数理解析研究所講究録 1384 巻 2004 年 121-128

On Generalized Radix Representations

By S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner

Kyoto, October 1, 2003.

Research partially supported by the Hungarian Academy of Sciences and Japan Society for the Promotion of Science.

You can find details and more material about the here studied kind of questions in the following papers:

[1] S. Akiyama and A. Pethő, *On canonical number systems*, Theor. Comp. Sci., 270 (2002), 921–933.

[2] S. Akiyama, H. Brunotte and A. Pethő, *Cubic CNS polynomials, notes on a conjecture of W.J. Gilbert*, J. Math. Anal. and Appl., **281** (2003), 402–415.

[3] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *On a generalization of the radix representation - a survey*, Fields Institute Communications, to appear.

[4] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő and J. Thuswaldner, *Generalized radix representations and dynamical systems I*, Acta Math. Hungar., submitted

1. Radix representation and two generalizations

Positive base: Let $g \geq 2$ be an integer. Then every $n \in \mathbf{Z}$ can be represented in the form

$$n = \pm \sum_{j=0}^{\ell} n_i g^i, \quad 0 \leq n_i < g.$$

Negative base: V. Grünwald (1885): Let $g \leq -2$ be an integer. Then every $n \in \mathbf{Z}$ can be represented in the form

$$n = \sum_{j=0}^{\ell} n_i g^i, \quad 0 \leq n_i < g.$$

1.1. β representation

A. Rényi (1957): Let $\beta > 1$ be a real number and $\mathcal{A} = \{0, 1, \cdots, \lfloor\beta\rfloor\}$ be the set of digits. Then each $\gamma \in [0, \infty)$ can be represented by

$$\gamma = a_m\beta^m + a_{m-1}\beta^{m-1}\cdots \tag{1}$$

with $a_i \in \mathcal{A}$. This β-representation is usually not unique.

Assuming however that

$$0 \leq \gamma - \sum_{i=n}^{m} a_i\beta^i < \beta^n \tag{2}$$

hold for all $n \leq m$ the β-representation become unique. For $\gamma \in [0, 1)$ this greedy expansion can be given by the β-transformation

$$T_\beta(\gamma) = \beta\gamma - \lfloor\beta\gamma\rfloor$$

This concept is the topics of a lot of research:

- Description of the representations of 1, Erdős, Joó, Horváth.
- Characterization of univoque numbers, i.e. those β for which 1 has a unique representation, Daróczy, Kátai, Komornik, Loretti, Allouche, Cosnard.
- Connections with fractals: Rauzy, Thurston, Akiyama.
- Characterization of those β which leads to finite or eventually finite representations, Bertrand, Schmidt, Frougny, Solomyak, Hollander.
- Connection with radix representations based on linear recursive sequences, Zeckendorf, Fraenkel, Grabner, Tichy, Pethő.

1.2. CNS representation

Number rings: Knuth, Kátai, J. Szabó, B. Kovács, Gilbert (1960-1981):
Let $\mathbb{Z}_{\mathbf{K}}$ be the ring of integers of the algebraic number field $\mathbf{K}$.

$\{\alpha, \mathcal{N}\}$; $\alpha \in \mathbb{Z}_{\mathbf{K}}$, $\mathcal{N} = \{0, \ldots, |Norm(\alpha)|-1\}$ is called a *canonical number system* if every $\nu \in \mathbb{Z}_{\mathbf{K}}$ can be represented in the form

$$\nu = \sum_{j=0}^{\ell} n_i \alpha^i, \quad n_i \in \mathcal{N}.$$

1.3. CNS polynomials

Observation: If $\mathbb{Z}_{\mathbf{K}}$ is monogenic then $\mathbb{Z}_{\mathbf{K}} = \mathbb{Z}[\alpha]$ for some $\alpha \in \mathbb{Z}_{\mathbf{K}}$. This means $\mathbb{Z}_{\mathbf{K}} \equiv \mathbb{Z}[x]/P(x)\mathbb{Z}[x]$, where $P(x)$ is the minimal polynomial of α.
Moreover, $\{\alpha, \mathcal{N}\}$ is a CNS in $\mathbb{Z}_{\mathbf{Q}(\alpha)}$ means nothing else that every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element (a representative) such that its coefficients is bounded by $|p_0| - 1$.

