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Dynamics of Teichmüller modular groups and general topology of moduli spaces: Announcement

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§0. PREFACE

This is an announcement of the author's recent researches on dynamics of Teichmüller modular groups and general topology of moduli spaces. All theorems are stated without proof. A complete paper is intended to be published elsewhere.

We emphasize how reasonably the study on the action of Teichmüller modular groups can be generalized to purely topological consideration on the dynamics of isometry groups for complete metric spaces. In this general situation, the comparison of countability versus uncountability works as a fundamental machinery for our arguments. When we apply this principle to Teichmüller modular groups, countable compactness of Riemann surfaces can stand for the countable side. In the first part of this note, we collect several consequences deduced from this topological structure of Riemann surfaces. Then we apply more specific results based on the hyperbolic geometric structure on Riemann surfaces in order to focus on the feature of the dynamics of Teichmüller modular groups.

§1. TEICHMÜLLER SPACES AND MODULAR GROUPS

The Teichmüller space $T(R)$ of a hyperbolic Riemann surface R is the set of all equivalence classes of the pair (f,σ), where $f : R \to R_\sigma$ is a quasiconformal homeomorphism of R onto another Riemann surface R_σ of a complex structure σ. Two pairs (f_1,σ_1) and (f_2,σ_2) are defined to be equivalent if $\sigma_1 = \sigma_2$ and $f_2 \circ f_1^{-1}$ is isotopic to a conformal automorphism of $R_{\sigma_1} = R_{\sigma_2}$. Here and below the isotopy is considered to be relative to the ideal boundary at infinity. The equivalence class of (f,σ) is denoted by $[f,\sigma]$ or just by $[f]$ in brief.

A distance between equivalence classes $p_1 = [f_1]$ and $p_2 = [f_2]$ in $T(R)$ is defined by $d_T(p_1,p_2) = \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the isotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete metric on $T(R)$, which is called the Teichmüller distance.

The Teichmüller modular group $\mathrm{Mod}(R)$ of R (or the quasiconformal mapping class group) is the group of all isotopy classes of quasiconformal automorphisms of R. An element γ of $\mathrm{Mod}(R)$ acts on $T(R)$ from the left in such a way that $\gamma_* : [f,\sigma] \mapsto [f \circ \gamma^{-1}, \sigma]$, where γ also means a representative of the isotopy class. It is evident from definition that $\mathrm{Mod}(R)$ acts on $T(R)$ isometrically with respect to the Teichmüller distance. Let $\theta : \mathrm{Mod}(R) \to \mathrm{Isom}(T(R))$ be a homomorphism defined by $\gamma \mapsto \gamma_*$, where $\mathrm{Isom}(T(R))$ denotes the group of all isometric automorphisms of $T(R)$. Except for a few cases, θ is injective. In particular, if R is analytically infinite, then θ is injective. This was first proved by Earle, Gardiner and Lakic. Another proof was given by Epstein [E]. Furthermore, Markovic [M] proved that θ is surjective. Hence, we may identify $\mathrm{Mod}(R)$ with $\mathrm{Isom}(T(R))$ and denote $\gamma_* \in \mathrm{Isom}(T(R))$ simply by γ.

Hyperbolic geometric aspects of Riemann surfaces affect the structure of their Teichmüller spaces and modular groups. Certain moderate assumptions on the geometry make their analysis easier.

Definition. We say that a hyperbolic Riemann surfaces R satisfies the *bounded geometry condition* if the following three properties are satisfied:

(a) The injectivity radius at any point of R is uniformly bounded away from zero except for cusp neighborhoods;

(b) There exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R;

(c) R has no ideal boundary at infinity.

This condition is quasiconformally invariant and hence we may regard it as an assumption on the Teichmüller space $T(R)$. For example, every normal cover of an analytically finite Riemann surface satisfies the bounded geometry condition except the universal cover.

§2. Dynamics on complete metric spaces

In general, let $X = (X, d)$ be a complete metric space with a distance d, and $\mathrm{Isom}(X)$ the group of all isometric automorphisms of X. For a subgroup $\Gamma \subset \mathrm{Isom}(X)$, the orbit of $x \in X$ under Γ is denoted by $\Gamma(x)$ and the isotropy (stabilizer) subgroup of $x \in X$ in Γ is denoted by $\mathrm{Stab}_\Gamma(x)$. For an element $\gamma \in \mathrm{Isom}(X)$, the set of all fixed points of γ is denoted by $\mathrm{Fix}(\gamma)$.

For a subgroup $\Gamma \subset \mathrm{Isom}(X)$ and for a point $x \in X$, a point $y \in X$ is a ***limit point*** of x for Γ if there exists a sequence $\{\gamma_n\}$ of distinct elements of Γ such that $\gamma_n(x)$ converge to y. The set of all limit points of x for Γ is denoted by $\Lambda(\Gamma, x)$ and the *limit set* for Γ is defined by $\Lambda(\Gamma) = \bigcup_{x \in X} \Lambda(\Gamma, x)$. It is said that $x \in X$ is a *recurrent point* for Γ if $x \in \Lambda(\Gamma, x)$ and the set of all recurrent points for Γ is denoted by $\mathrm{Rec}(\Gamma)$. It is evident that $\mathrm{Rec}(\Gamma) \subset \Lambda(\Gamma)$ and these sets are Γ-invariant.

