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An Algebraic Foundation for Object-Oriented Euclidean Geometry

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Abstract

The conformal model of Euclidean geometry in Geometric Algebra provides a compact way to characterize Euclidean objects such as spheres, planes, circles, lines, etc. as blades. The algebraic structure of the model provides a 'grammar' for these objects and their relationships. In this rather informal paper we explore this grammar, developing a new geometric intuition to use it effectively. This results in the identification of two important construction products, the known meet and the new plunge. These provide compact specification techniques to parametrize operators and objects directly in terms of other objects.

1 Introduction

1.1 Euclidean Primitives as Subspaces

An elegant model for Euclidean geometry was introduced recently [8], called the 'conformal model' (since it can support conformal transformations as well, although we do not use those in this paper). The idea behind the conformal model is to embed the Euclidean space E^n into the Minkowski space $\mathbb{R}^{n+1,1}$, and the Euclidean metric into the inner product of that Minkowski space. Subspaces of $\mathbb{R}^{n+1,1}$ are blades in its geometric algebra, and easily interpretable as primitive objects in E^n. The operators of geometric algebra then organize the Euclidean geometry algebraically, and this results in useful 'data types' for elementary geometry with well-understood relationships. This technique is metric, and considerably extends the common, non-metric, homogeneous coordinate methods for modeling Euclidean geometry.

Extensive introductions to this 'conformal model of Euclidean geometry' are available [8][2]. Here we just briefly repeat the main points required for working with it. The paper provides a more intuitive understanding of the modeling method and its advantages. If you would like to play with it, we recommend our interactive tutorial running in GAViewer [2].

1.2 The conformal model in brief

First, we extend the Euclidean space with a point at infinity. We represent this as a particular vector of $\mathbb{R}^{n+1,1}$, and in this paper we denote this vector by the symbol ∞.

We represent a general Euclidean point $\mathcal{P}$ homogeneously by a vector $p \in \mathbb{R}^{n+1,1}$ (not just any vector, see below). We design the inner product of two such vectors p and q in $\mathbb{R}^{n+1,1}$ through their relationship to Euclidean points $\mathcal{P}$, $\mathcal{Q}$:

$$\frac{p}{\infty \cdot p} \cdot \frac{q}{\infty \cdot q} = -\tfrac{1}{2} d_E^2(\mathcal{P}, \mathcal{Q}),$$

where $d_E : E^n \times E^n \to \mathbb{R}$ is the Euclidean distance function. It follows that $p \cdot p = 0$, so Euclidean points are represented as *null vectors* in the Minkowski space. Note that since the distance between p and ∞ should be infinite, we have $\infty \cdot \infty = 0$, so it is a null vector as well.

The homogeneous nature of the representation implies that αp may be interpreted as $\mathcal{P}$ for any $\alpha \neq 0$. In this paper we use normalized points for convenience. We set the normalization as:

$$\infty \cdot p = -1, \quad \text{so}: \; p \cdot q = -\tfrac{1}{2} d_E^2(\mathcal{P}, \mathcal{Q}).$$

This representation of geometry is coordinate free. In particular there is no need to introduce a particular origin. Still, if we do choose to introduce the Euclidean origin $\mathcal{O}$ by a specific vector o in $\mathbb{R}^{n+1,1}$, then a point $\mathcal{P}$ in E^n which would traditionally be specified using the position vector $\mathbf{p}$ relative to this origin, can be embedded as the vector $p = o + \mathbf{p} + \frac{1}{2}\mathbf{p}^2\infty$ in $\mathbb{R}^{n+1,1}$ [8].[1]

We are especially interested in E^3, and when required use an orthogonal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and pseudoscalar $\mathbf{I}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$.

2 Basic Constructions

2.1 Products in geometric algebra

The fundamental product in geometric algebra is the geometric product of a vector space V^m. It is linear, associative, and scalar-valued for vectors in V^m. From it can be derived products that are very useful for the geometrically meaningful construction of basic elements.

The Grassmann exterior product, invented to represent the 'extended quantities' that k-dimensional subspaces are, is a natural part of geometric algebra. It is defined in terms of the geometric product as the *outer product*:

$$A \wedge B = \sum_{r,s} \langle \langle A \rangle_r \langle B \rangle_s \rangle_{r+s} \tag{1}$$

[1] Normalization varies between authors. Letting $\mathbf{p}$ become large, the dominant term is proportional to $+\infty$, and this is the reason for our choosing the normalization $\infty \cdot p = -1$.

where $\langle\cdot\rangle_r$ takes the grade r part of its argument. For vectors, it is anti-commutative, linear and extended to more factors by associativity. In this paper we mostly restrict ourselves to elements that are factorizable in terms of the outer product. These are called *blades*, and the number of factors is called the *grade* of the blade.

Two inner products can be defined as the adjoint to the outer product relative to a metric scalar product [3], resulting in

$$\begin{aligned}\text{left contraction: } A\rfloor B &= \sum_{r,s} \langle\langle A\rangle_r \langle B\rangle_s\rangle_{s-r} \\ \text{right contraction: } A\lfloor B &= \sum_{r,s} \langle\langle A\rangle_r \langle B\rangle_s\rangle_{r-s}\end{aligned}$$

If the order of the arguments is chosen well and scalar blades are avoided, we can also use the 'standard' inner product of Hestenes denoted by '$\cdot$', see [3].

