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IDEAS AND RESULTS FROM THE THEORY OF DIOPHANTINE APPROXIMATION

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In many areas of mathematics problems of small divisors, or exceptional sets on which certain desired qualities do not hold, appear. The obvious question that then arises is how large are these exceptional sets? This question leads to other questions regarding what do we mean by size. For example there exist many sets of Lebesgue measure zero which have positive Hausdorff dimension implying that although small they are still uncountable. Similarly how does one compare two sets of the same Hausdorff dimension — recent results using Hausdorff measure are one possibility.

Diophantine approximation began as a study of how closely real numbers could be approximated by rationals. The aim of this paper is to show how the classical results of real Diophantine approximation have been adapted and extended to deal with other kinds of approximation in other spaces. Three cases will be specifically discussed, those being the classical case; approximation by algebraic numbers; and Diophantine approximation on manifolds. There are many results regarding the latter but various important open problems remain. At the end of this article it will be shown that even by a simple translation the approximation properties of a manifold can change.

To introduce notation, the example of the classical set of $\psi-$ approximable points/linear forms will be considered. Let

$$W(m,n;\psi) = \{X \in \mathbb{R}^{mn} : |\mathbf{q}X - \mathbf{p}| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in \mathbb{Z}^n\}. \quad (1)$$

Here X is an $m \times n$ matrix, $\mathbf{q}$ is a row vector and $\mathbf{p}$ is a column vector. If the approximating function ψ is of the form $\psi(r) = r^{-\tau}$ then the set will be denoted by $W(m,n;\tau)$. Clearly when $m = 1$ the set $W(1,n;\psi)$ is the set of points $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ which satisfy the system of

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inequalities

$$\max_{i=1,\dots,n} |x_i - p_i/q| < \psi(q)/q$$

infinitely often. Thus, it is the set of points in $\mathbb{R}^n$ which are close to infinitely many rational points of the form $(p_1/q, \dots, p_n/q)$. On the other hand consider the case $n = 1$. In this case $W(m, 1; \psi)$ is the set of points $\mathbf{x} \in [0,1]^m$ which satisfy the inequality

$$|\mathbf{q}.\mathbf{x} - p| < \psi(|\mathbf{q}|)$$

for infinitely many vectors $\mathbf{q} \in \mathbb{Z}^m$ and integers $p \in \mathbb{Z}$ (here, the dot represents the usual scalar product between two vectors). Hence, this set consists of those points in $\mathbb{R}^m$ which are "close" (within $\psi(|\mathbf{q}|)/|\mathbf{q}|$) to infinitely many rational hyperplanes with equation $\mathbf{q}.\mathbf{x} = p$. It should be obvious that all rational points $(p_1/q, \dots, p_n/q)$ are in $W(1, n; \psi)$ for all ψ and all rational hyperplanes with equations $\mathbf{q}.\mathbf{x} = p$ are contained in $W(m, 1; \psi)$. The question then arises as to whether there is anything else in these sets. The size of $W(m, n; \psi)$ has been completely determined in terms of Lebesgue measure, Hausdorff dimension and Hausdorff measure as detailed below. If A is a set then the Lebesgue measure of A will be denoted $|A|$.

Theorem 1 (Khintchine–Groshev). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and supppose that for $m = 1$ and 2, $r^m \psi(r)^n$ is decreasing. Then*

$$|W(m, n; \psi)| = \begin{cases} \infty & \textit{if } \sum_{r=1}^{\infty} \psi(r)^n r^{m-1} = \infty, \\ 0 & \textit{if } \sum_{r=1}^{\infty} \psi(r)^n r^{m-1} < \infty. \end{cases}$$

The case $m = 1$ was proved by Khintchine in [23] and Groshev [20] proved the result for general m.

Theorem 2 (Dodson). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function and let λ be the lower order at infinity of $1/\psi$. Then*

$$\dim W(m, n; \psi) = \begin{cases} (m-1)n + \frac{m+n}{\lambda+1} & \textit{if } \lambda > \frac{m}{n}, \\ mn & \textit{if } \lambda \leq \frac{m}{n}. \end{cases}$$

The result was proved for $W(1, 1; \tau)$ by Jarnik [21] and independently by Besicovitch [12] and is commonly called the Jarnik–Besicovitch Theorem.