The following fact is proved in [F].

Proposition 2.1. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$*, the limit set* $\Lambda(\Gamma)$ *is coincident with* $\mathrm{Rec}(\Gamma)$ *and it is a closed set. Moreover,* $x \in X$ *is a limit point of* Γ *if and only if either the orbit* $\Gamma(x)$ *is not discrete or the isotropy subgroup* $\mathrm{Stab}_\Gamma(x)$ *consists of infinitely many elements.*

A limit point $x \in \Lambda(\Gamma)$ is called a *generic limit point* if $\Gamma(x)$ is not discrete, and a *fixed limit point* if $\mathrm{Stab}_\Gamma(x)$ is infinite. The set of all generic limit points is denoted by $\Lambda_0(\Gamma)$ and the set of all fixed limit points is denoted by $\Lambda_\infty(\Gamma)$. By Proposition 2.1, we see that $\Lambda(\Gamma) = \Lambda_0(\Gamma) \cup \Lambda_\infty(\Gamma)$, however the intersection can be non-empty. Furthermore $\Lambda_\infty(\Gamma)$ is divided into two disjoint subsets $\Lambda^1_\infty(\Gamma)$ and $\Lambda^2_\infty(\Gamma)$, which are also introduced in [F]. A limit point $x \in \Lambda_\infty(\Gamma)$ belongs to $\Lambda^1_\infty(\Gamma)$ if there is an element of infinite order in $\mathrm{Stab}_\Gamma(x)$ and otherwise to $\Lambda^2_\infty(\Gamma)$. In other words, $\Lambda^1_\infty(\Gamma) = \bigcup \mathrm{Fix}(\gamma)$, where the union is taken over all elements $\gamma \in \Gamma$ of infinite order.

Here we introduce several criteria for discontinuity and stability of Γ.

Definition. Let Γ be a subgroup of $\mathrm{Isom}(X)$. We say that Γ acts at $x \in X$

(a) *discontinuously* if $\Gamma(x)$ is discrete and $\mathrm{Stab}_\Gamma(x)$ is finite;
(b) *weakly discontinuously* if $\Gamma(x)$ is discrete;
(c) *stably* if $\Gamma(x)$ is closed and $\mathrm{Stab}_\Gamma(x)$ is finite;
(d) *weakly stably* if $\Gamma(x)$ is closed.

If Γ acts at every point x in X discontinuously, stably and so on, then we say that Γ acts on X discontinuously, stably and so on. The set of points $x \in X$ where Γ acts discontinuously is denoted by $\Omega(\Gamma)$ and called the *region of discontinuity* for Γ. The set of points $x \in X$ where Γ acts stably is denoted by $\Phi(\Gamma)$ and called the *region of stability* for Γ. There is an inclusion relation $\Omega(\Gamma) \subset \Phi(\Gamma)$.

The discontinuity is usually defined in another way, however, as the following proposition says, these definitions are all equivalent.

Proposition 2.2. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$ *and a point* $x \in X$*, the following conditions are equivalent:*

(1) Γ *acts at* x *discontinuously;*
(2) *There exists an open ball* U *centered at* x *such that the number of elements* $\gamma \in \Gamma$ *satisfying* $\gamma(U) \cap U \neq \emptyset$ *is finite;*
(3) x *is not a limit point of* Γ.

Hence the region of discontinuity $\Omega(\Gamma)$ *is coincident with* $X - \Lambda(\Gamma)$*, which is an open set.*

Similar statements hold for weak discontinuity.

Proposition 2.3. *For a subgroup* $\Gamma \subset \mathrm{Isom}(X)$ *and a point* $x \in X$*, the following conditions are equivalent:*

(1) Γ *acts at* x *weakly discontinuously;*
(2) *There exists an open ball* U *centered at* x *such that* $\gamma(U) = U$ *for every* $\gamma \in \mathrm{Stab}_\Gamma(x)$ *and* $\gamma(U) \cap U = \emptyset$ *for every* $\gamma \in \Gamma - \mathrm{Stab}_\Gamma(x)$*;*

(3) *x is not a generic limit point of* Γ.

Discontinuity and stability criteria mentioned above have obvious inclusion relations immediately known from their definitions. The following theorem says that a converse assertion holds under a certain countability assumption. This fact is based on the Baire category theorem and uncountability of perfect closed sets.

Theorem 2.4. *Assume that $\Gamma \subset \mathrm{Isom}(X)$ contains a subgroup G of countable index (that is, the cardinality of Γ/G is countable) such that G acts at $x \in X$ (weakly) discontinuously. If Γ acts at x (weakly) stably, then Γ acts at x (weakly) discontinuously (respectively). In particular, this claim is always satisfied if Γ itself is countable.*

While the region of discontinuity $\Omega(\Gamma)$ is always an open set, the region of stability $\Phi(\Gamma)$ becomes an open set under a certain condition upon Γ. This is also based on the Baire category theorem.