We can distribute the inner product with respect to the outer product:

$$a\cdot(B\wedge C) = (a\cdot B)\wedge C + \widehat{B}\wedge(a\cdot C) \tag{2}$$

where a is a vector, B and C general blades, and $\widehat{B} = \sum_r (-1)^r \langle B\rangle_r$, or

$$(A\wedge B)\rfloor C = A\rfloor(B\rfloor C)$$

(where the use of the left contraction is essential).

The *dual* of an element A of the geometric algebra of the space $\mathbb{R}^{n+1,1}$ is defined as geometric division by a the unit pseudoscalar $I_{n+1,1} = o\wedge\mathbf{I}_n\wedge\infty$, with $\mathbf{I}_n$ the pseudoscalar of E^n.

$$A^* = A\, I_{n+1,1}^{-1} = A\rfloor I_{n+1,1}^{-1}. \tag{3}$$

Please note that A^{**} is not necessarily equal to A, there is a sign difference of $(-1)^{n(n-1)/2}$. Duality laws allow conversion between the inner and outer product:

$$A\rfloor B^* = (A\wedge B)^* \quad \text{and} \quad A\wedge B^* = (A\rfloor B)^* \tag{4}$$

We also need the purely Euclidean dualization, which we denote by a subtly different star, as $\mathbf{A}^\star$ for a pure Euclidean blade $\mathbf{A}$. Pure Euclidean blades are always denoted in bold.

With the exception of the geometric product, these products are closed on blades: the product of two blades is again a blade [1]. There are also ways of using the geometric product as blade operations, e.g. in projection and the versor product. We get back to that in section 6.

2.2 Direct and dual representation

Consider an m-dimensional vector space V^m. In this space, k vectors $v_1\cdots v_k$ span a subspace $\mathcal{A}$. In geometric algebra, this k-dimensional subspace is directly characterized by the blade $A = v_1\wedge\cdots\wedge v_k$, in the sense that:

$$x\wedge A = 0 \iff x\in\mathcal{A} \tag{5}$$

Because of this correspondence we may talk about 'the subspace A', in particular the Euclidean point $\mathcal{P} \in E^n$ can be denoted by its representing vector $p \in \mathbb{R}^{n+1,1}$ without confusion. The anti-commutative and linear properties of the outer product provide us with orientations and magnitudes for the subspaces, but we do not emphasize those in this paper.

Taking the dual of the expression $x \wedge A = 0$ with respect to the pseudoscalar (volume blade) of the total algebra we work in, we see that the same subspace can also be characterized dually by A^*:

$$x \cdot A^* = 0 \iff x \in \mathcal{A} \tag{6}$$

We use both characterizations in this paper, and find it necessary to distinguish sharply between them, so please note the difference between the *direct* characterization eq.(5) and the *dual* characterization eq.(6) of a subspace.

2.3 Euclidean spheres and planes as vectors

The direct relationship between the Euclidean distance and the inner product makes specification of Euclidean primitives very straightforward. Some examples:

- **Midplane between points p and q**

$$x \text{ on the midplane} \iff d_E(x,p) = d_E(x,q) \implies$$

$$x \cdot p = x \cdot q \iff x \cdot (p-q) = 0$$

  So the vector $p - q$ is the dual representation of the midplane. Note that $(p-q)\cdot(p-q) = d_E^2(p,q)$, so this is not a null vector and therefore distinct from the representation of a point.

- **Sphere center c radius ρ**

$$x \text{ on the sphere} \iff d_E(x,c) = \rho \implies$$

$$x \cdot c = -\tfrac{1}{2}\rho^2 \iff x \cdot c - \tfrac{1}{2}\rho^2\, x \cdot \infty = x \cdot (c - \tfrac{1}{2}\rho^2\, \infty) = 0$$

  So $c - \frac{1}{2}\rho^2\,\infty$ is the dual sphere. Note that $(c - \frac{1}{2}\rho^2)\cdot(c - \frac{1}{2}\rho^2) = \rho^2$, so this is also not a null vector when $\rho \neq 0$.

- **Plane normal n, distance δ from the origin**
  The traditional Hesse normal form of a plane for Euclidean vectors can be converted to 1-blade from $\mathbb{R}^{n+1,1}$ dually representing that plane as follows:

$$\mathbf{x} \cdot \mathbf{n} = \delta \iff x \cdot (\mathbf{n} + \delta\,\infty) = 0$$

  So: $\mathbf{n} + \delta\,\infty$, with Euclidean $\mathbf{n}$, is a dual plane. However, δ is the distance to the (arbitrary) origin. We prefer a coordinate free form, as follows.