The convergence part of the Khintchine–Groshev theorem and the upper bound of the Hausdorff dimension in Theorem 2 are obtained with straightforward covering and counting arguments. In what follows (and above) the sets to which we are approximating will be called *resonant sets*; for example above, the resonant sets for $W(1,n;\psi)$ are the rational points with common denominator q and the resonant sets for $W(m,1;\psi)$ are the rational hyperplanes. To prove the divergence half of the Khintchine–Groshev theorem and to determine the lower bound for the Hausdorff dimension in Theorem 2 and indeed for any analogues or generalisations of these theorems, detailed information regarding the distribution of the resonant sets is needed. In fact, to extend or generalise these theorems to other spaces most of the work is in obtaining such information. To this end, various general ideas have been developed to consider these problems. In particular, we draw attention to the regular systems of Baker and Schmidt [1] and the ubiquitous systems of Dodson, Rynne and Vickers [19] leading to the local ubiquitous systems of Beresnevich, Dickinson and Velani [7]. Regular systems were developed in order to investigate approximation by algebraic numbers and ubiquitous system were developed when considering approximation by rational hyperplanes. Previous methods were not particularly useful when the resonant sets were not positively separated and had dimension ≥ 1. It has been shown that with a slight change, regular and ubiquitous systems are equivalent [25] when the resonant sets are zero dimensional (points). Many adaptations of these systems are now in use — the idea of a local ubiquitous system is presented below (in [7] the system is called *locally m–ubiquitous* where m is a measure on a space Ω, however in this paper only Lebesgue measure is used). For simplification, given the setting of this paper, the definition and the theorem following are not given in full generality. For further details the reader is referred to [7].

Let Ω denote a compact Lebesgue measurable set. Let R_α denote a resonant set indexed by α and let β_α be a weight assigned to this set. The set of all resonant sets will be denoted by $\mathcal{R}$ and γ denotes the dimension of each R_α. Let $J_n = \{\alpha : k^n \leq \beta_\alpha \leq k^{n+1}\}$ where $k > 1$ is some fixed constant. We will use $B(R_\alpha,\delta)$ to denote a δ–thickening of the resonant set R_α; i.e.

$$B(R_\alpha,\delta) = \{x \in \Omega : \text{dist}\,(x, R_\alpha) < \delta\}.$$

The system $(\mathcal{R},\beta)$ is said to be a *local ubiquitous system with respect to the function ρ* if there exists a constant $\kappa > 0$ such that for every ball $B \subset \Omega$

$$|B \cap \cup_{\alpha\in J_n} B(R_\alpha, \rho(k^n))| \geq \kappa |B|$$

for n sufficiently large. The main difference between the local ubiquitous systems for Lebesgue measure and the original ubiquitous systems in [19] is the range in J_n which means that a Khintchine–Groshev type theorem can be proved together with a Hausdorff measure result (see below); as is shown in [7] this restriction on the range of J_n can be somewhat relaxed. The definition in its full generality requires conditions on the measure (automatically satisfied for Lebesgue measure) and some intersection conditions, on the measure of a thickened resonant set with an arbitrary ball, which are satisfied in the cases we will be considering. Basically this definition means that the set of resonant sets with weights in a certain range, when thickened by an appropriate amount cover a given proportion of any ball. The set of points in Ω which lie within $\psi(\beta_\alpha)$ of infinitely many resonant sets R_α will be denoted by $\Lambda(\psi, \mathcal{R})$, that is

$$\begin{aligned}\Lambda(\psi,\mathcal{R}) &= \{x \in \Omega : \operatorname{dist}(x, R_\alpha) < \psi(\beta_\alpha) \text{ for infinitely many } R_\alpha \in \mathcal{R}\} \\ &= \cap_{N=1}^{\infty} \cup_{\beta_\alpha > N} B(R_\alpha, \psi(\beta_\alpha)).\end{aligned}$$