Theorem 2.5. *If $\Gamma \subset \mathrm{Isom}(X)$ contains a subgroup G of countable index such that G acts on X stably, then the region of stability $\Phi(\Gamma)$ is open. In particular, this claim is always satisfied if Γ itself is countable.*

§3. Closure equivalence

We consider quotient spaces of a metric space (X, d) by the group action of $\mathrm{Isom}(X)$. For an arbitrary subgroup Γ of $\mathrm{Isom}(X)$, we define two points x and y in X to be equivalent ($x \sim y$) if there exists a sequence of elements γ_n of Γ not necessarily distinct such that $\gamma_n(x)$ converge to y. In particular, all points in the orbit of Γ are mutually equivalent. It is easy to check that this satisfies the axiom of equivalence relation, which we call *closure equivalence*. In particular, the conditions $\overline{\Gamma(x_1)} \cap \overline{\Gamma(x_2)} \neq \emptyset$ and $\overline{\Gamma(x_1)} = \overline{\Gamma(x_2)}$ are both equivalent to $x_1 \sim x_2$.

The closure equivalence is stronger than the ordinary equivalence under the group action of Γ. The ordinary quotient space by Γ is denoted by X/Γ and the quotient space by the closure equivalence is denoted by $X//\Gamma$. The projections are denoted by $\pi_1 : X \to X/\Gamma$ and $\pi_2 : X \to X//\Gamma$ respectively. There is also a projection $\bar{\pi} : X/\Gamma \to X//\Gamma$ defined by $\pi_2 \circ (\pi_1)^{-1}$.

The pseudo-distance d induces pseudo-distances d_1 on X/Γ and d_2 on $X//\Gamma$ as

$$d_1(\pi_1(x), \pi_1(y)) := \inf\{d(x', y') \mid x' \in \Gamma(x),\ y' \in \Gamma(y)\};$$
$$d_2(\pi_2(x), \pi_2(y)) := \inf\{d(x', y') \mid x' \sim x,\ y' \sim y\}.$$

Here d_2 always becomes a distance in virtue of the way of defining the closure equivalence. Hence $(X//\Gamma, d_2)$ is a complete metric space.

A theorem on general topology says the following.

Theorem 3.1. *For a subgroup $\Gamma \subset \mathrm{Isom}(X)$ and a point $x \in X$, the following conditions are equivalent:*

(a) *Γ acts at x weakly stably;*

(b) *There exists no different point $\pi_1(y)$ from $\pi_1(x)$ such that $d_1(\pi_1(x), \pi_1(y)) = 0$;*
(c) *For every point $\pi_1(y)$ different from $\pi_1(x)$, there exists a neighborhood of $\pi_1(y)$ that separates $\pi_1(x)$;*
(d) *A point set $\{\pi_1(x)\}$ is closed in X/Γ.*

Corollary 3.2. *The quotient space X/Γ satisfies the first separation axiom if and only if the pseudo-distance d_1 on X/Γ is a distance. In this case, $\bar{\pi} : X/\Gamma \to X//\Gamma$ is a homeomorphism.*

§4. Dynamics of Teichmüller modular groups and moduli spaces

For an analytically finite Riemann surface R, the Teichmüller modular group $\mathrm{Mod}(R)$ acts on $T(R)$ discontinuously. Although $\mathrm{Mod}(R)$ has fixed points on $T(R)$, each orbit is discrete and each isotropy subgroup is finite. Hence an orbifold structure on the moduli space $M(R)$ is induced from $T(R)$ as the quotient space by $\mathrm{Mod}(R)$. However, this is not always satisfied for analytically infinite Riemann surfaces.

Hereafter, we assume that R is analytically infinite. We introduce the concepts (limit set etc.) defined in the previous sections to the Teichmüller space $X = T(R)$ with the Teichmüller distance $d = d_T$ and the group of all isometries $\mathrm{Isom}(X) = \mathrm{Mod}(R)$. Then the results in the previous sections are all applicable to this case. Moreover, the following property peculiar to $\mathrm{Mod}(R)$ (partially proved in [FST]) enables us to conclude more interesting consequences from Theorems 2.4 and 2.5.

Theorem 4.1. *For a free homotopy class c of a simple closed geodesic on R, set*

$$G = \{g \in \mathrm{Mod}(R) \mid g(c) = c\}.$$

Then G is a subgroup $\mathrm{Mod}(R)$ of countable index and it acts on $T(R)$ stably. Moreover, if $T(R)$ satisfies the bounded geometry condition, then G acts on $T(R)$ discontinuously.

Then Theorems 2.5 and 2.4 turn to be the following assertions respectively.

