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Biholomorphic maps between asymptotic Teichmüller spaces

Ege Fujikawa
Department of mathematical and Computing Sciences
Tokyo Institute of Technology

藤川 英華　東京工業大学 大学院情報理工学研究科

1 Introduction

Let R be a hyperbolic Riemann surface. The asymptotic Teichmüller space $AT(R)$ of R is a quotient space of the Teichmüller space $T(R)$, which was introduced by Gardiner and Sullivan [7] when R is the upper half-plane and by Earle, Gardiner and Lakic [1], [2], [6, Chap. 14] when R is an arbitrary hyperbolic Riemann surface.

In this note, we investigate basic properties of asymptotic Teichmüller spaces. In particular, we prove that if R is of analytically finite type, then $AT(R)$ consists of just one point. Furthermore, we prove that for a Riemann surface R and a Riemann surface R from which finitely many points are removed, their asymptotic Teichmüller spaces are biholomorphically equivalent.

An element of the Teichmüller modular group $\mathrm{Mod}(R)$ induces an isometric automorphism of $T(R)$. Similarly, an element of $\mathrm{Mod}(R)$ also induces an isomorphism of $AT(R)$. Such an isomorphism is called geometric and the set of all geometric isomorphisms of $AT(R)$ is denoted by $\mathcal{G}(R)$. We give a sufficient condition for $\mathcal{G}(R)$ to act on $AT(R)$ non-trivially. This condition is crucial for further observations of the action of geometric isomorphisms.

2 Preliminaries

2.1 Teichmüller space and Teichmüller modular group

Throughout this note, we assume that a Riemann surface R is hyperbolic. Namely, it is represented by a quotient space $\mathbf{H}/\Gamma$ of the upper half-plane $\mathbf{H}$ by a torsion free Fuchsian group Γ. We say that R is of the *analytically finite type* if it is compact except for finitely many punctures. Furthermore we say that R is of the *topologically finite type* if it is compact except for finitely many punctures and holes.

First we recall the definition of Teichmüller spaces and Teichmüller modular groups (see [12]). Fix a Riemann surface R. We say that two quasiconformal maps f_1 and f_2 on R are *equivalent* if there exists a conformal map h of $f_1(R)$ onto $f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The *Teichmüller space* $T(R)$ with the base Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal maps f. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1],[f_2]) = \log K(f)$, where f is an extremal quasiconformal map in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete metric on $T(R)$, which is called the Teichmüller distance.

We say that two quasiconformal automorphisms g_1 and g_2 of R are *equivalent* if $g_2 \circ g_1^{-1}$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The *Teichmüller modular group* $\mathrm{Mod}(R)$ is the set of all equivalence classes $[g]$ of quasiconformal automorphisms g of R. Every element $\chi = [g] \in \mathrm{Mod}(R)$ induces an automorphism χ_* of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is an isometry with respect to d_T. Let $\mathrm{Isom}(T(R))$ be the group of all orientation preserving isometric automorphisms of $T(R)$, which coincides with the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism $\iota_T : \mathrm{Mod}(R) \to \mathrm{Isom}(T(R))$ by $\chi \mapsto \chi_*$. With a few exceptional surfaces, ι_T is faithful. This was first proved in [2]. Other proofs were given by Epstein [4] and Matsuzaki [10]. Furthermore, it was proved by Markovic [9] that ι_T is surjective. Hence we can identify $\mathrm{Mod}(R)$ with $\mathrm{Isom}(T(R))$.

2.2 Asymptotic Teichmüller space

We say that a quasiconformal map f on R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset E of R such that the maximal dilatation f is less than $1+\epsilon$ on $R-E$. A Teichmüller equivalence class $[f] \in T(R)$ is called asymptotically conformal if it is represented by an asymptotically conformal map. The set of all asymptotically conformal classes in $T(R)$ is denoted by $T_0(R)$. It was proved in [2] that $T_0(R)$ is a closed and connected complex submanifold of $T(R)$.

We define the asymptotic Teichmüller space of R. We say that two quasiconformal maps f_1 and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal map h of $f_1(R)$ onto $f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every points of the ideal boundary fixed throughout. The *asymptotic Teichmüller space* $AT(R)$ with the base Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal maps f on R. Since a conformal map is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. Note that for two equivalence classes $[f_1]$ and $[f_2]$ in $T(R)$, $\pi([f_1]) = \pi([f_2])$ if and only if $[f_2 \circ f_1^{-1}] \in T_0(f_1(R))$. It was proved in [2] that the asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic, and it was proved by Earle, Markovic and Saric [3] that $T_0(R)$ and $AT(R)$ are contractible.

