- = 1$. Since S is unimodular and even, $S_- \cong \langle 2 \rangle$ and $S_+ \cong \langle -2 \rangle$. Then S_+ has elements with square -2 which is impossible for θ. Thus, $\operatorname{rk} S_- = 2$ and $\theta = -1$ on S. Then $S_- = S$ and $S_+ = \{0\}$. For this case, $Y = X/\tau = \Sigma_4$.

6 Geometric interpretation of invariants of integral involutions

Here we discuss the geometric interpretation of invariants. We first mention the invariants $(r(\varphi), a(\varphi), \delta(\varphi)) = (r, a, \delta)$. We denote by S_g an oriented surface of the genus g.

We have (see Theorem 3.10.6 in [18]) for the real part of $X_\varphi(\mathbb{R}) = X^\varphi$ of X with the real structure defined by φ the same result as for the holomorphic non-symplectic involution τ:

$$X_\varphi(\mathbb{R}) = \begin{cases} \emptyset & \text{if } (r,a,\delta) = (10,10,0); \\ T_1 \amalg T_1 & \text{if } (r,a,\delta) = (10,8,0); \\ T_g \amalg (T_0)^k & \text{otherwise;} \end{cases} \tag{6.1}$$

where $g = (22 - r - a)/2, \quad k = (r-a)/2$ and

$$X_\varphi(\mathbb{R}) \sim s_\varphi \pmod 2 \text{ in } H_2(X, \mathbb{Z}). \tag{6.2}$$

Now we have the following interpretation of the invariant δ_{x_-}.

Theorem 11 ([28],[22]) *Let X be a compact Kähler surface with an anti-holomorphic involution φ, and $X(\mathbb{R})$ be the fixed point set of φ. We assume that $H_1(X;\mathbb{Z}) = 0$ and $X(\mathbb{R}) \neq \emptyset$. We set $L = H_2(X;\mathbb{Z})$ and $L_\varphi = \{x \in L \mid \varphi_*(x) = -x\}$. Let C be a 1-dimensional complex submanifold of X with $\varphi(C) = C$, and $C(\mathbb{R})$ be the fixed point set of φ on C. Let $[C]$ $(\in L_\varphi)$ denote the homology classes represented by C. Then we have:*

$$[C]\cdot x \equiv 0 \pmod 2 \forall x \in L_\varphi \textit{ if and only if } [C(\mathbb{R})] = 0 \textit{ in } H_1(X(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}).$$

Remark 12 Following (5.8), we have $\delta_{[C]} = 0$ iff $[C]\cdot x \equiv 0 \pmod 2 \forall x \in L_\varphi$.

Proof of Theorem 11. Suppose that $[C(\mathbb{R})] \neq 0$ in $H_1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$. Then there exists an embedded circle D on $X(\mathbb{R})$ such that $[C(\mathbb{R})]\cdot[D] \neq 0$, where $[C(\mathbb{R})]\cdot[D]$ means the $\mathbb{Z}/2\mathbb{Z}$-intersection number in $X(\mathbb{R})$. We set $E_- = \{x \in H^2(X;\mathbb{Z}) \mid \varphi^*(x) = -x\}$. E_- is the Poincaré dual to L_φ. By Théorème 2.4 in [17], there exists a surjective canonical morphism

$$\varphi_X : E_- \to H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$$

such that

$$(\varphi_X(\gamma), \varphi_X(\gamma')) \equiv Q(\gamma,\gamma') \pmod 2 \ \ \forall\gamma,\gamma' \in E_-,$$

where $(\ ,\ )$ and Q are the forms induced by the cup products on $H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$ and $H^2(X;\mathbb{Z})$. Moreover, by Théorème 2.5 in [17], the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pic}(X)^G & \xrightarrow{c_1} & E_- \\ \alpha\downarrow & & \varphi_X\downarrow \\ H^1_{\mathrm{alg}}(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{i} & H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z}), \end{array}$$

where α is defined as in [17],p.562 (see also below), c_1 is the first Chern class map, and i is the inclusion map. In the sequel, A^P denotes the Poincaré dual element to a (co)homology class A. Since φ_X is surjective, there exists γ ($\in E_-$) such that $\varphi_X(\gamma) = [D]^P$ ($\in H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$). We consider the *divisor class* $[C]$ in $\mathrm{Pic}(X)^G$. Then, as is well known, its first Chern class $c_1([C])$ is the Poincaré dual to x_-. By the definition (see [17]) of α, we see

$$\alpha([C]) = \eta(C) = [C(\mathbb{R})]^P \ (\in H^1_{\mathrm{alg}}(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})).$$