A monic polynomial $P(x) = x^d + p_{d-1}x^{d-1} + \cdots + p_0$ is called CNS polynomial if every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element

$$a_0 + a_1 x + \cdots + a_k x^k \tag{3}$$

such that $0 \leq a_i < |p_0|$.

2. Comparison of the properties of greedy expansions and of CNS-polynomials

Let β be the root of $B(X) = X^d - b_1 X^{d-1} - \cdots - b_d \in \mathbb{Z}[X]$.
Let Fin(β) be the set of positive real numbers having finite greedy expansion with respect to β. We say that $\beta > 1$ has property (F) if

$$\text{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0, \infty).$$

Property (F)	**CNS-polynomials**
β is a Pisot number: $\beta > 1$, but its conjugates are < 1	the absolute value of all zeroes of $P(X)$ are larger than 1.
If $b_1 \geq \cdots \geq b_d \geq 1$, Frougny and Solomyak (1992)	If $p_{d-1} \leq \cdots \leq p_0$, $p_0 \geq 2$, B. Kovács (1981)
Characterization results if	
$b_1 > \lvert b_2\rvert + \cdots + \lvert b_d\rvert$, $b_d \neq 0$, Hollander (1996)	$p_0 > \lvert p_1\rvert + \cdots + \lvert p_{d-1}\rvert$, Akiyama, Pethő, (2002), Scheicher, Thuswaldner

3. Shift Radix Systems

Let $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d$. To $\mathbf{r}$ we associate the mapping
$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$: if $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ then let

$$\tau_{\mathbf{r}}(\mathbf{a}) = (a_2, \dots, a_d, -\lfloor \mathbf{ra} \rfloor),$$

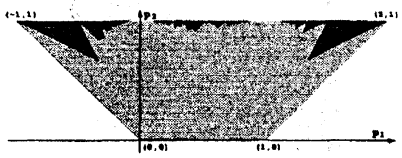
where $\mathbf{ra} = r_1 a_1 + \cdots + r_d a_d$, i.e. the inner product of $\mathbf{r}$ and $\mathbf{a}$.

Let $\mathbf{r}$ be fixed. We will show: $\mathbf{r}$ gives rise to a Pisot number β with property (F) as well as to a CNS-polynomial P iff

$$\textbf{for all } \mathbf{a} \in \mathbb{Z}^d \; \exists \, k > 0 \textbf{ with } \tau_{\mathbf{r}}^k(\mathbf{a}) = 0. \quad (4)$$

If (4) holds, we will call $\tau_{\mathbf{r}}$ a *shift radix system* (*SRS* for short). Hence SRS is a common generalization of greedy expansions with property (F) and CNS-polynomials.

It is clear that $\mathcal{D}_1 = [-1, 1]$ and $\mathcal{D}_1^0 = [0, 1)$. To illustrate the difficulty of the characterization problem of $\mathcal{D}_d^0$ we show an approximation of $\mathcal{D}_2^0$.



An approximation of $\mathcal{D}_2^0$.

4. Relation between SRS and CNS-polynomials

This is a more delicate question.

Let $P(x) = p_d x^d + p_{d-1} x^{d-1} + \cdots + p_0 \in \mathbb{Z}[x]$ with $p_d = 1$. Every coset of $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has an element of form

$$A_0 + A_1 x + \cdots + A_{d-1} x^{d-1}, \quad A_i \in \mathbb{Z}. \quad (5)$$

3.1. Relation between SRS and β-expansions

Two basic definitions. Let

$$\mathcal{D}_d^0 := \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d \exists k > 0 : \tau_{\mathbf{r}}^k(\mathbf{a}) = 0 \right\} \text{ and}$$
$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \,|\, \forall \mathbf{a} \in \mathbb{Z}^d : \{\tau_{\mathbf{r}}^k(\mathbf{a})\}_{k \geq 0} \text{ is ultimately periodic} \right\}.$$

Now we can formulate the connection between SRS and greedy expansions.

Theorem 1 (Hollander,1996). *Let $\beta > 1$ be a Pisot number with minimal polynomial $X^d - b_1 X^{d-1} - \cdots - b_{d-1} X - b_d$. Set*

$$r_1 := 1,$$
$$r_j := b_j \beta^{-1} + b_{j+1} \beta^{-2} + \cdots + b_d \beta^{j-d-1},$$

$(2 \leq j \leq d)$. Then β has property (F) if and only if $(r_d, \dots, r_2) \in \mathcal{D}_{d-1}^0$.