- **Plane with normal n, point p on it**
Since p should be on the plane, we have $p \cdot (\mathbf{n} + \delta\infty) = 0$ therefore, $\delta = -(p \cdot \mathbf{n})/(p \cdot \infty) = p \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$, so the blade dually representing the plane is proportional to (using eq.(2)):

$$-(p \cdot \infty)\mathbf{n} + (p \cdot \mathbf{n})\infty = p \cdot (\mathbf{n} \wedge \infty), \tag{7}$$

- **Sphere center c, point p on it**

$$c - \tfrac{1}{2}\rho^2 \infty = c + (p \cdot c)\infty = (p \cdot c)\infty - (p \cdot \infty)c = p \cdot (c \wedge \infty) \tag{8}$$

So: $p \cdot (c \wedge \infty)$ is the dual sphere with center c, through point p.

These are algebraic constructions of the primitives 'dual sphere' and 'dual plane'. We now use the algebra, guided by geometrical intuition, to construct more involved elements.

3 Meet them all

3.1 Incidence

Given two blades A and B in general position representing two subspaces, let us find the blade representing their common subspace. A vector x representing a point in that subspace satisfies:

$$x \cdot A^* = 0 \quad \text{and} \quad x \cdot B^* = 0.$$

If A^* and B^* are 'independent', these can be gathered using eq.(2) as:

$$x \cdot (B^* \wedge A^*) = 0$$

Therefore the dual representation of the intersection $A \cap B$ of the two blades is

$$(A \cap B)^* = B^* \wedge A^*$$

(for the reason of the change of order, see [3]) and the direct representation is obtained by duality eq.(4) as:

$$A \cap B = B^* \rfloor A \tag{9}$$

In both these equations, some care need to be taken in the duality, it should be done relative to the pseudoscalar of the smallest space containing both A and B, to guarantee the required 'independence' of A^* and B^*. Then $A \cap B$ is called the meet operation [7]. It is more general than just geometrical intersection, it can for instance also provide the distance between skew lines as their 'common blade' (which is then a scalar multiple of ∞). The meet is therefore more like a general 'incidence' operation than a classical geometric intersection.

We can use this meet operation to construct the dual representation of circles and lines:

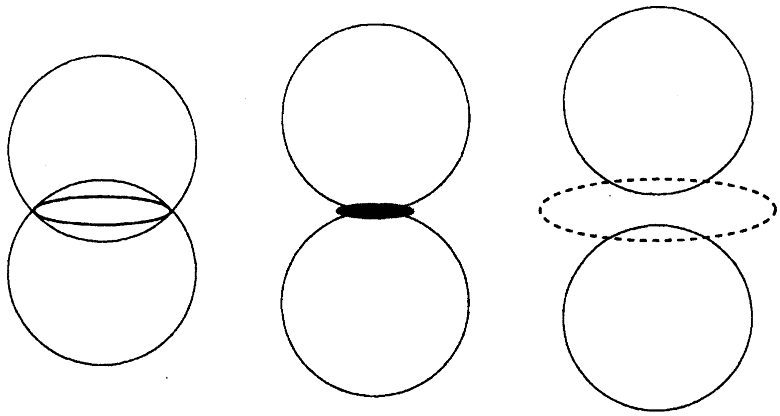

Figure 1: *Intersection of two spheres at increasing distances, leading to a real circle, a tangent 2-blade, and an imaginary circle.*

- **Circle with center c in plane with normal $\mathbf{n}$ and radius ρ**
  We construct this circle by intersecting the dual plane $c \cdot (\mathbf{n} \wedge \infty)$ and the dual sphere $c - \frac{1}{2}\rho^2\infty$, which immediately gives:
  $$(c - \tfrac{1}{2}\rho^2\infty) \wedge (c \cdot (\mathbf{n} \wedge \infty)) \tag{10}$$
  This is a translated version of a circle at the origin o which has the standard form $(o - \frac{1}{2}\rho^2\infty) \wedge \mathbf{n}$.

- **Point pairs are 0-spheres**
  The intersection of three spheres gives a *point pair*, with dual representation in standard form $(o - \frac{1}{2}\rho^2\infty) \wedge \mathbf{B}$ where $\mathbf{B}$ is a Euclidean bivector denoting the dual of the line carrier direction. A point pair is a sphere on a line, of course, so we would indeed expect to have it as a basic element.

- **lines and flat points**
  Intersecting two dual planes $\mathbf{m}$ and $\mathbf{n}$ (for convenience at the origin), gives the dual representation of a line through the origin as $\mathbf{n} \wedge \mathbf{m}$. We will meet the direct form in eq.(13). Intersecting three planes gives a blade proportional to $\mathbf{I}_3$. This is a 'dual flat point', note that both o and ∞ are contained in it (which is perhaps clearer from its direct representation $o \wedge \infty$).

Collectively, we call the elements obtained from spheres and their intersections: *rounds*, and those of planes and their intersections: *flats*. Of course intersecting flats with rounds also produces rounds.

3.2 Imaginary rounds

The **meet** operation in $\mathbb{R}^{n+1,1}$ always produces a blade, even when the geometric intersection in E^n becomes imaginary. Therefore the *real model* $\mathbb{R}^{n+1,1}$ contains the representation of *imaginary* spheres and circles. This apparent paradox is resolved when one realizes that only the squared distances occur in this model, and these are allowed to become negative in a very real way.