2.3 Boundary dilatation

For a quasiconformal map f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf\{K(f|_{R-E}) \mid E \subset R : \text{compact}\}$. Furthermore, for a point $\tau = [f] \in T(R)$, the *boundary dilatation* of τ is defined by $H(\tau) = \inf\{H^*(g) \mid g \in [f]\}$. Set $K_0(\tau) = \inf\{K(g) \mid g \in [f]\}$. Then clearly, $H(\tau) \leq K_0(\tau)$. A point $\tau \in T(R)$ is said to be a *Strebel point* if $H(\tau) < K_0(\tau)$. It was proved by Lakic [8] that the set of all Strebel points are open and dense in $T(R)$.

A distance between two points $\tau_1 = [[f_1]]$ and $\tau_2 = [[f_2]]$ in $AT(R)$ is defined by $d_{AT}(\tau_1, \tau_2) = \log H([f_2 \circ f_1^{-1}])$. Then d_{AT} is a complete metric on $AT(R)$, which is called the asymptotic Teichmüller distance. It was proved in [6, Chap. 15] that for any point $[[f]] \in AT(R)$, there exists an element $f_0 \in [[f]]$ such that $H([f]) = H^*(f_0)$. We call such f_0 asymptotically extremal.

3 Results

3.1 Biholomorphic maps

First we observe a modification of a quasiconformal map around a point.

Lemma 1 *Let R be a Riemann surface and p a point of R. For a quasiconformal map f of R, suppose that the Teichmüller equivalence class $[f]$ belongs to $T_0(R)$. Then the Teichmüller equivalence class $[f|_{R-\{p\}}]$ belongs to $T_0(R-\{p\})$.*

Proof. We take a sufficiently small constant $\epsilon > 0$ so that $U_\epsilon = \{q \in R \mid d(p,q) < \epsilon\}$ is simply connected. Since $[f] \in T_0(R)$, we may assume that f is an asymptotically conformal map. For the Beltrami coefficient μ of f and for $t \in [0,1]$, we set $\mu_t = (1-t)\mu$ on U_ϵ and $\mu_t = \mu$ on $R-U_\epsilon$. Let f_t be a quasiconformal map on R whose Beltrami coefficient is μ_t. Then f_t $(0 \leq t \leq 1)$ is a homotopy connecting $f_0 = f$ and f_1. We take a quasiconformal map $h_t : f_t(R) \to f(R)$ so that $h_t = f \circ f_t^{-1}$ on $f_t(R) - f_t(U_\epsilon)$ and h_t is conformal on $f_t(U_{\epsilon/2})$ and it satisfies $h_t \circ f_t(p) = f(p)$. Furthermore we take the h_t so that it is continuous on t and h_0 is the identity. Set $g_t := h_t \circ f_t : R \to f(R)$, which is a homotopy connecting $g_0 = f$ and g_1. Since $g_t(p) = f(p)$, we have $[g_t|_{R-\{p\}}] = [f|_{R-\{p\}}]$ in $T(R-\{p\})$. Since g_1 is conformal on $U_{\epsilon/2}$ and $g_1 = f$ on $R - U_\epsilon$, we see that $g_1|_{R-\{p\}}$ is asymptotically conformal. Thus $[f|_{R-\{p\}}] = [g_1|_{R-\{p\}}] \in T_0(R - \{p\})$. ∎

Lemma 1 immediately yields the following.

Corollary 2 *Let R be a Riemann surface of analytically finite type. Then $AT(R)$ is singleton.*

Proof. By definition, R is a compact Riemann surface $\overline{R}$ from which at most finitely many points $\{p_i\}_{i=1}^n$ are removed. We take an arbitrary Teichmüller equivalent class $[f] \in T(R)$. The quasiconformal map f of R extends to a quasiconformal map $\overline{f}$ of $\overline{R}$ and we have $[\overline{f}] \in T(\overline{R}) = T_0(\overline{R})$. Then by Lemma 1, we have $[\overline{f}|_{R-\{p_1\}}] \in T_0(\overline{R} - \{p_1\})$. Again by Lemma 1, we see that $[\overline{f}|_{R-\{p_1,p_2\}}] \in T_0(R - \{p_1, p_2\})$. By repeating this process, we conclude that $[f] \in T_0(R)$, which implies the assertion. ■

On a biholomorphic equivalence between asymptotic Teichmüller spaces, we have the following.