On the other hand, by the above commutative diagram, we see

$$\alpha([C]) = \varphi_X(c_1([C])) = \varphi_X(x_-^P).$$

Hence, we have $[C(\mathbb{R})]\cdot[D] = ([C(\mathbb{R})]^P, [D]^P) = (\varphi_X(x_-^P), \varphi_X(\gamma)) \equiv Q(x_-^P, \gamma) \pmod 2$ $= (x_-, \gamma^P)$. Thus we have $\delta_{x_-} = 1$. This completes the proof of the implication $\Longrightarrow$. The converse assertion can be proved by the same argument as the proof of Lemma 2 in [26]. Suppose that $[C(\mathbb{R})] = 0$ in $H_1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$. Then $X(\mathbb{R})$ and $C(\mathbb{R})$ satisfy the conditions a) and b) of Remark 2.2 in [14]. By that remark, Lemma 2.3 is applicable to the involution $\varphi : X \to X$ and C. Hence, we see $(x_-)_{\mathrm{mod}\ 2}$ ($\in H_2(X;\mathbb{Z}/2\mathbb{Z})$) is orthogonal to $\mathrm{Im}\alpha_2$. Since $H_1(X;\mathbb{Z}) = 0$, as in the proof of Lemma 3.7 in [15], we have $\mathrm{Im}\alpha_2 = \{x \in H_2(X;\mathbb{Z}/2\mathbb{Z}) \mid \varphi_*(x) = x\}$. Thus we have $\delta_{x_-} = 0$. □

By similar arguments as above, we also have:

Proposition 13 ([22]) *Let (X,τ,φ) be a real K3 surface with a non-symplectic involution of type (S,θ) with non empty $A = X^\tau$. Then, if A is dividing, then $\delta_{\varphi S} = 0$.*

Proof. Suppose that $[A(\mathbb{R})] = 0$ in $H_1(A;\mathbb{Z}/2\mathbb{Z})$. Then $A(\mathbb{R})$ and A satisfy the condition a) in Sect. 2.2 of [14]. Since $A(\mathbb{R})$ is a disjoint union of circles, $w_1(A(\mathbb{R})) = 0$, and the condition b) is also satisfied. Using the notation of [14], we obtain that $l(P_{A(\mathbb{R})})$ realizes the nulls of the group $H_2(P_A;\mathbb{Z}/2\mathbb{Z})$. Since $\tau\varphi = \varphi\tau$, we have $\tau(X(\mathbb{R})) = X(\mathbb{R})$, and $A(\mathbb{R}) = A \cap X(\mathbb{R})$.

The tangent bundle $T(A)$, which is real 2-dimensional, is isomorphic to the normal bundle $N(A)$ of A in X (because X is a K3 surface), and the tangent bundle $T(A(\mathbb{R}))$ is isomorphic to the normal bundle $N(A(\mathbb{R}))$ of $A(\mathbb{R})$ in $X(\mathbb{R})$. Since $l(P_{A(\mathbb{R})})$ realizes the nulls of the group $H_2(P_A,\mathbb{Z}/2\mathbb{Z})$, by Lemma 1 in Sec. 2.3 of [14], the class $[X(\mathbb{R})]$ in $H_2(X,\mathbb{Z}/2\mathbb{Z})$ is orthogonal to $\operatorname{Im}\alpha_2$ in $H_2(X,\mathbb{Z}/2\mathbb{Z})$ with respect to the intersection pairing where α_2 is the homomorphism in the Smith exact sequence for (X,τ) (see [14]). Since $H_1(X,\mathbb{Z}) = 0$, as in the proof of Lemma 3.7 in [15], we have $\operatorname{Im}\alpha_2 = \{x \in H_2(X,\mathbb{Z}/2\mathbb{Z}) \mid \tau_*(x) = x\}$. It follows, $\delta_{\varphi S} = 0$. □

Applying Donaldson's trick [7], like in [6], we can consider φ as a holomorphic involution and τ as an anti-holomorphic. Then the converse statement to Proposition 13 follows from Theorem 11. See also §9.

7 Connected components of moduli of real non-singular curves of bidegree (4, 4) on $\mathbb{H}$

Here we consider the case $S \cong U(2)$ and $\theta = -1$ on S. Then $Y = X/\{1,\tau\} = \mathbb{P}^1\times\mathbb{P}^1$ in the non-degenerate case. $A = X^\tau$ is a non-singular curve of bidegree $(4,4)$ in Y.

Let $S = \mathbb{Z}e_1 + \mathbb{Z}e_2$ where $e_1^2 = e_2^2 = 0$ and $(e_1, e_2) = 2$. The generators e_1 and e_2 of S are classes of preimages of pt $\times\, \mathbb{P}^1$ and $\mathbb{P}^1 \times$ pt respectively.

If Y is a hyperboloid, namely, has $(usual, usual)$ structure, then A is the zero set of a real bi-homogeneous polynomial $P(x_0 : x_1; y_0 : y_1)$ of bidegree $(4,4)$. We call a pair (A,φ) a ***positive curve***. For (A,φ), equivalently for $A^+ = \pi(X_\varphi(\mathbb{R}))$, we can choose P by the condition that $A = \{P = 0\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and $A^+ = \{P \geq 0\}$ on $\mathbb{R}\mathbb{P}^1 \times \mathbb{R}\mathbb{P}^1$. The polynomial P is defined up to $\mathbb{R}_{++} \times ((PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z})$. Here $\mathbb{R}_{++}$ denote all positive real numbers. Thus, classification of connected components of moduli of positive curves A^+ on a hyperboloid is equivalent to the description of connected components of

$$((\mathbb{R}^{25} - Discr)/\mathbb{R}_{++})\ /\ ((PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z})$$

where the discriminant *Discr* is the set of all polynomials giving singular (over $\mathbb{C}$) curves. The group $(PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}$ has 8 connected components and has dimension 6 over $\mathbb{R}$.