Let us intersect two (dual) spheres with equal radius ρ at opposite sides of the origin in the unit $\mathbf{e}_1$ direction:

$$(o - \mathbf{e}_1 + \tfrac{1}{2}(1-\rho^2)\infty) \wedge (o + \mathbf{e}_1 + \tfrac{1}{2}(1-\rho^2)\infty) = (o - \tfrac{1}{2}(\rho^2-1)\infty) \wedge (2\mathbf{e}_1) \quad (11)$$

When $\rho^2 > 1$ (see also Fig. 1a), we get a real circle, nicely factored in the outer product as the intersection of a real sphere and the dual plane $2\mathbf{e}_1$ which is the flat carrier of the circle.

But when we have $\rho^2 < 1$ (see also Fig. 1b), the resulting intersection circle becomes imaginary and is factored as the intersection of a plane and an imaginary sphere (of the dual form $o + \frac{1}{2}r^2\infty$, as opposed to $o - \frac{1}{2}r^2\infty$ for a real sphere).

3.3 Tangents

When we take $\rho^2 = 1$ in eq.(11) (see also Fig. 1c), the element $o \wedge \mathbf{e}_1$ results. This is the dual representation of a grade 2 Euclidean tangent at the point of intersection o. You may verify that the only point this contains is o; yet it has the (dual) direction aspect $\mathbf{e}_1$, so it is more than that point. Its direct representation is $(o \wedge \mathbf{e}_1)^* = o \wedge \mathbf{e}_1{}^\star(-1)^n = o \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ for E^3.

Thus 'tangent elements' are a natural part of this model of Euclidean geometry, even without differentiation.

3.4 Attitudes

When we compute the **meet** of two parallel planes at distance δ such as $\mathbf{n}$ and $\mathbf{n} + \delta\infty$, we obtain

$$\mathbf{n} \wedge (\mathbf{n} + \delta\infty) = \delta\, \mathbf{n} \wedge \infty.$$

This blade is proportional to their distance and the (dual) direction $\mathbf{n}$ of the planes. This is a pure (dual) direction element, it is translation invariant and rotation covariant [4]. The only 'point' it contains is ∞.

We call such elements *attitudes*. They are the generalization of the 'points at infinity' vectors in the traditional homogeneous model of E^n, and capable of denoting higher dimensional direction elements.

3.5 That's all

More detailed analysis shows that we have now found all the basic elements we can obtain by combining the basic spheres and planes. Denoting the (arbitrary) origin by o, and a purely Euclidean blade by $\mathbf{E}$ (the carrier direction) or its Euclidean dual $\mathbf{E}^\star$, these take the standard forms (see [4]):

dual rounds:	$(o - \frac{1}{2}\rho^2\infty) \wedge \mathbf{E}^\star$	(real when $\rho^2 > 0$, imaginary when $\rho^2 < 0$)
rounds:	$(o + \frac{1}{2}\rho^2\infty) \wedge \mathbf{E}$	(real when $\rho^2 > 0$, imaginary when $\rho^2 < 0$)
dual flats	$\mathbf{E}^\star$	
flats:	$o \wedge \mathbf{E} \wedge \infty$	
dual tangents:	$o \wedge \mathbf{E}^\star$	
tangents:	$o \wedge \mathbf{E}$	
dual attitudes:	$\mathbf{E}^\star \wedge \infty$	
attitudes:	$\mathbf{E} \wedge \infty$	

All of these can be brought to general position by a translation versor (see section 6), which has the effect $o \mapsto p$ (so it translates over $\mathbf{p}$), $\infty \mapsto \infty$ (since the point at infinity is translation invariant), $\mathbf{E} \mapsto -(\mathbf{E}\infty)\lfloor p = \mathbf{E} + (\mathbf{E}\lfloor p)\infty = \mathbf{E} + (p\rfloor\widehat{\mathbf{E}})\infty = \mathbf{E} + (p \cdot \widehat{\mathbf{E}})\infty$ (as we saw for a vector in eq.(7)).

In representing these elements, we strive to the factorized 'standard form'

$$\text{[optional location and size]} \wedge \text{[direction element]} \wedge \text{[optional } \infty\text{]}. \qquad (12)$$

The 'direction' factor is a purely Euclidean blade (or its translated version). It can be a 0-blade, i.e. a scalar.

4 Taking the plunge

There is an operation which is in a sense dual to the **meet**, and which is a useful manner of constructing objects. While the **meet** constructs a representation of an object *in common* with given elements, the new operation of 'plunging' (our term) constructs the representation of an object that hits the other elements *perpendicularly*. This is therefore a truly metric operation (the **meet** is not).[2]

Two blades A and B are orthogonal to each other when their inner products are zero (or equivalently, the inner products of their duals):

$$A \perp B \iff A \cdot B = 0 = B^* \cdot A^*$$

(where we can use the inner product as long as A and B are not 0-blades).

[2]We coined the term 'plunge' (which according to Webster's may be etymologically related to 'plumb') to give the feeling of this perpendicular dive into its arguments. We could have called it 'co-incidence' because of its dual relationship to 'incidence', to please algebraists.