Theorem 3 *Let R be a Riemann surface and p a point of R. Then the asymptotic Teichmüller spaces $AT(R)$ and $AT(R - \{p\})$ are biholomorphically equivalent.*

Proof. Every quasiconformal map f of $R - \{p\}$ extends to a quasiconformal map $\overline{f}$ of R. Since the map of $T(R - \{p\})$ onto $T(R)$ defined by $[f] \mapsto [\overline{f}]$ is holomorphic (see [12, §5.3]) and the projection $\pi : T(R) \to AT(R)$ is holomorphic, the map $\psi : AT(R-\{p\}) \to AT(R)$ defined by $[[f]] \mapsto [[\overline{f}]]$ is holomorphic. We will prove that ψ is injective. Suppose that $[[\overline{f}]] = [[id]]$ in $AT(R)$. Then $[\overline{f}] \in T_0(R)$. By Lemma 1, we have $[f] \in T_0(R - \{p\})$. Thus $[[f]] = [[id]]$ in $AT(R - \{p\})$, which means that ψ is injective. ■

For a Riemann surface R of topologically finite type with n boundary components, the asymptotic Teichmüller space $AT(R)$ is biholomorphically equivalent to the product space $AT(\mathbf{D})^n$ of the asymptotic Teichmüller of the unit disk $\mathbf{D}$ in $\mathbf{C}$. This was proved by Miyachi [11].

3.2 Geometric isomorphisms on $AT(R)$

Similar to the action of the Teichmüller modular group $\mathrm{Mod}(R)$ on $T(R)$, every element $\chi = [g] \in \mathrm{Mod}(R)$ induces an automorphism χ_* of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is an isometry with respect to d_{AT}. Let $\mathrm{Isom}(AT(R))$ be the group of all orientation preserving isometric automorphisms of $AT(R)$. Then we have a homomorphism $\iota_{AT} : \mathrm{Mod}(R) \to \mathrm{Isom}(AT(R))$ by $\chi \mapsto \chi_*$. It is different from the case of ι_T that the homomorphism ι_{AT} is not faithful for any hyperbolic Riemann surface R. Indeed, let $[g_0] \in \mathrm{Mod}(R)$ be a Dehn twist along a simple closed geodesic c on R. Since $[g_0]$ has a representative that is the identity outside of the collar of c, we see that $[g_0] \in \ker \iota_{AT}$, whereas $[g_0] \neq [id]$ as an element of $\mathrm{Mod}(R)$. Hence ι_{AT} is not faithful. Thus we define the *geometric isomorphism group* by

$$\mathcal{G}(R) = \mathrm{Mod}(R)/\ker \iota_{\mathrm{AT}}.$$

We call an element of $\mathcal{G}(R)$ geometric isomorphism and denote the equivalence class of $[g] \in \mathrm{Mod}(R)$ in $\mathcal{G}(R)$ by $[[g]]$.

We give a sufficient condition for $[g] \notin \ker \iota_{\mathrm{AT}}$, namely $[[g]]$ acts non-trivially on $AT(R)$. For a non-trivial simple closed curve c, let $\ell(c)$ be the hyperbolic length of the geodesic that is homotopic to c, and d the hyperbolic distance on R.

Theorem 4 *Let g be a quasiconformal automorphism of R. Suppose that there exist a sequence $\{c_n\}_{n=1}^{\infty}$ of simple closed geodesics on R and a positive constant δ independent of n such that $d(p, c_n) \to \infty$ for a point $p \in R$ and*

$$\left| \frac{\ell(g(c_n))}{\ell(c_n)} - 1 \right| \geq \delta$$

for all n. Then the class $[g] \in \mathrm{Mod}(R)$ is not asymptotically conformal. Namely, the action of $[[g]] \in \mathcal{G}(R)$ on $AT(R)$ is not trivial.

A proof of Theorem 4 is given in the author's forthcoming paper [5].

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