Theorem 3. *Let Ω be a subset of $\mathbb{R}^d$ and suppose that $(\mathcal{R}, \beta)$ is a locally ubiquitous system with respect to ρ and that ψ is a decreasing function. Assume that either*

$$\limsup_{i\to\infty} \frac{\psi(k^i)}{\rho(k^i)} > 0 \tag{2}$$

or

$$\sum_{i=1}^{\infty} \left(\frac{\psi(k^i)}{\rho(k^i)}\right)^{d-\gamma} = \infty \tag{3}$$

Then

$$|\Lambda(\psi, \mathcal{R})| = |\Omega|.$$

Let f be a dimension function such that $r^{-d} f(r) \to \infty$ as $r \to 0$ and is decreasing and furthermore, suppose that $r^{-\gamma} f(r)$ is increasing. Let

$$G = \limsup_{r\to\infty} f(\psi(r))\psi(r)^{-\gamma}\rho(r)^{\gamma-d}.$$

If $G < \infty$ and there exists a constant c such that $\rho(k^i) \leq c\rho(k^{i+1})$ then $\mathcal{H}^f(\Lambda(\psi, \mathcal{R})) = \infty$ if

$$\sum_{i=1}^{\infty} \frac{f(\psi(k^i))}{\psi(k^i)^{\gamma}\rho(k^i)^{d-\gamma}} = \infty.$$

Also, if $G = \infty$ then $\mathcal{H}^f(\Lambda(\psi, \mathcal{R})) = \infty$.

In the classical case it is not difficult to show that the set of rational hyperplanes with equations $\mathbf{q}X = \mathbf{p}$ are locally ubiquitous with respect to $\rho(r) = r^{-(m+n)/n}$. Here, as the set of ψ–approximable numbers is invariant under integer translations, Ω is taken as $[0,1]^{mn}$. The dimension of the hyperplanes is $\gamma = (m-1)n$ and the weight is $|\mathbf{q}|$. Also, $W(m,n;\psi)$ can be written as $\Lambda(\phi,\mathcal{R})$ where $\mathcal{R}$ is the set of such hyperplanes and $\phi(r) = \psi(r)/r$. Hence, the above theorem immediately implies the divergence half of the Khintchine–Groshev theorem and also shows that the Hausdorff measure of $W(m,n;\tau)$ is infinite at the critical dimension $(m-1)n + \frac{m+n}{\tau+1}$ (this is the case when the dimension function $f(r) = r^s$ where $s = (m-1)n + \frac{m+n}{\tau+1}$); this was first proved in [17].

Now, consider the case of approximation by algebraic numbers. Here, the resonant sets will be algebraic numbers α of degree $\leq n$ and the weight of each resonant set will be its height $H(\alpha)$ (the maximum of the coefficients of its minimal polynomial). Let $K_n(\tau)$ denote the set of $x \in [0,1]$ such that the inequality

$$|x-\alpha| < (H(\alpha))^{-(n+1)\tau}$$

is satisfied for infinitely many algebraic numbers α of degree $\leq n$. Clearly $K_1(\tau) = W(1,1;2\tau-1)$. Approximation by algebraic numbers was originally discussed by Baker and Schmidt [1] when they determined that the Hausdorff dimension of $K_n(\tau)$ was $1/\tau$ for $\tau > 1$. Since then a Khintchine–Groshev theorem has been obtained by Bernik (convergence) [9] and Beresnevich (divergence) [3]; the latter obtained a "best regular system" which was then used by Bugeaud [13] to solve the question of Hausdorff measure, (also determined in [7]). This "best regular system" shows that the set of algebraic numbers with degree at most n is a local ubiquitous system with respect to the function $\rho(r) = r^{-(n+1)}$. Obviously every algebraic number is the root of some integer polynomial so the set $K_n(\tau)$ is closely related to the set of x for which the inequality $|P(x)| < H(P)^{-v}$ is satisfied for infinitely many integer polynomials P of height $H(P)$ and degree n, see [10] for details (if $|P(x)|$ is small then x must be close to a root of P). Rewriting this, shows that this is the same problem as considering the set of points lying on the Veronese curve $\{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_i = x_1^i\}$ which are also in $W(n,1;v)$ and this leads to the question of Diophantine approximation on manifolds.