Theorem 4.2. *The region of stability $\Phi(\Gamma)$ for a subgroup Γ of $\mathrm{Mod}(R)$ is an open subset of $T(R)$.*

Theorem 4.3. *If $T(R)$ satisfies the bounded geometry condition, then the (weak) stability of a subgroup Γ of $\mathrm{Mod}(R)$ is equivalent to the (weak) discontinuity (respectively). If Γ is countable, then this is valid without any assumption on $T(R)$.*

Remark that one cannot remove the assumptions on $T(R)$ and Γ in Theorem 4.3. Namely, there is an example of an uncountable subgroup $\Gamma \subset \mathrm{Mod}(R)$ which acts on $T(R)$ stably but not discontinuously. For instance, let R have a sequence of mutually disjoint, simple closed geodesics $\{c_n\}_{n=1}^{\infty}$ with the geodesic lengths $\ell(c_n)$ tend to 0 and Γ be a subgroup of $\mathrm{Mod}(R)$ represented by the composition of the

Dehn twists along $\{c_n\}$. Then the orbit $\Gamma(p)$ for every $p \in T(R)$ is closed but not discrete.

If $\operatorname{Mod}(R)$ is countable, then the geometry of R is much more restricted by this assumption itself and we have a stronger result than Theorem 4.3. This is given in [ ].

Theorem 4.4. *If $\Gamma = \operatorname{Mod}(R)$ is countable, then it acts discontinuously on $T(R)$, namely, $\Lambda(\Gamma) = \emptyset$.*

Next we consider the moduli space of a Riemann surface R. No matter how the action of $\operatorname{Mod}(R)$ is far from discontinuity, the moduli space $M(R)$ is a topological space by the quotient topology induced by the projection $\pi_1 = \pi_M : T(R) \to M(R) = T(R)/\operatorname{Mod}(R)$. We call $M(R)$ the *topological* moduli space. Moreover a pseudo-distance $d_1 = d_M$ on $M(R)$ is induced from the Teichmüller distance $d = d_T$ on $T(R)$.

We define two subregions in $M(R)$: an open subregion $M_\Omega(R) = \Omega(\Gamma)/\Gamma$ is the *geometric* moduli subspace and an open subregion $M_\Phi(R) = \Phi(\Gamma)/\Gamma$ is the *metric* moduli subspace. The $M_\Phi(R)$ is the maximal open subset of $M(R)$ where the restriction of the pseudo-distance d_M becomes a distance.

The *contracted* moduli space $M_*(R)$ is a complete metric space, which is the quotient by the closure equivalence with the projection

$$\pi_2 = \pi_{M_*} : T(R) \to M_*(R) = T(R)//\operatorname{Mod}(R).$$

The distance $d_2 = d_{M_*}$ is induced from $d = d_T$. Let $\bar{\pi} : M(R) \to M_*(R)$ be the canonical projection. The inverse image $\bar{\pi}^{-1}(s)$ for $s \in M_*(R)$ is the closure $\overline{\{\sigma\}} \subset M(R)$ for any point $\sigma \in \bar{\pi}^{-1}(s)$.

If $\operatorname{Mod}(R)$ acts on $T(R)$ weakly stably, then the contracted moduli space $M_*(R)$ is nothing but the topological moduli space $M(R)$ and the pseudo-distance d_M is coincident with the distance d_{M_*} under the homeomorphism $\bar{\pi}$. However, if it does not act weakly stably, the projection $\bar{\pi} : M(R) \to M_*(R)$ is non-trivial and d_M is not a distance on $M(R)$.

Finally, we give an example of a quotient space defined by a proper subgroup Γ of $\operatorname{Mod}(R)$. Let Γ be a subgroup of $\operatorname{Mod}(R)$ consisting of all elements γ that are freely isotopic to the identity of R, where R is assumed to have the ideal boundary at infinity. It is clear that Γ is normal in $\operatorname{Mod}(R)$. Also Γ acts on $T(R)$ weakly stably. Then $T(R)/\Gamma = T(R)//\Gamma$ is the reduced Teichmüller space $T^\#(R)$, $d_1 = d_2$ is the reduced Teichmüller distance $d^\#$, and $\operatorname{Mod}(R)/\Gamma$ is the reduced Teichmüller modular group $\operatorname{Mod}^\#(R)$, which acts on $(T^\#(R), d^\#)$ isometrically.

§5. Classification of the modular transformations

For an analytically finite Riemann surface R, there are two kinds of classification of the elements of $\operatorname{Mod}(R)$ related to each other: one is topological classification due to Thurston and the other is analytical classification due to Bers [B]. The latter can be regarded as a generalization of the type of the isometric automorphisms of the hyperbolic plane (space), and classifies the elements of $\operatorname{Mod}(R)$ as follows.

Definition. An element γ of $\mathrm{Mod}(R)$ is called

(a) *elliptic* if γ has a fixed point on $T(R)$;
(b) *parabolic* if $\inf_{p\in T(R)} d_T(\gamma(p),p) = 0$ but γ has no fixed point on $T(R)$;
(c) *hyperbolic* if $\inf_{p\in T(R)} d_T(\gamma(p),p) > 0$.

When the Riemann surface R is analytically infinite, the topological classification of $\mathrm{Mod}(R)$ is no more effective, whereas the analytical classification still works reasonably. Hence, even in this case, we adopt the definition as above to classify the elements of $\mathrm{Mod}(R)$.