If Y has one of spin structures, then $Discr$ has codimension two. Hence the moduli of positive curves and the moduli of curves are connected.

For $(S,\theta)=(U(2),-1)$, we have $G=\{\text{identity},\ g\}\cong\mathbb{Z}/2\mathbb{Z}$, where $g(e_1)=e_2$ and $g(e_2)=e_1$.

We have $s=2$, $p=0$. $A_{S_-}:=S_-^*/S_-=S^*/S$ is generated by $e_1^*=\frac{1}{2}e_2$ and $e_2^*=\frac{1}{2}e_1$, and hence it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and $l(A_{S_-})=2$. We have $A_{S_+}=0$, $H_+=0$ and $H=H_-$ ($\subset S/2S$) is one of the following 5 subgroups;

$$0,\ [e_1],\ [e_2],\ [h]=[e_1+e_2],\ S/2S=[e_1,e_2],$$

where we set $h=e_1+e_2$ and consider e_1, e_2 and h mod $2S$. Since $H_+=0$, we have $q_\rho=(-q_{S_-})|H_-$.

From such considerations, we see that genus of an integral involutions (L,φ,S) of the type $(U(2),-1)$ satisfying (RSK3) is determined by the data $(r,a,H,\delta_{\varphi S},v)$. Moreover, in this case, each genus determines the unique isomorphism class (see [22]).

Using Theorem 9, we obtain all possible data

$$(r,a,H,\delta_{\varphi S},v) \tag{7.1}$$

are given in Figures (図) 1 and 2.

Since $G=\{\text{identity},\ g\}$, the triplets $(r,a,[e_1])$ and $(r,a,[e_2])$ represent the same isomorphism class for each Type (0, Ia and Ib). The other different triplets represent different isomorphism classes.

The relations between related involutions are as follows. Since $\delta(S)=0$ and $\theta=-1$ on S, then $s_\theta=0 \mod 2L$. Hence, by Theorem 10, we have:

$$r(\varphi)+r(\tau\varphi)=20;\quad a_{H(\varphi)}+a_{H(\tau\varphi)}=2;\quad a(\varphi)-a_{H(\varphi)}=a(\tau\varphi)-a_{H(\tau\varphi)}; \tag{7.2}$$

$$H(\tau\varphi)=H(\varphi)^\perp \text{ w. r. t. } b_{S_-} \text{ on } A_{S_-};\quad \delta_{\varphi S}=\delta_{\tau\varphi S};\quad s_\varphi\equiv s_{\tau\varphi}\mod 2L. \tag{7.3}$$

Hence, involutions of Type 0 (resp. Type Ib) with $H=0$ (11 (resp. 39) classes) are related to involutions of Type 0 (resp. Type Ib) with $H=S/2S$. Involutions of Type 0 (resp. Type Ia, Type Ib) with $H=[h]$ (11 (resp. 12, 36) classes) are related to involutions of Type 0 (resp. Type Ia, Type Ib) with $H=[h]$. (More precisely, the class (r,a) is related to the class with $(20-r,a)$, too.) Involutions of Type 0 (resp. Type Ia, Type Ib) with $H=[e_i]$ (9 (resp. 11, 30) classes) are related to involutions of Type 0 (resp. Type Ia, Type Ib) with $H=[e_i]$. (More precisely, the class (r,a) is related to the class $(20-r,a)$.)