Let $\mathbb{Z}'[x] = \{A(x) \in \mathbb{Z}[x] \; : \; \deg A < d\}$ and

$$T(A) = \sum_{i=0}^{d-1} (A_{i+1} - q p_{i+1}) x^i,$$

where $A_d = 0$ and $q = \lfloor A_0 / p_0 \rfloor$.

Then $T : \mathbb{Z}'[x] \to \mathbb{Z}'[x]$ and

$$A = a_0 + x T(A), \text{ with } a_0 = A_0 - q p_0.$$

This backward division process can become:

- divergent $A(X) = -1$ for $P(X) = X^2 + 4X + 2$
$T_{X^2+4X+2}^k(-1) = -1, X + 4, -2X - 8, 4X + 16, \dots$ or
- ultimately periodic $A(X) = -1$ for $P(X) = X^2 - 2X + 2$
$T_{X^2-2X+2}^k(-1) = -1, X - 2, X - 1, X - 1, \dots$ or
- can terminate after finitely many steps $A(X) = -1$ for $P(X) = X^2 + 2X + 2$
$-1 = 1 + x^2 + x^3 + x^4$.

Let

$$\Pi(P) = \{A \ : \ T_P^\ell(A) = A \text{ for some } \ell > 0\}$$

denote the set of periodic points of the mapping T_P.
We always have $0 \in \Pi(P)$. With help of this set we define

$$\begin{aligned} \mathcal{C}_d^0 &= \{(p_0, p_1, \dots, p_{d-1}) \in \mathbf{Z}^d : \\ &\quad \Pi(X^d + p_{d-1}X^{d-1} + \cdots + p_0) = \{0\}\} \quad \text{and} \\ \mathcal{C}_d &= \{(p_0, p_1, \dots, p_{d-1}) \in \mathbf{Z}^d : \\ &\quad T_{X^d + p_{d-1}X^{d-1} + \cdots + p_0} \text{ has only finite orbits}\}. \end{aligned}$$

Clearly, we have $\mathcal{C}_d^0 \subset \mathcal{C}_d$.
The elements of $\mathcal{C}_d^0$ will be called CNS polynomials.

It is convenient to replace T_P by the conjugate mapping
$\tilde{T}_P : \mathbf{Z}^d \to \mathbf{Z}^d$ defined as

$$\tilde{T}_P(\mathbf{A}) = (A_1 - qp_1, \dots, A_{d-1} - qp_{d-1}, -qp_d),$$

where $\mathbf{A} = (A_0, \dots, A_{d-1})$ and $q = \lfloor A_0/p_0 \rfloor$.

4.1. Affect of a new representation

H. Brunotte (2000) and K. Scheicher and J. Thuswaldner (2001) observed that the basis transformation

$$\{1, x, \dots, x^{d-1}\} \to \{w_1, \dots, w_d\},$$
$$w_j = \sum_{i=d-j+1}^{d} p_i x^{i+j-d-1}$$

of R implies a nicer and much better applicable transformation than $\tilde{T}_P$ is. Indeed, if

$$\begin{aligned} A(x) &= \sum_{j=1}^{d} A_j w_j, \quad \text{then} \\ \tilde{T}_P(A) &= -tw_d + \sum_{j=1}^{d-1} A_{j+1} w_j, \text{ where} \\ t &= \left\lfloor \frac{p_1 A_d + \cdots + p_d A_1}{p_0} \right\rfloor. \end{aligned}$$

Hence, $\tilde{T}_P$ implies the mapping $\tau_P : \mathbf{Z}^d \to \mathbf{Z}^d$

$$\tau_P(\mathbf{A}) = \left(A_2, \dots, A_d, -\left\lfloor \frac{p_1 A_d + \cdots + p_d A_1}{p_0} \right\rfloor\right)$$

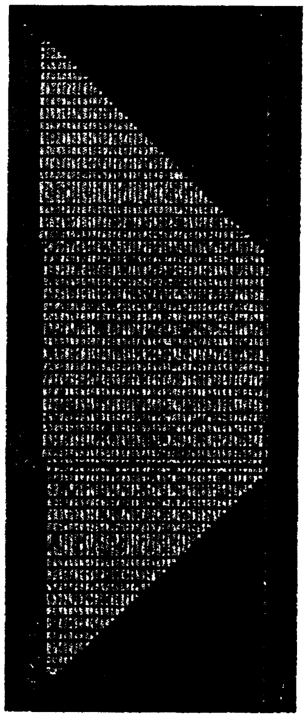
where $\mathbf{A} = (A_1, \dots, A_d)$. The mapping τ_P will be called *Brunotte's mapping*.