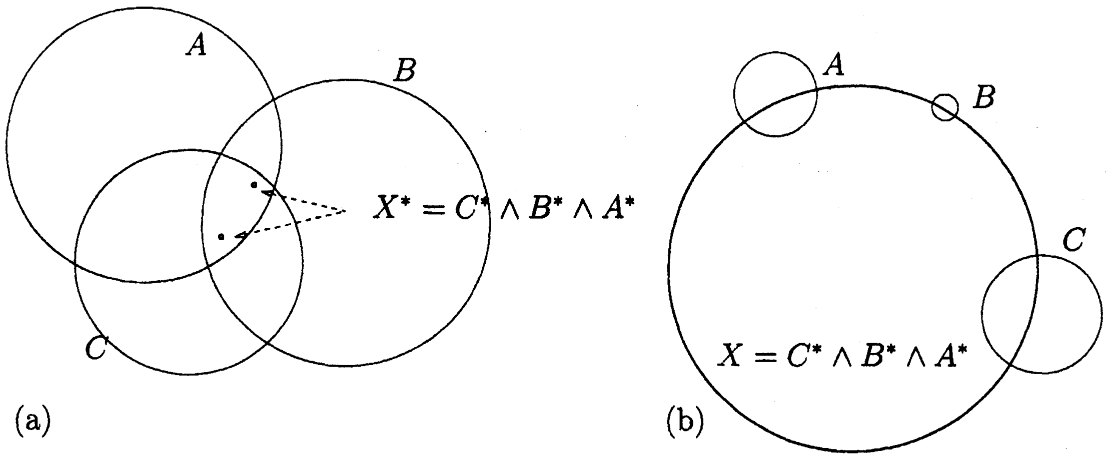

Figure 2: *(a) The* meet *of three intersecting spheres is the real point pair with dual representation $C^* \wedge B^* \wedge A^*$. (b) The* plunge *of three non-intersecting spheres is the circle with direct representation $C^* \wedge B^* \wedge A^*$. (If you interpret picture (b) as 2D, this is the* plunge *of 3 circles, at least if you take duality relative to the pseudoscalar of the containing plane. What happens to (a) in 2D?)*

Let us use this to construct the plunge operation. Suppose we have two blades A, B, and are looking for the Euclidean object X that is perpendicular to each. We therefore need to satisfy $X \cdot A = 0$ and $X \cdot B = 0$. Dualizing this we get $X \wedge A^* = 0$ and $X \wedge B^* = 0$. If A and B are independent blades, the simplest blade X satisfying these equations is:

$$X = B^* \wedge A^*$$

(again, don't mind the order, which only differs possibly by a sign from the reverse). If A and B are spheres, this is a point-pair, i.e. a 1-dimensional sphere. In fact, we can use eq.(11) and our classification to note that the radius of the point pair is real when $\rho < 1$. But that is perhaps a difficult example. The formula easily generalizes to the plunge of three spheres, as in Fig 2b, which is the circle:

$$X = C^* \wedge B^* \wedge A^*$$

(if A, B, C are in general position).

The outcome is interesting. The *direct* representation of the object perpendicular to other objects is the outer product of their *duals.* Compare this to the meet, in which the *dual* representation of the object in common to other objects is the outer product of their *duals.* Algebraically they are closely related, geometrically they have quite a different feeling to them.

Shrinking the spheres to points, you see that you get a circle through those points. So we suddenly realize that *what we have called points are in fact small* *dual spheres!* To 'pass through' a point means to cut its corresponding *direct* sphere perpendicularly, i.e. to **plunge** into it. This neatly unifies the point description with the spheres in one consistent scheme. We should in fact rephrase our earlier usage of points – we now realize that those were all *dual points.* A direct point at the origin is of the form $o^* = o \wedge \widehat{\mathbf{I}}_n^{-1} = o \wedge \mathbf{I}_3$ for E^3, denoting location o (the origin) and $\mathbf{I}_3$ (all directions presents). A general direct point is the translated version of this.

We can also interpret an object containing mixed terms, such as

$$p \wedge \mathbf{e}_1 \wedge \infty. \tag{13}$$

By the previous analysis, it contains the points p and ∞, and should be orthogonal to $\mathbf{e}_1{}^*$, which is the $(\mathbf{e}_2 \wedge \mathbf{e}_3)$-plane through the origin. Obviously this is the line through p in the $\mathbf{e}_1$ direction, and a direct construction of the line in the form of eq.(12). Including another Euclidean vector factor gives a plane. Removing the Euclidean factor gives the representation of a flat of dimension zero, i.e. a 'flat point'.

Because of the duality relationship between direct and dual flats (see section 3.5), when the objects change position or size, their round **plunge** becomes real just when their **meet** becomes imaginary, and vice versa. Also, they are dual sets on a sphere encompassing both, e.g. complex equator to real poles.

5 Parametrizing objects through objects

5.1 Flat points are hairy

Now let us revisit the object $p \cdot (c \wedge \infty)$ from eq.(8). It was a dual sphere through the point p, with center c. Dualizing this, we find that it is the direct object

$$S = p \wedge (c \wedge \infty)^*.$$

We see that this indeed contains p (it plunges into p since it is a direct representation of the form '$p\wedge$'), and that we can think of it as being perpendicular to the object $c \wedge \infty$. Since the result has to be the sphere, this suggests the intuitive picture of Fig. 3a: the object $c \wedge \infty$ has 'hairs' extending to infinity, and our object S cuts them orthogonally. These hairs then help to construct an object consisting of points equidistant to c. This gives a better intuition of the 'flat point' $c \wedge \infty$.

