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Trace fields of genus 3 surfaces with regular fundamental polygons

愛知工業大学 中村 豪 (Gou Nakamura)
Aichi Institute of Technology

1. Introduction

Let $\Gamma \subset \mathrm{SL}(2,\mathbf{R})$ be a Fuchsian group. The trace field $\mathrm{tr}(\Gamma)$ of Γ is the field generated over $\mathbf{Q}$ by the traces of elements in Γ. In [5] M. Näätänen and T. Kuusalo determined the trace fields of all Fuchsian groups of signature $(2;0)$ with a regular polygon as a fundamental polygon. In the present paper we shall consider the trace fields for the case of signature $(3,0)$ analogously.

2. Regular fundamental polygons and trace fields

By Euler's formula we see that there are 4 regular polygons to be a compact surface of genus three.

1. 30-gon with each angle $2\pi/3$,
2. 20-gon with each angle $\pi/2$,
3. 14-gon with each angle $2\pi/7$,
4. 12-gon with each angle $\pi/6$.

By using a computer we can show the side-pairing patterns for each polygon.

Theorem 1. There exist 927 side-pairing patterns for 30-gon, 297 for 20-gon, 112 for 14-gon and 82 for 12-gon up to mirror images.

The following is mentioned for the case of $(2,0)$ in [5].

Lemma 2. Let Γ be a Fuchsian group of signature $(3;0)$ with a regular $2n$-gon as a fundamental polygon $(n=6,7,10,15)$. Then Γ is a subgroup of the triangle group Λ_n of type $(2, 2n/(n-5), 2n)$.

Proposition 3.(cf. Hilden, Lozano and Montesinos-Amilibia [3]) Let Λ_n^2 be the subgroup of Λ_n generated by the squares of the elements of Λ_n. Then it follows that

$$\mathrm{tr}(\Lambda_n^2) \subset \mathrm{tr}(\Gamma) \subset \mathrm{tr}(\Lambda_n).$$

Proposition 4.(cf. Hilden, Lozano and Montesinos-Amilibia [3])

$$\mathrm{tr}(\Lambda_n) = \mathbf{Q}\left(\cos\frac{\pi}{2n}, \cos\frac{(n-5)\pi}{2n}, \cos\frac{\pi}{2}\right) = \mathbf{Q}\left(\cos\frac{\pi}{2n}\right),$$

$$\mathrm{tr}(\Lambda_n^2) = \mathbf{Q}\left(\cos\frac{\pi}{n}, \cos\frac{(n-5)\pi}{n}, \cos\frac{\pi}{2n}\cos\frac{(n-3)\pi}{2n}\cos\frac{\pi}{2}\right) = \mathbf{Q}\left(\cos\frac{\pi}{n}\right).$$

We denote by C_k the k-th side of the regular $2n$-gon. Suppose that the polygon is centered at the origin such that the middle points of C_n and C_{2n} are real.

Lemma 5. Let F_n be a hyperbolic translation of the regular $2n$-gon identifying a pair of opposite sides C_n and C_{2n}. Then the diagonal entries of F_n are equal to $1+4\cos^2(\pi/n)$.

A proof of this lemma is analogous to that of Lemma 2.1 in [5].

Definition 6. A side-pairing T of the regular $2n$-gon is the composite $T = R_n^k F_n R_n^{-l}$ of F_n and the

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rotation R_n arround the origin by angle π/n. Then T is said to be odd or even if $k-l$ is odd or even, respectively.

Theorem 7. Let Γ be a Fuchsian group of signature (3;0) with a regular $2n$-gon as a fundamental polygon. Then $\mathrm{tr}(\Gamma) = \mathbf{Q}(\cos(\pi/n))$ if all side-pairings are even, and $\mathrm{tr}(\Gamma) = \mathbf{Q}(\cos(\pi/(2n)))$ if some side-pairing is odd.

See Theorem 2.2 in [5] for a proof.

By considering the side-pairings for each polygons we have the following:

Theorem 8. The polygons only with even side-pairings are listed as follows:

$2n$	Side-pairings	Trace field
30	$P_{313}, P_{314}, P_{315}, P_{316}, P_{317}, P_{318}, P_{397}, P_{398}, P_{399}, P_{400}, P_{401}, P_{402}, P_{403}, P_{404}, P_{405}, P_{406}, P_{407}, P_{408}, P_{409}, P_{410}, P_{494}, P_{495}, P_{496}, P_{497}, P_{498}, P_{499}, P_{500}, P_{509}, P_{510}, P_{511}, P_{512}, P_{513}, P_{514}, P_{568}, P_{569}, P_{570}, P_{571}, P_{586}, P_{587}, P_{588}, P_{589}, P_{590}, P_{591}, P_{737}, P_{738}, P_{739}, P_{740}, P_{741}, P_{742}, P_{833}, P_{834}, P_{835}, P_{836}, P_{837}, P_{838}, P_{839}, P_{840}, P_{841}, P_{842}, P_{843}, P_{844}, P_{845}, P_{846}, P_{847}, P_{848}, P_{849}, P_{850}, P_{851}, P_{852}, P_{853}$	$\mathbf{Q}(\cos(\pi/15))$
20	Side-pairings in Figure 1	$\mathbf{Q}(\cos(\pi/10))$
14	Side-pairings in Figure 2	$\mathbf{Q}(\cos(\pi/7))$
12	Side-pairings in Figure 3	$\mathbf{Q}(\cos(\pi/6))$

Here, P_j denotes the 30-gon endowed with j-th side-pairing pattern in [6].