Theorem 2. *Let $P(X) := X^d + p_{d-1}X^{d-1} + \cdots + p_1 X + p_0 \in \mathbf{Z}[X]$. Then $P(X)$ is a CNS polynomial (or belongs to $\mathcal{C}_d^0$) if and only if* $\mathbf{r} = \left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \dots, \frac{p_1}{p_0}\right) \in \mathcal{D}_d^0$.

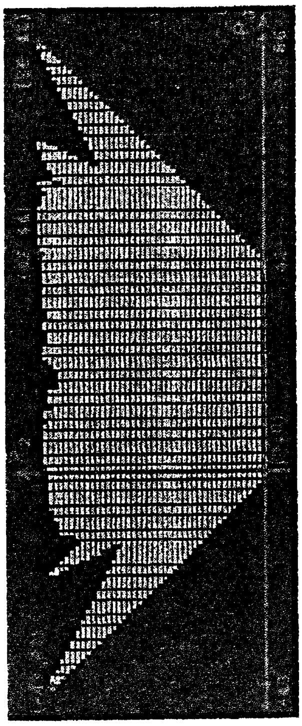
4.2. $\mathcal{C}_d^0$ for small d's

- $\mathcal{C}_1^0 = \{p_0 : p_0 \geq 2\}$, V. Grünwald
- $\mathcal{C}_2^0 = \{(p_0, p_1) : -1 \leq p_1 \leq p_0, p_0 \geq 2\}$, Kátai, Szabó, B. Kovács, Gilbert.
- Conjecture of Gilbert, 1981: $(p_0, p_1, p_2) \in \mathcal{C}_3^0$ if and only if

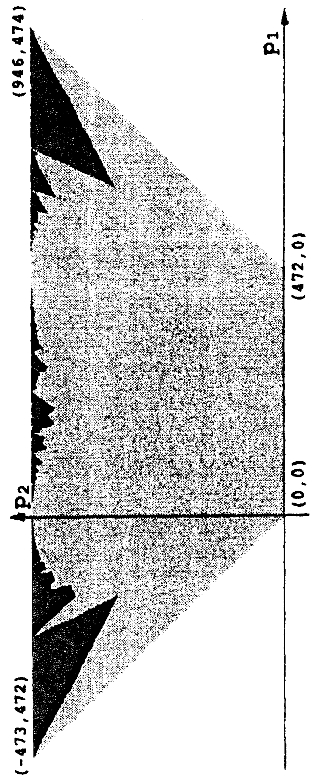
(*i*) $p_0 \geq 2$,
(*ii*) $p_2 \geq 0$,
(*iii*) $p_1 + p_2 \geq -1$,
(*iv*) $p_1 - p_2 \leq p_0 - 2$,
(*v*) $p_2 \leq \begin{cases} p_0 - 2, & \text{if} \quad p_1 \leq 0, \\ p_0 - 1, & \text{if} \quad 1 \leq p_1 \leq p_0 - 1, \\ p_0, & \text{if} \quad p_1 \geq p_0. \end{cases}$



Visualization of Gilbert's conjecture, $p_0 = 44$.



C_3^0 for $p_0 = 44$.



C_3^0 for $p_0 = 474$.

5. Basic properties of SRS

For a matrix M denote the spectral norm by $||M||$. For a vector v, $||v||$ shall denote the Euclidean norm.

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{D}_d$ let

$$R := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -r_1 & -r_2 & \cdots & \cdots & -r_d \end{pmatrix}. \qquad (6)$$

Lemma 1. *Let $d \in \mathbf{N}$. If $\mathbf{r} = (r_1, \dots, r_d) \in \mathcal{D}_d$ then the spectral radius of R is less than or equal to 1.*

In the opposite direction we get the following result.