An elliptic element γ of $\mathrm{Mod}(R)$ is realized as a conformal automorphism of the Riemann surface corresponding to the fixed point of γ. In the case where R is analytically finite, an elliptic element of $\mathrm{Mod}(R)$ is of finite order because every conformal automorphism of an analytically finite Riemann surface is of finite order. However, in the case where R is analytically infinite, an elliptic element of $\mathrm{Mod}(R)$ can be of infinite order.

In the analytically finite case, if $\gamma \in \mathrm{Mod}(R)$ is of finite order, then we conversely know that γ is elliptic from the Nielsen theorem. Furthermore, by the solution of the Nielsen realization problem due to Kerchhoff [K], we have the following equivalent conditions on a subgroup of $\mathrm{Mod}(R)$ not necessarily cyclic.

Proposition 5.1. *Let R be an analytically finite Riemann surface and Γ a subgroup of the Teichmüller modular group* $\mathrm{Mod}(R)$. *Then the following conditions are equivalent.*

(1) *Γ is a finite group.*
(2) *Γ has a common fixed point on $T(R)$.*
(3) *For every/some point $p \in T(R)$, the orbit $\Gamma(p)$ is a bounded set in $T(R)$.*

We consider generalization of this fact to the analytically infinite case. However, we do not have to restrict ourselves to finite groups in this case. We propose the following conjecture as the generalization of the Nielsen realization problem.

Conjecture 5.2. A subgroup Γ of $\mathrm{Mod}(R)$ has a common fixed point on $T(R)$ if the orbit $\Gamma(p)$ is bounded for every/some $p \in T(R)$. In particular, $\gamma \in \mathrm{Mod}(R)$ is elliptic if the orbit $\langle\gamma\rangle(p)$ is bounded.

Let $\Delta \to R$ be the universal cover of a Riemann surface R and H_R the corresponding Fuchsian group acting on the unit disk Δ. Let Γ be a subgroup of $\mathrm{Mod}(R)$ and assume that the orbit $\Gamma(p)$ is bounded for some $p \in T(R)$. We lift a representative of each $\gamma \in \Gamma$ to Δ as a quasiconformal automorphism and extend it to a quasisymmetric homeomorphism of the boundary $\partial\Delta$. In this way, we have a group H of quasisymmetric homeomorphisms that contains the Fuchsian group H_R as a normal subgroup. Since the orbit $\Gamma(p)$ is bounded, we see that there exists a uniform bound for the quasisymmetric constants of all elements of H, namely, H is a quasisymmetric group. Then the above conjecture follows from the next.

Conjecture 5.3. For a quasisymmetric group H acting on the unit circle $\partial\Delta$, there exists a quasisymmetric homeomorphism f of $\partial\Delta$ such that fHf^{-1} is the restriction of a Fuchsian group to $\partial\Delta$.

A partial solution to this problem is given by Hinkkanen [H]. If H extends to Δ as a quasiconformal group $\tilde{H}$, then by Tukia [T1], we can always find a quasiconformal homeomorphism $\tilde{f}$ that conjugates $\tilde{H}$ to a Fuchsian group. However, the barycentric extension E does not have quasiconformal naturality $E(h_1h_2) = E(h_1)E(h_2)$ for instance; it is difficult to find a quasiconformal extension of H as a group.

A quasisymmetric group H is a convergence group. By celebrated results due to Tukia [T2] and Gabai [G], every convergence group acting on $\partial\Delta$ is homeomorphically conjugate to a Fuchsian group by f. In other words, one has an extension of H to a group $\tilde{H}$ of homeomorphisms of Δ. The above conjecture actually asks whether this homeomorphism f can be taken to be quasisymmetric for the quasisymmetric group H. In case R is analytically finite, f automatically becomes quasisymmetric, and hence the Nielsen realization problem has an affirmative answer as a special case of this problem.

Next we look at the orbit of a cyclic group of $\mathrm{Mod}(R)$ and raise a problem to characterize it in terms of the type of the modular transformation. For an elliptic transformation $\gamma \in \mathrm{Mod}(R)$ of finite order, the orbit under $\Gamma = \langle\gamma\rangle$ is finite. However, For $\gamma \in \mathrm{Mod}(R)$ of infinite order, we can prove the following.

Theorem 5.4. *Let $\gamma \in \mathrm{Mod}(R)$ be an elliptic transformation of infinite order. Then the cyclic group $\Gamma = \langle\gamma\rangle$ does not act weakly stably on $T(R)$.*

For a parabolic or hyperbolic modular transformation γ, we do not know whether the orbit is discrete or not. As a conjecture, we expect that $\langle\gamma\rangle(p)$ is discrete for every $p \in T(R)$. In other words, comparing with the bounded orbit conjecture above, we have the indiscrete orbit conjecture as follows.

Conjecture 5.5. A modular transformation $\gamma \in \mathrm{Mod}(R)$ is elliptic if $\langle\gamma\rangle(p)$ is not discrete for some $p \in T(R)$.

§6. Isolated points of the limit sets

We begin investigating the dynamics of Teichmüller modular groups by finding an isolated point of the limit set. This problem itself is not affect the succeeding arguments, however, it opens up an interesting group theoretical problem. We will discuss this topic in the next section.