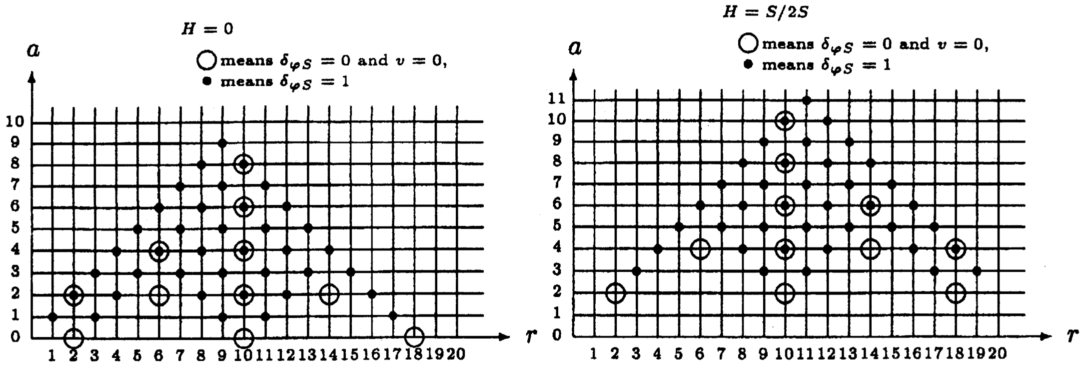

図 1: ℍ: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = 0$ and $H = S/2S$

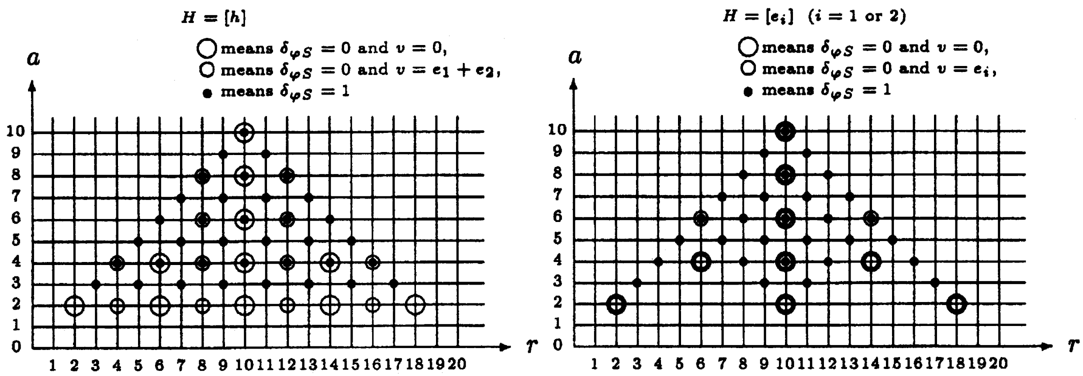

図 2: ℍ: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h]$ and $H = [e_i]$ $(i = 1$ or $2)$

Thus, there are $50 (= 11 + 39)$ classes (i.e., connected components of moduli of positives curves of bidegree $(4,4)$ with $(S, \theta) = (U(2), -1)$) with $H = 0$ (or $H = S/2S$ respectively), $59 (= 11 + 12 + 36)$ classes with $H = [h]$, and $50 (= 9 + 11 + 30)$ classes with $H = [e_i]$.

Moreover, if we identify related involutions, there are $50 (= 11 + 39)$ classes (i.e., connected components of moduli of real non-singular curves of bidegree $(4,4)$ with $(S, \theta) = (U(2), -1)$) with $H = 0$ (or $H = S/2S$), $34 (= 8 + 6 + 20)$ classes with $H = [h]$, and $32 (= 7 + 8 + 17)$ classes with $H = [e_i]$.

If two positive curves are in one connected component of moduli or are related, the real structure on Y stays the same. By (6.1) and (6.2), the A^+ is empty, if and only if $(r, a, \delta_\varphi) = (10, 10, 0)$. It follows that the component $(r, a, H, \delta_{\varphi S}, \delta_\varphi) = (10, 10, [h], 0, 0)$ corresponds to the real structure (spin, spin), the component $(10, 10, [e_i], 0, 0)$ corresponds to the real structure (usual, spin) (or (spin, usual)). And all the

$\langle 1\langle m\rangle \sqcup n\rangle$, where $(m,n) = (9,0),\ (5,4),\ (1,8),\ (8,0),\ (5,3),\ (4,4),\ (1,7)$; or $m \geq 1,\ n \geq 0$ and $m+n \leq 7$.
$\langle m\rangle$, where $0 \leq m \leq 9$.
$\langle 1\langle 1\rangle \sqcup 1\langle 1\rangle\rangle$
$\langle l_1 + l_2,\ m,\ l_1 + l_2,\ n\rangle$, where $0 \leq m \leq n$ and $m+n \leq 8$.
$\langle 4(l_1 + l_2)\rangle$
$\langle l_1,\ m,\ l_1,\ n\rangle$, where $(m,n) = (0,8),\ (4,4),\ (0,7),\ (3,4)$; or $0 \leq m \leq n$ and $m+n \leq 6$.
$\langle 4(l_1)\rangle$
$\langle 2(l_1 + 2l_2)\rangle$

表 1: All isotopy types of real non-singular curves of bidegree $(4,4)$ on a hyperboloid.

remaining components correspond to the real structure (usual, usual) (namely, hyperboloid). The component $(10, 10, S/2S, 0, 0)$ consists of empty A^+ on hyperboloid.

The isotopy classification of real non-singular curves of bidegree $(4,4)$ on a hyperboloid was obtained by Gudkov [8]. Zvonilov [34] clarified all the *complex schemes* of curves of bidegree $(4,4)$ on hyperboloid and ellipsoid, where *complex schemes* mean real schemes (i.e. real isotopy types) with dividingness and their complex orientations (if dividing). The notions: *torsion* (s,t) $(\in \mathbb{Z}\times\mathbb{Z})$ of a connected component of $A(\mathbb{R})$, *oval* or *non-oval* and *odd (even) branch* are well-known. See [8], [25], [26]. We quote the isotopy classification of non-singular curves of bidegree $(4,4)$ on hyperboloid from [34] in TABLE(表) 1 where we use notations due to Viro [32] and Zvonilov [34], and l_1 and l_2 denote non-ovals with torsions $(1,0)$ and $(0,1)$ respectively.