To state the problem generally we will now consider points which are restricted to some m–dimensional submanifold M embedded in Euclidean space $\mathbb{R}^n$. Here the two main types of approximation which arise have very different characteristics depending on whether the resonant sets are rational points (simultaneous approximation) or rational hyperplanes intersected with the manifold (dual approximation). First the case of dual approximation will be considered as the results that exist here are very similar to those of the classical set although the proving of them was a great deal more difficult. Let $L(M;\psi)$ denote the set of dually ψ–approximable points lying on the manifold M embedded in $\mathbb{R}^n$; that is

$$L(M;\psi) = \{\mathbf{x} \in M : |\mathbf{q}.\mathbf{x} - p| < \psi(|\mathbf{q}|) \text{ for infinitely many } \mathbf{q} \in \mathbb{Z}^n, p \in \mathbb{Z}\}, \quad (4)$$

again if $\psi(r) = r^{-\tau}$ then the set will be denoted $L(M;\tau)$. A good account of many of the results can be found in [10]. Since this book was published there have been some major advances. First, it was shown by Kleinbock and Margulis [24] that the Lebesgue measure of $L(M; n+\varepsilon)$ was zero for all non–degenerate manifolds if $\varepsilon > 0$ proving that if M is non–degenerate then it is extremal. This solved a long standing conjecture of Sprindžuk. Things advanced further when a complete Khintchine–Groshev theorem for such manifolds was proved [4], [11] and [6]. Obviously the Lebesgue measure of M is zero if $m < n$ so instead we take the Lebesgue measure induced on the manifold. Regarding other results, R. C. Baker [2] showed that the Hausdorff dimension of the $L(\Gamma;\tau)$ with $\tau > 2$ was $3/(\tau+1)$ for any planar curve Γ which has non–zero curvature almost everywhere. There also exists a lower bound for the Hausdorff dimension of any extremal manifold [14] but the upper bound remains an open problem. A Hausdorff measure result for non–degenerate manifolds was obtained in [7] — the results in [24] were enough to show that the resonant sets (the intersection of hyperplanes, with equations $\mathbf{q}.\mathbf{x} = p$, with the manifold M so they have dimension $m-1$ and weight $|\mathbf{q}|$) were locally ubiquitous with respect to the function $\rho(r) = r^{-(n+1)}$. The Hausdorff measure was shown to be infinite on the divergence of the appropriate volume sum (as in Theorem 3).

Turning to simultaneous Diophantine approximation on manifolds another example of the type of question which might arise is: for which $x \in \mathbb{R}$ are the inequalities $|x - p/q| < \psi(q)$ and $|x^2 - r/q| < \psi(q)$ simultaneously satisfied infinitely often? This is equivalent to asking which points lying on the parabola with equation $y = x^2$ are also in $W(1,2;\phi)$ where $\phi(r) = \psi(r)/r$. The results for simultaneous approximation on manifolds are rather more curious than in the dual case and much less is known. First we define the set

$$S(M;\psi) = \{x \in M : \max|qx_i - p_i| < \psi(q) \text{ for infinitely many } \mathbf{p} \in \mathbb{Z}^n, q \in \mathbb{Z}\}; \quad (5)$$