Here we give necessary conditions for a point $p \in T(R)$ to be an isolated limit point of a subgroup Γ of $\mathrm{Mod}(R)$. Without loss of generality, we may assume that p is the origin $o \in T(R)$.

Theorem 6.1. *Assume that $o \in T(R)$ is an isolated point of the limit set $\Lambda(\Gamma)$ of a subgroup $\Gamma \subset \mathrm{Mod}(R)$. Then the isotropy subgroup $\mathrm{Stab}_\Gamma(o)$, which is also* *regarded acting on R as a group of conformal automorphisms, satisfies the following conditions.*

(1) $\mathrm{Stab}_\Gamma(o)$ *is an infinite group but does not contain an element of infinite order. In other words,* $o \in \Lambda^2_\infty(\Gamma)$.
(2) *Every subgroup G of* $\mathrm{Stab}_\Gamma(o)$ *is of either finite order or finite index.*
(3) *For every infinite subgroup G of* $\mathrm{Stab}_\Gamma(o)$, *the Teichmüller space $T(R/G)$ of the orbifold R/G is a singleton.*

We cannot tell whether an isolated limit point exists or not. In the next section, we see that an abstract group having these properties actually exists and can be realized as a group of conformal automorphisms of a Riemann surface. Then we examine the dynamics of the isotropy subgroup. Via the Bers embedding of Teichmüller spaces, this is related to the study of isometric linear operators on Banach spaces. See [FM].

§7. Burnside groups and Tarski monsters

A finitely generated group G is called a *periodic group* if the order of each element of G is finite, and *bounded periodic group* if the order is uniformly bounded. For integers $m \geq 2$ and $n \geq 2$, let F_m be a free group of rank m and $F_m^{(n)}$ the characteristic subgroup of F_m generated by all the elements of the form f^n for $f \in F_m$. Then the quotient group $B(m,n) = F_m/F_m^{(n)}$ is an m-generator group all of whose elements become the identity by n-times composition. This is called a *Burnside group* or a *free periodic group*. It is easy to see that, for every bounded periodic group G, there exists a Burnside group $B(m,n)$ for some positive integers m and n such that G is the image of a homomorphism of $B(m,n)$. For $m = 2$, it had been known that $B(2,2)$, $B(2,3)$, $B(2,4)$ and $B(2,6)$ are finite groups. On the other hand, Novikov and Adjan [NA] finally proved the following.

Proposition 7.1. *For all sufficiently large odd $n \in \mathbb{N}$, the Burnside group $B(2,n)$ is an infinite group.*

As a problem to seek a stronger example, Smidt asked whether there is a finitely generated, infinite group G all of whose proper subgroups are finite. To this problem, the strongest example was given for which all of proper subgroups are contained in a cyclic subgroup of order n. This is obtained as a quotient group of $B(m,n)$ by certain extra relations. See Ol'shanskii [O] and Adjan and Lysionok [AL] among other works. Such a group is sometimes called a *Tarski monster.*

Proposition 7.2. *For all sufficiently large odd $n \in \mathbb{N}$, there exists a 2-generator Tarski monster of exponent n.*

The Burnside group $B(m,n)$ and its quotient can be realized as a group of conformal automorphisms of a Riemann surface. Indeed, since the fundamental group of an $(m+1)$-times punctured sphere is isomorphic to the free group F_m, a covering Riemann surface R corresponding to the subgroup $F_m^{(n)}$ has the covering transformation group $B(m,n) = F_m/F_m^{(n)}$. This means that a subgroup of $\mathrm{Aut}(R)$ is isomorphic to $B(m,n)$.

From this argument, we have a hopeful candidate for providing an isolated limit point of $\mathrm{Mod}(R)$, which satisfies all necessary conditions given in Theorem 6.1.

Lemma 7.3. *Let R be a Riemann surface that covers the three-times punctured sphere with the covering transformation group isomorphic to a Tarski monster of 2 generators. Then the isotropy subgroup $\mathrm{Stab}_\Gamma(o)$ for $\Gamma = \mathrm{Mod}(R)$, which is identified with $\mathrm{Aut}(R)$, satisfies the four conditions presented in Theorem 6.1*

We conjecture that, in the circumstances of Lemma 7.3, $o \in T(R)$ is an isolated limit point of $\mathrm{Mod}(R)$. We look for conversely what happens if this conjecture is not valid. Let f be a quasiconformal automorphism of the unit disk Δ that is a lift of a quasiconformal homeomorphism of R corresponding to a limit point $p \neq o$ of $\mathrm{Stab}_\Gamma(o)$. Then fMf^{-1} is a quasiconformal group, where M is the Fuchsian group of the three-times punctured sphere. By the assumption that p is a limit point and other consideration, we can choose generators of fMf^{-1} so that their maximal dilatations are arbitrarily close to 1. One may think that this rarely happens, from which we can seek a way of solving the conjecture. However, as is seen in Theorem 5.4, this can happen if $\mathrm{Stab}_\Gamma(o)$ contains an infinite cyclic group. Hence a solution of the conjecture seems heavily depending on the group structure of $\mathrm{Stab}_\Gamma(o)$.