Let $A(\mathbb{R})$ be a curve on hyperboloid, and A^+ and A^- be the halfs of $\mathbb{RP}^1 \times \mathbb{RP}^1 \setminus A(\mathbb{R})$. (If $A^+ = \pi(X_\varphi(\mathbb{R}))$, then $A^- = \pi(X_{\tau\varphi}(\mathbb{R}))$.) When a curve $A(\mathbb{R})$ on hyperboloid has only ovals or $A(\mathbb{R}) = \emptyset$, A^+ or A^- contains the outermost component. Thus we divide isotopy types of positive curves A^+ into the following 4 cases: **(i)** $A(\mathbb{R})$ has only ovals or $A(\mathbb{R}) = \emptyset$, and A^+ contains the outermost component. (In this case we say A^+ is *outer*.) **(ii)** $A(\mathbb{R})$ has only ovals or $A(\mathbb{R}) = \emptyset$, and A^+ does not contain the outermost component. (In this case we say A^+ is *inner*.) **(iii)** $A(\mathbb{R})$ has even branches. **(iv)** $A(\mathbb{R})$ has odd branches.

The subgroup $H = H_-$ is determined by the invariants δ_{e_1}, δ_{e_2} and δ_h. For the geometric interpretation of the invariant δ_{x_-}, we have Theorem 11. Thus, we have $H = 0$ if and only if A^+ is outer, $H = S/2S$ if and only if A^+ is inner, $H = [h]$ if and only if $A(\mathbb{R})$ has even branches, $H = [e_1]$ if and only if $A(\mathbb{R})$ has odd branches with odd s, and $H = [e_2]$ if and only if $A(\mathbb{R})$ has odd branches with odd t.

When A is a dividing curve on hyperboloid with non-ovals, we define the number $\hat{l}$ ($\in \mathbb{Z}/2\mathbb{Z}$) as follows (see [24]): Fix a complex orientation of $A(\mathbb{R})$. When the number of non-ovals of $A(\mathbb{R})$ is 2, we define $\hat{l} = 0$ if the complex orientations of the 2 non-ovals are different in $\mathbb{RP}^1 \times \mathbb{RP}^1$, $\hat{l} = 1$ if otherwise. When the number of non-ovals of $A(\mathbb{R})$ is 4, we fix a non-oval E and define $\hat{l} = 0$ if the number of non-ovals whose complex orientations are the same as E in $\mathbb{RP}^1 \times \mathbb{RP}^1$ is even, $\hat{l} = 1$ if odd.

The following proposition is a corollary of Proposition 13. But we can prove it independently by some results in [24].

Proposition 14 ([27], [11], [22]) *Let A be a non-singular real curve of bidegree $(4,4)$ on hyperboloid. If A is dividing, then the positive curves (A,φ) satisfies $\delta_{\varphi S} = 0$.*

Moreover, we have the following interpretation of v when A is a dividing curve with non-ovals on hyperboloid:
$v = 0$ if and only if $\hat{l} = 0$,
$v = h \pmod{2S}$ if and only if $A(\mathbb{R})$ has even branches and $\hat{l} = 1$,
$v = e_1 \pmod{2S}$ if and only if $A(\mathbb{R})$ has odd branches with odd s and $\hat{l} = 1$,
$v = e_2 \pmod{2S}$ if and only if $A(\mathbb{R})$ has odd branches with odd t and $\hat{l} = 1$.

By the geometric interpretations above and Zvonilov's classification [34], we see that the complex scheme of A is unique for each value of the invariant (7.1), equivalently, each connected component of moduli of positive curves A^+ on a hyperboloid. Recall that $A(\mathbb{R})$ is an empty curve on hyperboloid if $H = S/2S$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$, Y has the (spin,spin) structure if $H = [h]$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$, and Y has the (usual,spin) (or (spin,usual)) structure if $H = [e_i]$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$.

Moreover, there exists (due to Zvonilov [34]) a dividing curve on hyperboloid for every connected component with $\delta_{\varphi S} = 0$. Hence, the opposite statement of Proposition 14 is true:

Theorem 15 ([22]) *Let A be a non-singular real curve of bidegree $(4,4)$ on hyperboloid. Then, A is dividing or $A(\mathbb{R}) = \emptyset$, if and only if the positive curves (A, φ) (or A^+) has $\delta_{\varphi S} = 0$.* □

It turns out that if $\delta_{\varphi S} = 0$ and $H = [h]$, then (r, a) determines v. But when $\delta_{\varphi S} = 0$ and $H = [e_i]$, (r, a) does not always determine v. As stated above, when $\delta_{\varphi S} = 0$, $H = [e_i]$ and $(r, a, v) \neq (10, 10, 0)$, $v = 0$ if and only if $\hat{l} = 0$.

Thus, we finally get

Theorem 16 ([22]) *A connected component of moduli of a positive real non-singular curve A^+ of bidegree $(4,4)$ on hyperboloid is defined by the isotopy type of $A^+ \subset \mathbb{RP}^1 \times \mathbb{RP}^1$ (up to the action of $(PGL(2,\mathbb{R}) \times PGL(2,\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}$), dividingness of $A(\mathbb{R})$ in $A(\mathbb{C})$, and by the invariant $\hat{l} \mod 2$ defined by the complex orientation (if $A(\mathbb{R})$ has odd branches and is dividing).*