as before if $\psi(r) = r^{-\tau}$ the set will be denoted $S(M;\tau)$. Recent results indicate that the Hausdorff dimension will have a different formula depending on the size of τ. In [8] a Khintchine–Groshev theorem was proved for planar curves with non–zero curvature almost everywhere. It was also shown that the Hausdorff dimension of $S(\Gamma;\tau)$ where Γ is such a curve is $(2-\tau)(1+\tau)$ when $1/2 \le \tau \le 1$. (For $\tau \le 1/2$, Dirichlet's Theorem implies that the set is of full measure.) On the other hand, for $\tau > 1$ different results are obtained for different curves, unlike in the dual case where they all had the same properties. For example, it is not difficult to show (using Wiles' Theorem) that if Γ is the curve satisfying the equation $x^n + y^n = 1$ for $n > 2$ then $S(\Gamma;\tau)$ is empty for $\tau > n-1$. Also, if $\tau > 1$ it has been shown that if Γ represents the parabola (given by $y = x^2$) [5] or the unit circle centred at the origin [15] then $\dim S(\Gamma,\tau) = 1/(1+\tau)$. Similarly, when Γ is a polynomial curve of degree n the Hausdorff dimension is $2/[n(\tau+1)]$ [16] for $\tau > n-1$. Although together these results completely solve the problem for the parabola and the circle the obvious questions are what happens for other curves and in the case of polynomial curves of higher degree what happens in the middle range $1 \le \tau \le n-1$? These questions are as yet unanswered. As can be seen the results in simultaneous Diophantine approximation on manifolds depend very much on the arithmetic properties of the manifold whereas in the dual case they depend solely on the geometric properties (curvature for example). To illustrate these peculiarities we will investigate polynomial curves in $\mathbb{R}^2$ and show that even when one manifold is a simple translate of another its simultaneous approximation properties may change.

Let $\Gamma = \{(x,y) \in [0,1] \times I : y = P(x)\}$ where P is an nth degree integer polynomial, and $I \subset \mathbb{R}$ is a suitable interval. The results below can be proved for any box (rather than $[0,1] \times I$) but restricting x to $[0,1]$ reduces some technical calculations. The remainder of the results in this paper will show that when Γ is translated by some vector $\mathbf{a} = (\alpha,\beta)$ then its approximation properties change depending on how well approximable the vector $\mathbf{a}$ is. As an example let $n = 2$, $P(x) = x^2$ and $\mathbf{a} = (0,\alpha)$; i.e. we are considering points on a parabola shifted vertically by a distance α. Let $\omega(\alpha) = \sup(\tau : \alpha \in W(1,1;\tau)\}$. Hence, if $\tau > \omega(\alpha)$ then $\alpha \notin W(1,1;\tau)$. Let $\Gamma(\alpha) = \{(x,y) \in [0,1]^2 : y = x^2 + \alpha\}$, then we are interested in the set

$$\begin{aligned} S(\Gamma(\alpha),\tau) &= \{(x,y) \in \Gamma(\alpha) : |x - p/q| < q^{-\tau}, |y - r/q| < q^{-\tau} \\ &\qquad \text{for infinitely many } p,q,r \in \mathbb{Z}\}. \end{aligned} \tag{6}$$

Lemma 1. *Assume $\tau > 1$. $S(\Gamma(\alpha),\tau) = \emptyset$ for $\tau > 2\omega(\alpha) + 1$.*

As already mentioned, for the parabola Γ with equation $y = x^2$ it is known that $\dim(S(\Gamma,\tau)) = 1/(\tau+1)$ so that $S(\Gamma,\tau)$ is an uncountable set for all $\tau > 1$. It is well known that for almost all $\alpha \in [0,1]$, $\omega(\alpha) = 1$. Hence, this lemma implies that $S(\Gamma(\alpha), \tau)$ is empty if $\tau > 3$ for almost all α.

Proof. Let $(x, y) \in S(\Gamma(\alpha), \tau)$ so that

$$\begin{aligned} x &= \frac{p}{q} + \varepsilon \\ y &= \frac{r}{q} + \eta \end{aligned}$$

where $\varepsilon = \varepsilon(p/q), \eta = \eta(r/q)$ and $\varepsilon, \eta = o(q^{-\tau-1})$, for infinitely many $p, q, r \in \mathbb{Z}$. Then

$$\frac{r}{q} + \eta = \frac{p^2}{q^2} + 2\varepsilon\frac{p}{q} + \varepsilon^2 + \alpha.$$