§8. Exceptional limit points and density of generic limit points

We wish to claim that the set $\Lambda_0(\Gamma)$ of the generic limit points for $\Gamma \subset \mathrm{Mod}(R)$ is dense in $\Lambda(\Gamma)$. However, since an isolated limit point is not in the closure of $\Lambda_0(\Gamma)$ for instance, we have to make a certain modification to justify this density problem.

We have seen in Theorem 6.1 that if $p \in \Lambda(\Gamma)$ is an isolated limit point of $\Gamma \subset \mathrm{Mod}(R)$, then the orbifold $R/\mathrm{Stab}_\Gamma(p)$ has no moduli. This property forces the isotropy subgroup $\mathrm{Stab}_\Gamma(p)$ to satisfy certain algebraic conditions. However, even if the condition of no moduli is removed, an isotropy subgroup is still possible to keep the same algebraic conditions. In this case, there appears a locus of limit points in the Teichmüller space. We define these points as exceptional.

Definition. A limit point $p \in \Lambda(\Gamma)$ is *exceptional* if $p \notin \Lambda_0(\Gamma)$ and if there exists a neighborhood U of p in $T(R)$ such that $U \cap \Lambda(\Gamma) \subset \Lambda^2_\infty(\Gamma)$. The set of all exceptional limit points is called the exceptional set and denoted by $E(\Gamma)$.

By this definition and Theorem 6.1, it is clear that

$$\{\text{isolated points}\} \subset E(\Gamma) \subset \Lambda^2_\infty(\Gamma).$$

However, we do not know yet the existence of exceptional limit points, not to mention isolated limit points. Similarly to the case of an isolated limit point, the isotropy subgroup for an exceptional limit point has a distinguished property.

Proposition 8.1. *For an exceptional limit point $p \in E(\Gamma)$, the isotropy subgroup* $\mathrm{Stab}_\Gamma(p)$ *contains a finitely generated infinite group G whose proper subgroups are all finite.*

The following theorem and its corollary provide an easier test for an exceptional limit point.

Theorem 8.2. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *If $\Lambda(\Gamma) = \Lambda_\infty(\Gamma)$, then they are coincident with $\Lambda^2_\infty(\Gamma)$. More generally, for an open subset U in $T(R)$, if $U \cap \Lambda(\Gamma) = U \cap \Lambda_\infty(\Gamma)$, then they are coincident with $U \cap \Lambda^2_\infty(\Gamma)$.*

Corollary 8.3. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *If $p \in \Lambda(\Gamma) - \Lambda_0(\Gamma)$ has a neighborhood U such that $U \cap \Lambda(\Gamma) \subset \Lambda_\infty(\Gamma)$, then p belongs to $E(\Gamma)$.*

Now we can formulate the density of generic limit points in the following form. This is the best possible assertion if we respect the existence of exceptional limit points.

Theorem 8.4. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *Then $\Lambda_0(\Gamma)$ is dense in $\Lambda(\Gamma) - E(\Gamma)$.*

Also we can add the following characterization in the general conditions for weak discontinuity given in Proposition 2.3.

Proposition 8.5. *Let Γ be a subgroup of* $\mathrm{Mod}(R)$. *Then Γ acts weakly discontinuously on $T(R)$ if and only if $\Lambda(\Gamma) = E(\Gamma)$.*

We conjecture that the condition $\Lambda(\Gamma) = E(\Gamma)$ above is equivalent to the condition $\Lambda(\Gamma) = \Lambda_\infty(\Gamma)$, which is equivalent to $\Lambda(\Gamma) = \Lambda^2_\infty(\Gamma)$ by Theorem 8.2. We extend this problem to Conjecture 9.2 in the next section.

§9. Fixed limit points are not dense

We prove that the set of the fixed limit points are not dense in the limit set. This gives a contrast to the nature of familiar dynamics such as Kleinian groups and iteration of rational maps. Strictly speaking, there may exist exceptional cases where the above statement is not true, for example, the case where $\Lambda(\Gamma)$ is coincident with the exceptional set $E(\Gamma)$. Hence certain restriction to the limit set is necessary to justify the statement. We state it in the following form.

Theorem 9.1. *The set $\Lambda^1_\infty(\Gamma)$ is not dense in the limit set $\Lambda(\Gamma)$.*

Since the closure of $\Lambda^1_\infty(\Gamma)$ is invariant under Γ, this result in particular implies that $\Lambda(\Gamma)$ contains a smaller Γ-invariant closed subset properly.

A stronger assertion than Theorem 9.1 is expected to be true, which will be a best possible result. However, there still remain some technical problems to prove it. A main concern is a fact that a fixed limit point can be a generic limit point at the same time.

Conjecture 9.2. If $\Lambda(\Gamma) - E(\Gamma)$ is not empty, then $\Lambda_\infty(\Gamma)$ is not dense in $\Lambda(\Gamma)$.