Lemma 2. *Let $\mathbf{r} \in \mathbf{R}^d$ such that the spectral radius ρ of the matrix R given above is less than 1. Then $\mathbf{r} \in \mathcal{D}_d$.*

It is not to hard to prove the following statement

Theorem 3. *The sets $\mathcal{D}_d$ and $\mathcal{D}_d^0$ are Lebesgue measurable.*

5.1. Convexity property of $\tau_{\mathbf{r}}$

Theorem 4. *Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathbf{R}^d$ and $\mathbf{a} \in \mathbf{Z}^d$ be such that $\tau_{\mathbf{r}_1}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}(\mathbf{a})$. Let $\mathbf{s}$ be any convex linear combination of $\mathbf{r}_1, \dots, \mathbf{r}_k$. Then we have $\tau_{\mathbf{s}}(\mathbf{a}) = \tau_{\mathbf{r}_1}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}(\mathbf{a})$.*

Corollary 1. *Let $\mathbf{r}_1, \dots, \mathbf{r}_k \in \mathbf{R}$ have the same period, i.e. $\tau_{\mathbf{r}_1}^{\ell}(\mathbf{a}) = \cdots = \tau_{\mathbf{r}_k}^{\ell}(\mathbf{a}), \ell = 0, \dots, q$ and $\mathbf{a} = \tau_{\mathbf{r}_1}^{q}(\mathbf{a})$. Then if $\mathbf{s}$ is lying in the convex hull of $\mathbf{r}_1, \dots, \mathbf{r}_k$ the mapping $\tau_{\mathbf{s}}$ is periodic and has the same period as $\tau_{\mathbf{r}_1}$.*

For example, it is easy to check that for the plane vectors $r_1 = \left(\frac{381}{254}, \frac{253}{254}\right), r_2 = \left(\frac{421}{254}, \frac{253}{254}\right)$ and $r_3 = \left(\frac{344}{254}, \frac{176}{254}\right)$ the corresponding mappings have the same period $(-2, 1); 3, -2, 1, 1, -2$, hence the corresponding mapping for any point lying in the rectangle r_1, r_2, r_3 have this period.

5.2. Brunotte's algorithm

To decide $\mathbf{r} \in \mathcal{C}_d^0$ Brunotte gave an algorithm, which was realized independently by Scheicher and Thuswaldner. We give here a generalization for $\mathcal{D}_d^0$.

Theorem 5. *Suppose that there exists a set $E \subset \mathbf{Z}^d$ satisfying*
(i) E contains $2d$ elements of the form $(0, \dots, 0, \pm 1, 0, \dots, 0)$.
(ii) $\tau_{\mathbf{r}}(E) \cup \tau_{\mathbf{r}}^*(E) \subset E$, where $\tau_{\mathbf{r}}^*(\mathbf{x}) = -\tau_{\mathbf{r}}(-\mathbf{x})$.
(iii) For each $\mathbf{a} \in E$ there is some $k > 0$ such that $\tau_{\mathbf{r}}^k(\mathbf{a}) = 0$.
Then $\mathbf{r} \in \mathcal{D}_d^0$.

An example: Let $\mathbf{r} = \left(\frac{2}{5}, -\frac{1}{5}\right)$.
Starting from $E_0 = \{(\pm 1, 0), (0, \pm 1)\}$ and using that
$\tau_{\mathbf{r}}(1,0) = (0,0), \tau_{\mathbf{r}}(-1,0) = (0,1), \tau_{\mathbf{r}}(0,1) = (1,1)$,
$\tau_{\mathbf{r}}(0,-1) = (-1,0)$ we get
$E_1 = \tau_{\mathbf{r}}(E_0) \cup \tau_{\mathbf{r}}^*(E_0) = E_0 \cup \{(0,0), (1,1), (-1,-1)\}$.
Now
$\tau_{\mathbf{r}}(1,1) = (1,0), \tau_{\mathbf{r}}(-1,-1) = (-1,1)$, hence we may take
$E_2 = \tau_{\mathbf{r}}(E_1) \cup \tau_{\mathbf{r}}^*(E_1) = E_1 \cup \{(1,-1), (-1,1)\}$.
Finally because
$\tau_{\mathbf{r}}(-1,1) = (1,1), \tau_{\mathbf{r}}(1,-1) = (-1,0)$, we get that
$E = E_2$ proves $\mathbf{r} \in \mathcal{D}_2^0$.