Hence

$$q^2\alpha - rq - p^2 = o((q^2)^{\frac{(1-\tau)}{2}})$$

which is impossible for infinitely many $p, q, r \in \mathbb{Z}$ if $\tau > 2\omega(\alpha) + 1$. Therefore $S(\Gamma(\alpha), \tau) = \emptyset$ as required. $\square$

More generally we can prove the following theorem. Let $\Gamma(\mathbf{a}) = \{(x, y) \in [0,1] \times I : y = P(x + \alpha) + \beta\}$ where $\mathbf{a} = (\alpha, \beta)$ and P is an nth degree integer polynomial. Also let

$$v(\mathbf{a}) = \begin{pmatrix} P(\alpha) + \beta \\ P'(\alpha) \\ P''(\alpha)/2! \\ \vdots \\ P^{(n-1)}(\alpha)/(n-1)! \end{pmatrix}.$$

For a vector $\mathbf{v} \in [0,1]^n$ let $\omega(\mathbf{v}) = \sup\{\tau : \mathbf{v} \in W(n, 1; \tau)\}$. Again, for almost all vectors $\mathbf{v} \in [0,1]^n$, $\omega(\mathbf{v}) = n$.

Theorem 4. *Assume $\tau > n-1$ If $\tau > n\omega(v(\mathbf{a})) + n - 1$ then $S(\Gamma(\mathbf{a}), \tau) = \emptyset$*

Again, it has already been mentioned that if Γ is the curve with equation $y = P(x)$ then $\dim(S(\Gamma, \tau)) = 2/[n(\tau + 1)]$ so that the set is uncountable for all $\tau > n - 1$.

Proof. Let $\mathbf{x} \in S(\Gamma(\mathbf{a}), \tau)$ so that

$$\begin{aligned} x &= \frac{p}{q} + \varepsilon \\ y &= \frac{r}{q} + \eta, \end{aligned}$$

where $\varepsilon = \varepsilon(p/q)$, $\eta = \eta(r/q)$ and $\varepsilon, \eta = o(q^{-\tau-1})$. We have $y = P(x+\alpha)+\beta = \frac{r}{q}+\eta$. There are various ways of rearranging this equation and one such is

$$q^n[P(\alpha)+\beta] \;+\; q^{n-1}pP'(\alpha)+\frac{q^{n-2}p^2P''(\alpha)}{2!}+\cdots+\frac{qp^{n-1}P^{(n-1)}(\alpha)}{(n-1)!}$$
$$+\; p^n - rq^{n-1} = \eta q^n + R(p/q,\alpha,\varepsilon)$$

where $R(p/q,\alpha,\varepsilon)$ consists of the remaining terms all of which contain ε. The RHS of this equation is $o(q^{n-\tau-1})$. Let $\mathbf{q}_o = (q^n, pq^{n-1}, \ldots, qp^{n-1})$ then the equation (which must be satisfied infinitely often) can be rewritten

$$\mathbf{q}_o.v(\mathbf{a}) - (rq^{n-1}-p^n) = o(q^{n-\tau-1}).$$

As $|\mathbf{q}_o| \leq q^n$ we have

$$\mathbf{q}_o.v(\mathbf{a}) - (rq^{n-1}-p^n) = o(|\mathbf{q}_o|^{(n-\tau-1)/n})$$

which is impossible if $\tau > n\omega(v(\mathbf{a}))+n-1$. Hence $S(\Gamma(\mathbf{a}),\tau)=\emptyset$ as required. $\square$

Clearly there are many more questions to be considered such as is $n\omega(v(\mathbf{a}))+n-1$ best possible — almost certainly not given that the $\mathbf{q}_o$ is of such a particular form. The other obvious question is what happens in the middle range $1 \leq \tau \leq n\omega(v(\mathbf{a}))+n-1$ if it exists and can anything be said regarding curves which are not polynomial? Similar questions can be asked regarding higher dimensional manifolds — in particular polynomial surfaces embedded in $\mathbb{R}^d$. These and other questions are the subject of ongoing work.

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