8 Connected components of moduli of real non-singular curves in $|-2K_{\Sigma_1}|$ on Σ_1

Here we consider the case $S \cong \langle 2\rangle \oplus \langle -2\rangle$ and $\theta = -1$ on S. Then we have $Y = X/\{1, \tau\} = \Sigma_1$. For the exceptional section of Σ_1 with square -1, we denote by e its preimage on X. Then $e^2 = -2$. We consider the contraction $\Sigma_1 \to \mathbb{P}^2$ of the exceptional section and denote by h the preimage in X of a line $l \subset \mathbb{P}^2$. Then $h^2 = 2$, $h \cdot e = 0$. We have $S = \mathbb{Z}h + \mathbb{Z}e$. Since $\theta = -1$, we have $S_+ = \{0\}$, $s = 2$, $p = 0$ and we see G is trivial. The group A_{S_-} is generated by $h^* = \frac{1}{2}h$ and $e^* = -\frac{1}{2}e$, and hence $A_{S_-} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $l(A_{S_-}) = 2$. The characteristic element of q_{S_-} is $h + e \pmod{2S}$. We have $A_{S_+} = 0$, $H_+ = 0$. Since $\Gamma_- = 0$, $H = H_-$ ($\subset S/2S$) is one of the following 5 subgroups:

$$0,\ [h],\ [e],\ [h+e],\ S/2S = [h, e],$$

where we consider h, e mod $2S$. Since $H_+ = 0$, we have $q_\rho = (-q_{S_-})|H_-$.

We see that the genus (hence, isomorphism class for Σ_1 case like $\mathbb{H}$ case) of an integral involutions (L, φ, S) of type $(\langle 2\rangle \oplus \langle -2\rangle, -1)$ satisfying (RSK3) is determined by the data

$$(r, a, H, \delta_{\varphi S}, v), \tag{8.1}$$

Using Theorem 9, we give all these possible data in Figures (図) 3 — 5.

By Theorem 10 about related involutions, we have

$$\begin{aligned} &r(\varphi) + r(\tau\varphi) = 20, \quad a(\varphi) - a_{H(\varphi)} = a(\tau\varphi) - a_{H(\tau\varphi)}, \\ &a_{H(\varphi)} + a_{H(\tau\varphi)} = 2, \quad H(\tau\varphi) = H(\varphi)^\perp \text{ w.r.t. } b_{S_-}, \\ &\delta_{\varphi S} = \delta_{(\tau\varphi)S}, \quad s_\varphi + s_{\tau\varphi} = h + e \mod 2L. \end{aligned} \tag{8.2}$$

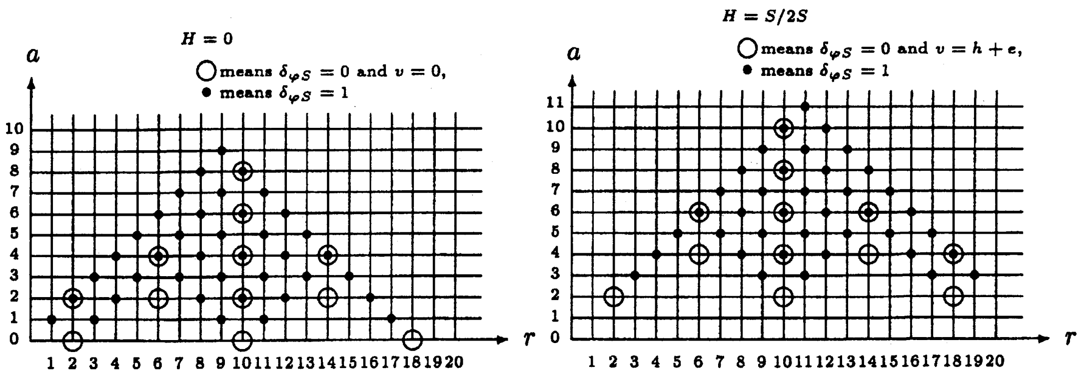

図 3: Σ_1: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = 0$ and $H = S/2S$