5.2 Midplane parametrization revisited

In Fig. 3b, we see a similar explanation for the construction of the midplane between two points p and q, which is $q - p$, or written multiplicatively and dualized:

$$(q - p)^* = (\infty \cdot (p \wedge q))^* = \infty \wedge (p \wedge q)^*.$$

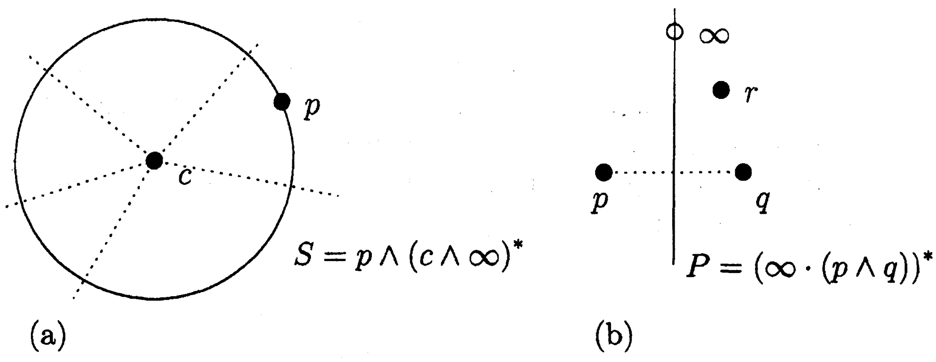

Figure 3: *A direct parametrization of dual spheres and planes by (dual) points, see text.*

Our 'grammar' yields immediately that the direct representation contains ∞ and cuts the point pair $p \wedge q$ perpendicularly, a fair description of the midplane if we imagine a 'hairline' between p and q as in the figure.

If we replace ∞ with a finite point r, we get $(r \cdot (p \wedge q))^*$. This is a sphere, and we leave it to you to explore its geometry in Fig. 3b.

5.3 Direct and dual spheres

We now show the relationship between the direct representation of a sphere as the plunge of four points $S = a \wedge b \wedge c \wedge d$, and the dual representation by a center point m and a point a on it which is $s = a \cdot (m \wedge \infty)$.

We realize that the center should be the intersection of the midplanes of three point pairs. These midplanes are dually represented as $b - a$, $c - a$ and $d - a$, and the dual of their intersection is their outer product. However, this is not merely the center m, since ∞ is also on all planes. Therefore:

$$(m \wedge \infty)^* = (b - a) \wedge (c - a) \wedge (d - a).$$

This helps use relate the two immediately. The dual representation gives $0 = x \cdot s$, and we dualize this and rearrange:

$$\begin{aligned} 0 &= (x \cdot (a \cdot (m \wedge \infty)))^* \\ &= x \wedge (a \cdot (m \wedge \infty))^* \\ &= x \wedge (a \wedge (m \wedge \infty)^*) \\ &= x \wedge (a \wedge (b - a) \wedge (c - a) \wedge (d - a)) \\ &= x \wedge (a \wedge b \wedge c \wedge d) \end{aligned}$$

This is of the form $0 = x \wedge S$, so we have found the direct representation $S = a \wedge b \wedge c \wedge d$. Done!

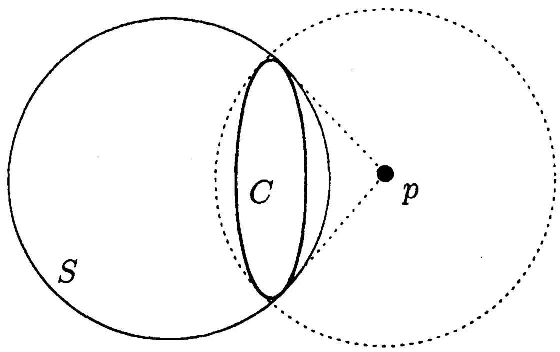

Figure 4: *Construction of a contour circle.*

It is rather satisfying that such geometrically involved computations can be done so simply in this conformal model of Euclidean geometry, without introducing coordinates.

5.4 A contour circle

Let us construct the 'contour' of a sphere S as seen from a point p, i.e. the circle C of points where the invisible part of the sphere borders the visible part, see Figure 4. Obviously, p and S should be enough to parametrize C.

The construction is based on the idea that a sphere S_p through that circle, with p at its center, plunges into S perpendicularly. So in dual form (using lower case for the duals), $s_p = s \cdot$ (something). We also know that p is the center of s_p, which means it should plunge into the flat point $p \wedge \infty$ (as in Fig. 3a). So $s_p = s \cdot (p \wedge \infty)$. Note that this generalizes eq.(8), since s is now a general dual sphere, not merely a point. Then the circle we are looking for is obtained by the meet of S_p with S, which is done as:

$$C = (s \wedge (s \cdot (p \wedge \infty)))^* = s \cdot (s \wedge (p \wedge \infty)^*).$$

This is a pleasantly coordinate-free parametrization of the sought-for object. The advantage of such parametrizations appears when using the full geometric calculus, in which we can directly differentiate such expressions to their constituents in a coordinate-free manner[6].