6. Lifting theorem

Let $d \in \mathbf{N}$ and

$$(a_{1+j}, \dots, a_{d+j}) \in \mathbf{Z}^d, \qquad (0 \le j \le L-1), \quad (7)$$

with $a_{L+1} = a_1, \dots, a_{L+d} = a_d$.
For which $\mathbf{r} = (r_1, \dots, r_d) \in \mathbf{R}^d$ these vectors form a period π of $\mathcal{D}_d$? By the definition of $\tau_{\mathbf{r}}$ this is the case if and only if the inequalities

$$0 \le r_1 a_{1+j} + \cdots + r_d a_{d+j} + a_{d+j+1} < 1 \quad (8)$$

hold simultaneously for all $0 \le j \le L-1$. They define a (possibly degenerated) polyhedron, which will be denoted by $\mathcal{P}(\pi)$.

Since $\mathbf{r} \in \mathcal{D}_d^0$ if and only if $\tau_{\mathbf{r}}$ has 0 as its only period we conclude that

$$\mathcal{D}_d^0 = \mathcal{D}_d \setminus \bigcup_{\pi \ne 0} \mathcal{P}(\pi)$$

where the union is extended over all families of vectors π of the shape (7). We call the family of (non-empty) polyhedra corresponding to this choice the *family of cutout polyhedra of $\mathcal{D}_d^0$*.

Let π be a period of $\mathcal{C}_d$ or $\mathcal{D}_d$ which corresponds to a non-degenerate cutout polyhedron. Then we call π a *non-degenerate period*. We will show that we can "lift" a non-degenerate period to higher dimensions.

Definition 1. *Let*

$$\pi : (a_1, \dots, a_d); a_{d+1}, \dots, a_L \qquad (9)$$

be a non-degenerate period of length L of $\mathcal{C}_d$ or $\mathcal{D}_d$. Then we call

$$l(\pi) : (a_1, a_2, \dots, a_{d+1}); a_{d+2}, \dots, a_L \qquad (10)$$

the lift of π to $d+1$.

Note that π and $l(\pi)$ have the same period length L.

Theorem 6 (Lifting Theorem). *Let $d \geq 1$ be an integer.*

(i) Let $p_0 \geq 2$ and let π be a non-degenerate period for $\mathcal{C}_d$. Then π is also a non-degenerate period for $\mathcal{D}_d$. More precisely, there exist $p_1, \ldots, p_{d-1} \in \mathbf{Z}$ such that $(p_0, \ldots, p_{d-1}) \in \operatorname{int}(\mathcal{P}'(\pi))$ and

$$\left(\frac{1}{p_0}, \frac{p_{d-1}}{p_0}, \cdots, \frac{p_1}{p_0}\right) \in \operatorname{int}(\mathcal{P}(\pi)).$$

(ii) Let π be a non-degenerate period of $\mathcal{D}_d$. Then its lift $\lambda := l(\pi)$ is a non-degenerate period of $\mathcal{C}_{d+1}$ for each sufficiently large p_0. More precisely, for all $(r_1, \ldots, r_d) \in \operatorname{int}(\mathcal{P}(\pi))$ there exists $\varepsilon > 0$ such that for all $(p_0, \ldots, p_d) \in \mathbf{Z}^{d+1}$ with

$$\max_{1 \leq k \leq d} \left| \frac{p_{d+1-k}}{p_0} - r_k \right| < \varepsilon$$

we have $(p_0, \ldots, p_d) \in \operatorname{int}(\mathcal{P}'(\lambda))$.

7. Long periods

Consider the following family of edges.

$$\begin{array}{llcll}
\alpha_k: & (-k-1, -n+k) & \to & (-n+k, k+1) & \\
 & & & (0 \leq k \leq n-1), & \\
\beta_k: & (-n+k, k+1) & \to & (k+1, n+1-k) & \\
 & & & (0 \leq k \leq n-1), & \\
\gamma_0: & (1, n+1) & \to & (n+1, 1), & \\
\gamma_k: & (k+1, n+1-k) & \to & (n+1-k, -k) & \\
 & & & (1 \leq k \leq n-1), & \\
\gamma_n: & (n+1, 1) & \to & (1, -n), & \\
\delta_k: & (n+1-k, -k) & \to & (-k, -n-1+k) & \\
 & & & (1 \leq k \leq n-1). &
\end{array}$$

With these edges we form the cycle

$$\zeta_n : \alpha_0\beta_0\gamma_0\gamma_n\alpha_{n-1}\beta_{n-1}\gamma_{n-1}\delta_{n-1}\alpha_{n-2}\cdots\alpha_1\beta_1\gamma_1\delta_1.$$

Note that δ_1 ends up in $(-1, -n)$. In this node α_0 starts. Thus ζ_n is indeed a cycle. We wonder whether there exists $\mathbf{r} := (x_n, y_n) \in \mathcal{D}_2$ such that $\tau_{\mathbf{r}}$ has ζ_n as a non-degenerate period. This is done in the following result.