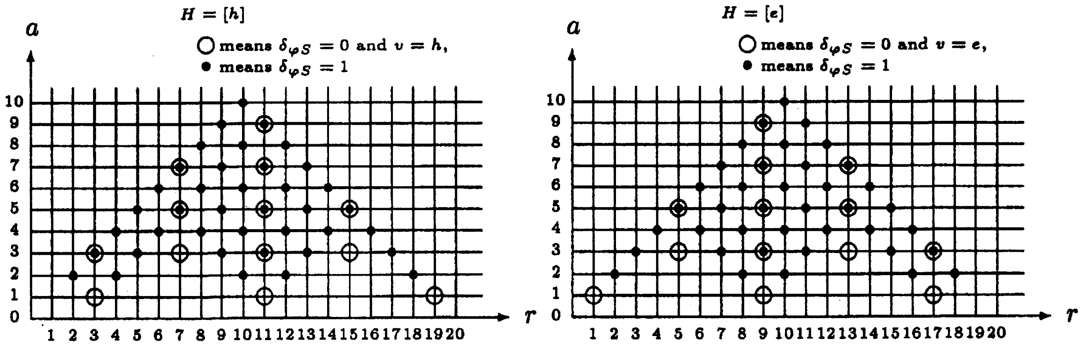

図 4: Σ_1: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h]$ and $H = [e]$

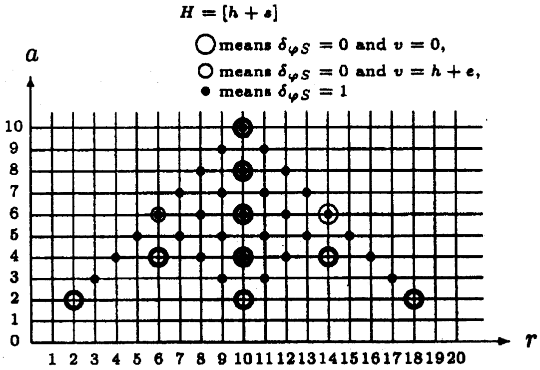

図 5: Σ_1: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h + e]$

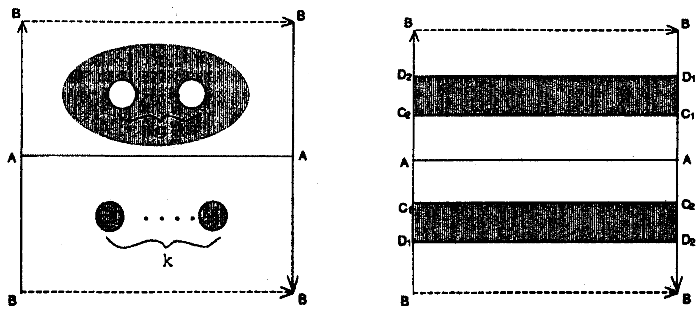

図 6: $H = S/2S$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 10, 0, h+e)$ and $= (10, 10, 0, h+e)$.

Thus, integral involutions of Type 0 with $H = 0$ (12 classes) are related to involutions of Type Ia with $H = S/2S$. Involutions of Type 0 with $H = [h+e]$ (10 classes) are related to involutions of Type Ia with $H = [h+e]$. Involutions of Type Ia with $H = [h]$ (13 classes) are related to involutions of Type Ia with $H = [e]$. Involutions of Type Ib with $H = 0$ (39 classes) are related to involutions of Type Ib with $H = S/2S$. Involutions of Type Ib with $H = [h]$ (39 classes) are related to involutions of Type Ib with $H = [e]$. Finally, the class $(r, a, H = [h+e], \mathrm{Type\,Ib})$ is related to $(20 - r, a, H = [h+e], \mathrm{Type\,Ib})$. (There are 30 classes of Type Ib with $H = [h+e]$.)

Moreover, if we identify related involutions, there are $35(= 12 + 10 + 13)$ classes with $\delta_{\varphi S} = 0$ and $95(= 39 + 39 + 17)$ classes with $\delta_{\varphi S} = 1$.

Let us consider the geometric interpretation of the above calculations. We denote by s the exceptional section of Σ_1 with $s^2 = -1$ and by c the fiber of the natural fibration $\pi : \Sigma_1 \to \mathbb{P}^1$. The contraction of s (as an exceptional curve of the first kind) gives the natural morphism $p : \Sigma_1 \to \mathbb{P}^2$, and we denote $P = p(s)$. A non-singular curve $A \in |-2K_{\Sigma_1}|$ gives then a curve $A_1 = p(A)$ of degree 6 in $\mathbb{P}^2$ with only one singular point P which is a quadratic singular point resolving by one blow-up. A small deformation of A or A_1 (in the same connected component of moduli) makes P non-degenerate. Using Bezoute theorem, one can easily draw all in principle possible pictures of A_1. For example, one can find these pictures in Figure 1 of [13]. Lifting these pictures to Σ_1 and using (6.1), (6.2) and Theorem 11 applied to both positive curves A^+ and A^-, we get from Figures (図) 3 — 5 all pictures of A^+ on Σ_1 up to isotopy. For example, see Figure (図) 6. The interval AA denotes $s(\mathbb{R})$, the real part of the exceptional section s, and the interval BB denotes the $p^{-1}(l(\mathbb{R}))$ of a real projective line $l \subset \mathbb{P}^2$ which does not contain P.

We have the following geometric interpretation of the invariants δ_h and δ_e (see (5.8)) of A^+ if $A^+ \neq \emptyset$. We have $\delta_h = 0$ (equivalently, $h \mod 2 \in H$), if and only if homotopically $l(\mathbb{R}) \subset A^-$ (i.e. some deformation of $l(\mathbb{R})$ is contained in A^-). Similarly, $\delta_e = 0$, if and only if homotopically $s(\mathbb{R}) \subset A^-$. The invariants δ_h and δ_e for both positive curves A^+ and A^- are sufficient to find the group H.

Thus, we get the isotopy classification of real non-singular curves $A \in |-2K_{\Sigma_1}|$. In this classification we don't care about position of $A(\mathbb{R})$ with respect to the real part $s(\mathbb{R})$ of the exceptional section (one can see that more delicate classification in [13]).