5.5 Another circle parametrization

Parametrizations can also be done with partial specifications, such as making the circle through the point p, passing through it in the direction $\mathbf{e}_1$, and perpendicular to the plane $\mathbf{e}_2$. The answer is proportional to $C = p \wedge (p \cdot (\mathbf{e}_1 \infty)) \wedge \mathbf{e}_2$, almost immediate once you recognize the middle part as the translation of the directional element to location p.

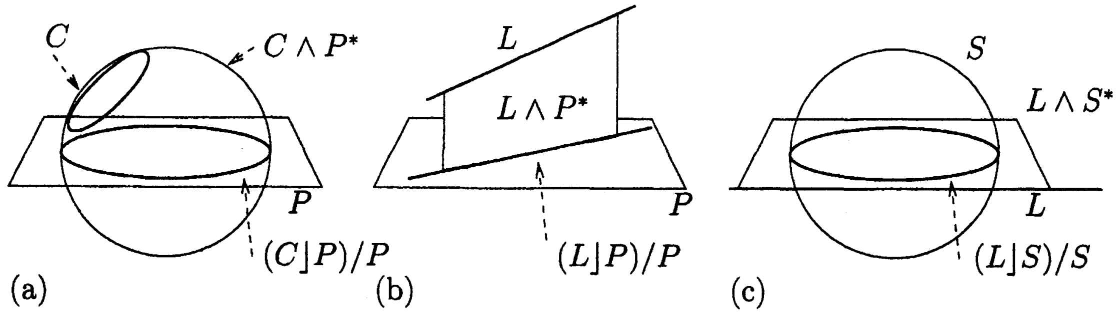

Figure 5: *Projections in the conformal model of Euclidean geometry.*

6 Euclidean operators on blades

This paper is concerned with objects represented by blades, and thus far we have used the inner and outer product and duality to construct blades out of elementary blades. The meet and plunge are just geometrically meaningful applications of those same basic products. But geometric algebra also has methods to construct blades from blades using the geometric product, and these are geometrically significant as well. We discuss two: Euclidean projections and Euclidean transformations.

6.1 Euclidean projections

In geometric algebra, the projection of a blade X onto a blade P is another blade:

$$X \mapsto (X\rfloor P)/P, \tag{14}$$

where the division uses the geometric product. Applying this to the conformal model, we the find expected projection behavior if X is a flat (as we'll show below when discussing Fig. 5b).

But when X is a round, it is not the projection you might expect. For instance, consider Fig. 5a, the projection of a circle onto a plane. You may have hoped for an ellipse, but an ellipse is not represented by a blade in our model, and therefore cannot be the outcome of eq.(14). Instead, the result of the projection of a circle onto a plane is a circle.

To explain this effect, we analyze the projection formula, trying to write it in terms of meet and plunge. We bring it into the direct form of the meet eq.(9), by replacing the geometric product by a contraction with the inverse, see [3]:

$$(X\rfloor P)\,P^{-1} = (X\rfloor P)\rfloor P^{-1} \;\propto\; (X \wedge P^*)^* \rfloor P^{-1} \;\propto\; P \cap (X \wedge P^*).$$

So modulo a sign and a magnitude, this is proportional to the meet of P with $X \wedge P^*$. The latter is the plunge of X into P: it contains X and hits P perpendicularly.

So for instance in the case of projecting a circle onto a plane, we indeed get the construction of Fig. 5a, resulting in a circle.

It is instructive to see what happens for flats, for instance when X is a line L and P a plane as in Fig. 5b. Now $L \wedge P^*$ is the plane containing L perpendicular to P, which is itself a flat. Therefore the meet with P now produces the expected line on the plane which is the projection of the line in the usual sense.

Figure 5c shows that when P is a sphere and L a line, we get a great circle on the sphere, which is indeed a sensible interpretation of what it would be to project a line on a sphere. Obviously, these examples generalize to the other elements we have treated. You can even project a tangent vector onto a sphere (and the result is the point pair in which the plunging circle containing the tangent vector meets the sphere).

In summary, the operation of 'projection' generalizes from flats in a sensible, but somewhat unusual manner, providing a fundamental operation that seems to be new to Euclidean geometry. Let us find applications for it!

6.2 Euclidean transformations

For Euclidean geometry, we are obviously interested in the Euclidean transformations. In the conformal model, these are represented as versor products, using a versor V on a vector x as:

$$x \quad \mapsto \quad \widehat{V} x V^{-1}, \tag{15}$$

and naturally extended to blades as outermorphisms. Versors transform linear combinations of geometric products covariantly [4]. All of our constructions are of this form, so structure preservation under Euclidean transformations is guaranteed in the conformal model.

The versors themselves are not necessarily blades, though they are certainly elements of the geometric algebra. The versors of the Euclidean transformations can be constructed from our blades as the ratios of flats.