We wish to choose a limit point $p \in \Lambda_\infty^1(\Gamma)$ such that $\mathrm{Stab}_\Gamma(p)$ itself is cyclic, in other words, there is no extra element that fixes p. This is always possible by the following lemma based on Epstein [E], where it was used to find a point $p \in T(R)$ that is not fixed by any element of $\mathrm{Mod}(R)$. Since $\mathrm{Mod}(R)$ may be uncountable in general, the number of the fixed point loci for elliptic elements of $\mathrm{Mod}(R)$ can be uncountable. Then the Baire category theorem does not work, which is the reason why we need an extra argument here. In this lemma, the countability of the loci comes from the number of the simple closed geodesics on R.

Lemma 9.3. *For a subgroup Γ of $\mathrm{Mod}(R)$, there exist a countable number of proper subsets $\{V_i\}_{i=1}^\infty$ such that $\bigcup_{\gamma\in\Gamma}\mathrm{Fix}(\gamma)$ is contained in $\bigcup_{i=1}^\infty V_i$. Moreover, for an elliptic element $g \in \Gamma$ of infinite order,*

$$\mathrm{Fix}(g) \cap \bigcup_{\gamma\in\Gamma-\langle g\rangle} \mathrm{Fix}(\gamma)$$

is contained in $\bigcup \mathrm{Fix}(g) \cap V_{i'}$, where the union is taken over all $i' \in \mathbb{N}$ such that $V_{i'}$ does not contain $\mathrm{Fix}(g)$.

Another argument for the proof of Theorem 9.1 involves finding a limit point of a cyclic group $\langle g\rangle$ of infinite order that is not lie on the closure $\overline{\Lambda_\infty(g)}$. In [FM], this is proved for a particular Riemann surface. Here we prove it more generally as follows.

Lemma 9.4. *Let R be a Riemann surface that admits a conformal automorphism $g \in \mathrm{Aut}(R)$ of infinite order. Assume that there is a simple closed geodesic c such that $\{g^i(c)\}_{i\in\mathbb{Z}}$ are mutually disjoint to each other. Then, for every neighborhood U of the origin $o \in T(R)$, there exists a generic limit point $q \in \Lambda_0(g) \cap U$ for the cyclic group $\langle g\rangle \subset \mathrm{Mod}(R)$ that does not belong to the closure $\overline{\Lambda_\infty(g)}$.*

The combination of Lemmata 9.3 and 9.4 yields Theorem 9.1.

§10. Topology of the moduli space

We investigate general topological structure of the moduli space of an analytically infinite Riemann surface. First we have the following theorem concerning the orbit of $\mathrm{Mod}(R)$ in $T(R)$.

Theorem 10.1. *For every point $p \in T(R)$ and for every subgroup Γ of $\mathrm{Mod}(R)$, the orbit $\Gamma(p)$ is nowhere dense in $T(R)$.*

Since the topological moduli space $M(R)$ may fail to satisfy the first separation axiom, the closure of a point set may become larger. However, the above theorem implies that the closure cannot be so large in the following sense.

Corollary 10.2. *For every point $\sigma \in M(R)$, the closure $\overline{\{\sigma\}}$ of the point set does not have interior points.*

Next we consider the metric completion $\overline{M_\Phi(R)}^d$ of the metric moduli subspace $M_\Phi(R)$ with a distance d. Here d is the path metric on $M_\Phi(R)$ induced by the pseudo-distance d_M on $M(R)$. The restriction of the projection $\bar{\pi} : M(R) \to M_*(R)$ to $M_\Phi(R)$ extends to a continuous map $\phi : \overline{M_\Phi(R)}^d \to M_*(R)$. We expect that ϕ is a bijective isometry. In order to prove this claim, we formulate the following.

Conjecture 10.3. For every subgroup $\Gamma \subset \mathrm{Mod}(R)$, the region of stability $\Phi(\Gamma)$ is dense in $T(R)$ and is connected in each open subset of $T(R)$.

Hereafter, we assume that $T(R)$ satisfies the bounded geometry condition, under which $\Phi(\Gamma) = \Omega(\Gamma)$ by Theorem 4.3, and prove the above conjecture.

Fujikawa [F] proved that, if R satisfies the bounded geometry condition, then $\Lambda(\Gamma)$ is a proper subset of $T(R)$ for a subgroup $\Gamma \subset \mathrm{Mod}(R)$. Extending this result, we have the following.

Theorem 10.4. *If $T(R)$ satisfies the bounded geometry condition, then, for a subgroup Γ of $\mathrm{Mod}(R)$, the limit set $\Lambda(\Gamma)$ is nowhere dense in $T(R)$.*

On the other hand, we can prove the connectivity of $\Omega(\Gamma)$ everywhere.

Theorem 10.5. *If $T(R)$ satisfies the bounded geometry condition, then, for a subgroup Γ of $\mathrm{Mod}(R)$, $\Omega(\Gamma) \cap U$ is connected for every open subset U of $T(R)$.*

As immediate consequences from these theorems, we have desired results under the bounded geometry assumption.

Corollary 10.6. *If $T(R)$ satisfies the bounded geometry condition, then $M_\Phi(R) = M_\Omega(R)$ is a connected open dense subset of $M(R)$.*

Corollary 10.7. *Assume that $T(R)$ satisfies the bounded geometry condition. In this case, the map $\phi : \overline{M_\Phi(R)}^d \to M_*(R)$ is a bijective isometry.*

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