We have the following interpretation of the invariant $\delta_{\varphi S}$: *one has $\delta_{\varphi S} = 0$ if and only if the curve A is dividing, i.e., $A(\mathbb{R}) = 0$ in $H_1(A(\mathbb{C}), \mathbb{Z}/2)$, equivalently, $A(\mathbb{R})$ divides $A(\mathbb{C})$ in two connected parts or $A(\mathbb{R}) = \emptyset$.*

One direction follows from Proposition 13 like Proposition 14 for the hyperboloid. For the opposite direction, it would be enough to construct a dividing curve A on Σ_1 in each case when $\delta_{\varphi S} = 0$ because we know that invariants (8.1) define the connected components of moduli. It should follow from known results about real curves of degree 6 (e.g., see [12] and [13]).

Thus, we finally get

Theorem 17 ([22]) *A connected component of moduli of a positive real non-singular curve $A \in |-2K_{\Sigma_1}|$ is defined by the isotopy type of $A^+ \subset \Sigma_1(\mathbb{R})$ and by the dividingness of $A(\mathbb{R})$ in $A(\mathbb{C})$ (equivalently, by the invariant $\delta_{\varphi S}$). All these possibilities are presented in Figures* (図) 3 — 5 *and in* (8.2).

9 A vista —Hyperkahler structures of K3 surfaces—

Let X be a K3 surface and ω be a nowhere vanishing holomorphic 2-form on X. Let τ be a holomorphic involution on X with $\tau^*\omega = -\omega$ (i.e., τ is non-symplectic). Let A be the fixed point set of τ on X. We assume that $A \neq \emptyset$. (A is a complex 1-dimensional submanifold of X.) We set $L = H_2(X;\mathbb{Z})$, $L_\tau = \{x \in L \mid \tau_*(x) = -x\}$ and $L^\tau = \{x \in L \mid \tau_*(x) = x\}$. Let φ be an anti-holomorphic involution on X with $\varphi^*\omega = \overline{\omega}$. We assume $\varphi_*(L^\tau) = L^\tau$. Then we have $\tau\varphi = \varphi\tau$ and $\varphi(A) = A$. Let $A(\mathbb{R})$ be the fixed point set of φ on A. We assume $A(\mathbb{R}) \neq \emptyset$. Let $X(\mathbb{R})$ be the fixed point set of φ on X. Then $X(\mathbb{R})$ is an orientable closed surface and $\tau(X(\mathbb{R})) = X(\mathbb{R})$. We see τ is orientation-reversing on $X(\mathbb{R})$. Hence, the homology class $[X(\mathbb{R})]$ represented by $X(\mathbb{R})$ in L belongs to L_τ.

Under the assumptions above, we want to prove the following assertion by means of Theorem 11 as we mentioned in the bottom of §6.

Assertion 18 *Under the assumptions above,*

$$[X(\mathbb{R})] \cdot x \equiv 0 \pmod 2 \forall x \in L_\tau \ \textit{ if and only if } \ [A(\mathbb{R})] = 0 \ \textit{ in } H_1(A; \mathbb{Z}/2\mathbb{Z}).$$

Remark 19 Let S be the isomorphism class of L^{τ}. We set $\theta := \varphi_*|S$. Then recall that (S, θ) is the type of (X, τ, φ). The conditon that $[X(\mathbb{R})] \cdot x \equiv 0 \pmod 2 \forall x$ in L_{τ} is equivalent to the condition that $\delta_{\varphi S} = 0$. The condition $[A(\mathbb{R})] = 0$ in $H_1(A; \mathbb{Z}/2\mathbb{Z})$ means $A(\mathbb{R})$ is dividing. Thus Assertion 18 says that the invariant $\delta_{\varphi S}$ describes the dividingness of $A(\mathbb{R})$.

Outline of an argument to "prove" Assertion 18 We use some facts on hyperkähler manifolds. (See, for example, the lecture notes [10], p.3 and also [6].) There exists a hyperkähler metric g on X. Thus there exist three complex structures I, J and K on X, such that g is Kähler with respect to all three of them and such that $K = I \circ J = -J \circ I$. Thus I is orthogonal with respect to g and the Kähler form $g(I(\ ),\ )$ is closed (similarly for J and K). We set $P = g(I(\ ),\ ), Q = g(J(\ ),\ )$, and $R = g(K(\ ),\ )$. The holomorphic 2-form on (X, I) can be given as $Q + iR$ (For this, [10] refers to a book written in 2003).

We want to check the following:

(*1) We may consider the K3 surface X in Assertion 18 has the complex structure I above.

(*2) We may consider that

$$\tau^*(P) = P,\ \tau^*(Q) = -Q,\ \tau^*(R) = -R,$$

$$\varphi^*(P) = -P,\ \varphi^*(Q) = Q \text{ and } \varphi^*(R) = -R.$$

Now we consider the K3 surface (X, J). Since $\tau^*(Q) = -Q$, we see that τ is anti-holomorphic on (X, J). (*2) also implies that φ is holomorphic on (X, J) and non-symplectic. Hence the fixed point set $X(\mathbb{R})$ of φ is a complex 1-dimensional submanifold of (X, J) ([19], p.1424). The fixed point set of τ on $X(\mathbb{R})$ is $A(\mathbb{R})$. Thus we can apply Theorem 11 to the K3 surface (X, J), the anti-holomorphic involution τ and the complex curve $X(\mathbb{R}) =: C$. Thus we might be able to prove Assertion 18. This idea is suggested by Professor K.-I. Yoshikawa. The author would like to thank him.

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