Theorem 7. *Fix $n \in \mathbf{N}$, $n > 3$, and set $\mathbf{r} = (x_n, y_n) \in \mathbf{R}^2$ with*

$$x_n := 1 - \frac{1}{2n^2} + z_n \quad \text{and} \quad y_n := -\frac{2n+1}{2n(n+1)} + u_n,$$

where $|z_n|, |u_n| < 1/n^4$. Then ζ_n is a non-degenerate period of $\tau_{\mathbf{r}}$.

Since we can select n arbitrarily large and the length of the period ζ_n is $4n+1$ the previous theorem implies that there exist non-degenerate periods of arbitrarily large length for $\mathcal{D}_2$.

By a direct application of the Lifting Theorem we obtain.

Theorem 8. *Let $d \geq 2$ be an integer, fix $n \in \mathbf{N}$, $n > 3$. Then there exist some $\mathbf{r} \in \mathbf{R}^d$ such that $l^{d-2}(\zeta_n)$ is a non-degenerate period of $\tau_{\mathbf{r}}$. Since we can select n arbitrarily large and the length of the period $l^{d-2}(\zeta_n)$ is $4n+1$ this implies that there exist non-degenerate periods of arbitrarily large length of $\mathcal{D}_d$ and $\mathcal{C}_{d+1}$.*

Corollary 2. *Fix $n \in \mathbf{N}$, $n > 3$, and set $d \geq 2$ and $\mathbf{r} = (0, \ldots, 0, x_n, y_n) \in \mathbf{R}^d$ with x_n, y_n as in Theorem 7. Then $l^{d-2}(\zeta_n)$ is a period of $\tau_{\mathbf{r}}$.*

8. Critical points

Definition 2. *Let* $\mathrm{x} \in \mathcal{D}_d$.

- *If there exists an open neighborhood of* x *which contains only finitely many cutout polyhedra then we call* x *a* **regular point**.
- *If each open neighborhood of* x *has nonempty intersection with infinitely many cutout polyhedra then we call* x *a* **weak critical point** *for* $\mathcal{D}_d$.
- *If for each open neighborhood* U *of* x *the set* $U \setminus \mathcal{D}_d^0$ *can not be covered by finitely many cutout polyhedra then* x *is called a* **critical point**.

We will show the existence of critical points for each $d \geq 2$. This shows that there is no way to characterize either of the sets $\mathcal{D}_d^0$ by finitely many cutouts if $d \geq 2$.

Lemma 3. *Let* x *be a weak critical point for* $\mathcal{D}_d$. *Then* $\mathrm{x} \in \partial\mathcal{D}_d$.

Lemma 4. *Let* $\{x_n\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *be sequences with* $x_n < 1$, $y_n < 0$, $\lim x_n = 1$, $\lim y_n = 0$ *and* $1 - x_n = o(y_n)$. *Let* $\{a_m\}_{m\geq 1}$ *be a sequence of integers such that* $|a_i| < K$ *for some constant* K. *Then there exists* $N \in \mathbb{N}$ *such that*

$$0 \leq a_{i-1}x_n + a_i y_n + a_{i+1} < 1 \qquad (11)$$

can not hold for all i *if* $n \geq N$ *unless* $a_i = 0$ *for all* i *large enough. Thus nonzero periods whose elements are bounded by* K *can not occur in* $\mathcal{D}_d$ *for* $\tau_{(0,\ldots,0,x_n,y_n)}$ *if* n *is large enough.*

Theorem 9. *Let* $d \geq 2$. *Then* $\mathrm{K}_d := (0, \ldots, 1, 0) \in \mathbb{R}^d$ *is a critical point of* $\mathcal{D}_d$.

Problem 1. *Characterize the critical points of* $\mathcal{D}_d$. *Can one show that for a given* d *there exist only finitely many critical points?*