- **translation versor: the ratio of two flat points**

  $$(q \wedge \infty)/(p \wedge \infty) = (q \wedge \infty)\,(p \wedge \infty) = 1 - (q - p)\infty = 1 - (\mathbf{q} - \mathbf{p})\infty.$$

  This is the versor for a translation over $2(\mathbf{q} - \mathbf{p})$.

- **rotation versor: the ratio of two non-parallel planes**
  If we give the planes a common point p for convenience, we get:

  $$(p \cdot (\mathbf{m} \wedge \infty))/(p \cdot (\mathbf{n}^{-1} \wedge \infty)) = \cdots = p \cdot (\mathbf{m}\mathbf{n}^{-1}\infty)$$

  Putting p at the origin, we indeed find the well-known Euclidean versor $-\mathbf{m}\mathbf{n}^{-1}$, a quaternion which turns over twice the angle between $\mathbf{m}$ and $\mathbf{n}$.

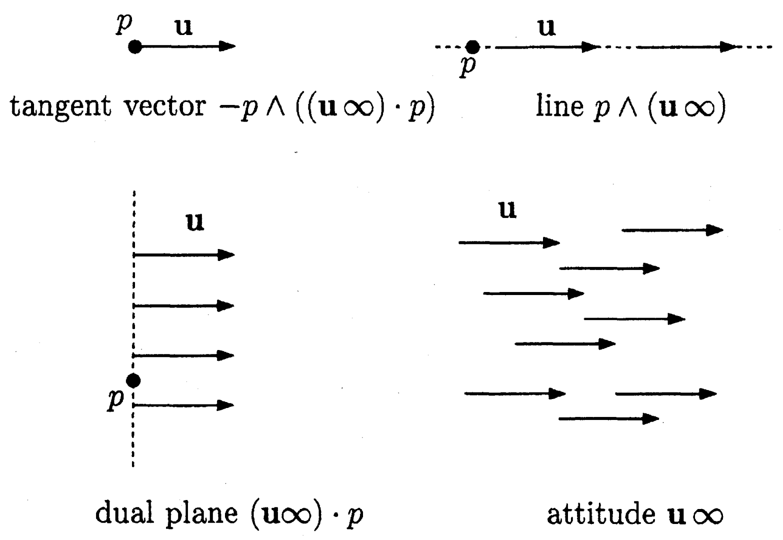

Figure 6: *Euclidean vector related elements of geometry and their algebraic forms. These map onto the classical concepts: tangent vector, (relative) position vector, normal vector, direction vector.*

- **general rigid body motion: the ratio of two lines**
  Rigid body motions are screws. Computing them as a ratio of lines gives:
  $$(q \wedge \mathbf{m} \wedge \infty)/(p \wedge \mathbf{n} \wedge \infty) = (q \wedge \mathbf{m} \wedge \infty)\,(p \wedge \mathbf{n}^{-1} \wedge \infty) = \cdots =$$
  $$\mathbf{m}\,\mathbf{n}^{-1} + (\mathbf{m} \cdot (\mathbf{n}^{-1} \wedge p) - \mathbf{n}^{-1} \cdot (\mathbf{m} \wedge q)) \wedge \infty - (p+q) \wedge \mathbf{m} \wedge \mathbf{n}^{-1} \wedge \infty,$$
  although it is more convenient to analyze this versor by considering it as the exponent of a bivector [8].

So the geometric product of blades is also a geometrically significant construction, even though it is not a blade: it represents an operator rather than an object. It is in fact a great advantage of geometric algebra over Grassmann-Cayley algebra that both elements and operators reside naturally in the same algebraic framework.

7 Conclusion

We have shown that the conformal model can generate an algebraic 'language' for Euclidean geometry. In this language, the elementary objects of Euclidean geometry occur as 'nouns', and the meet, plunge and projection as 'verbs'. Euclidean transformations are in a sense 'adjectives' or 'adverbs': they result in a modified object of the same kind. The direct parametrization of the constructions of these elements in terms of their constituent objects motivated the term 'object-oriented' in the title.

But of course in computer science, 'object-oriented' has a different meaning, so the title is also meant as a suggestive teaser. For we begin to see how this algebraic language for Euclidean geometry also specifies the data structures and the permissible operations on them. For instance, in classical graphics programming based in linear algebra, there are many vector-related concepts that should be given separate data structures since they transform differently under translations and rotations. Figure 6 indicates how each of these are different objects in the conformal model, constructed precisely so that they transform automatically correctly under the Euclidean versors. This is a structural clean-up of the language to express geometry, probably solving many common programming errors caused by the confusion of the traditional vector concept [5]. And we know how to extend this to the elements of higher grade, too: geometric algebra makes this automatic. We also saw that the language is powerful enough to contain not only the objects, but also operators and transformations acting upon them.

The pleasure of this new language is best experienced interactively, in a program that allows one to type in the formulas and immediately depict the corresponding elements on the screen. Several free programs exists, of which we prefer our own GAViewer [2]. You will find that doing geometry this way provides compact ways of encoding constructions, yields new insights into Euclidean geometry, and is just plain unadulterated coordinate-free fun.

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