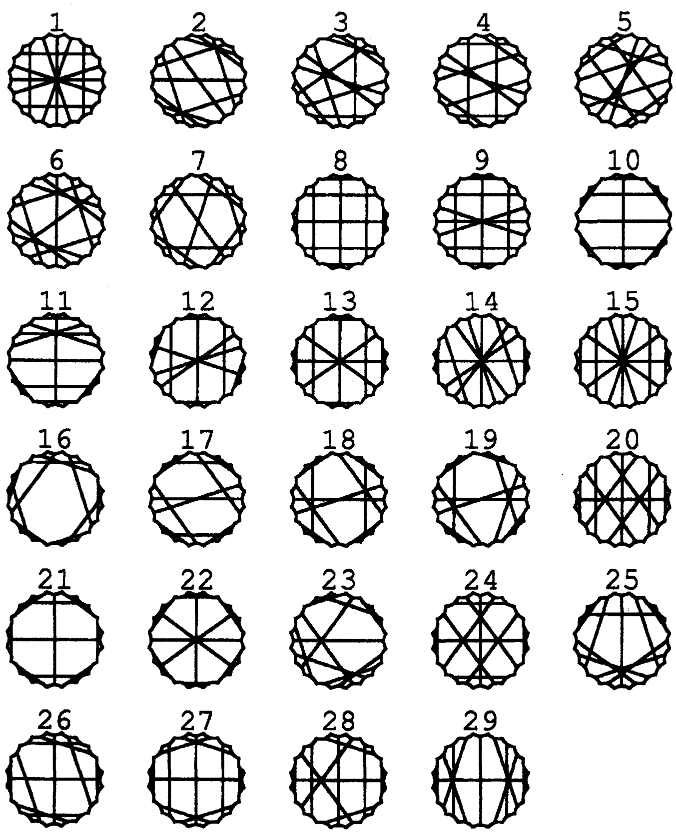

Figure 1: 20-gons only with even side-pairings

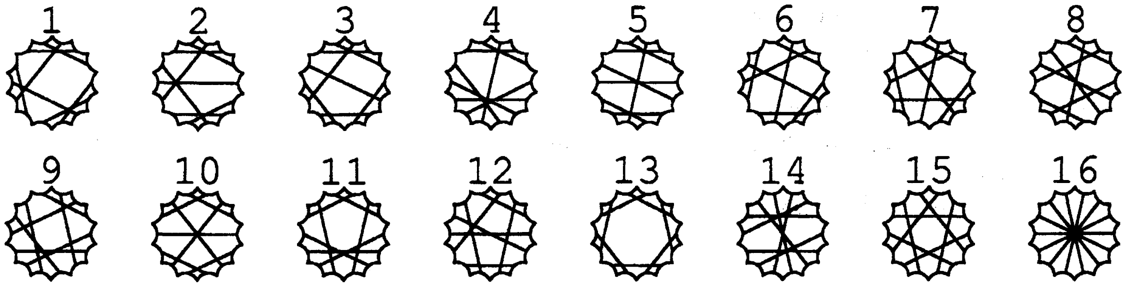

Figure 2: 14-gons only with even side-pairings

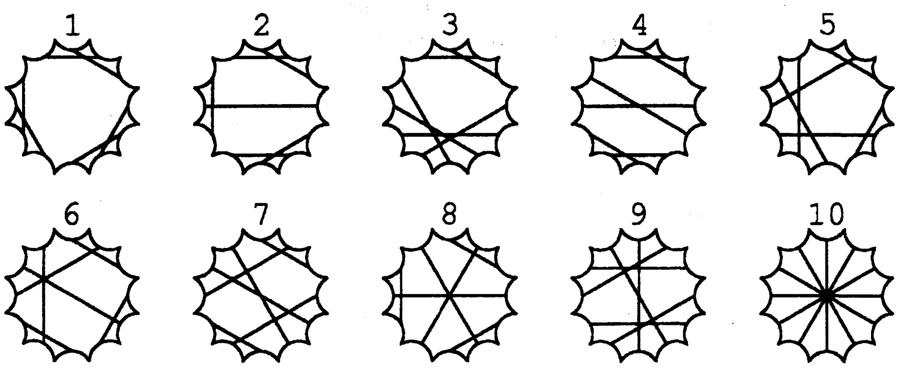

Figure 3: 12-gons only with even side-pairings

An extremal surface of genus g in the sense of C. Bavard has the regular $(12g-6)$-gon as a fundamental polygon. We see that every extremal surface of genus 3 admitting two extremal disks has the trace field $\mathbf{Q}(\cos(\pi/30))$ (see Figure 9).

References

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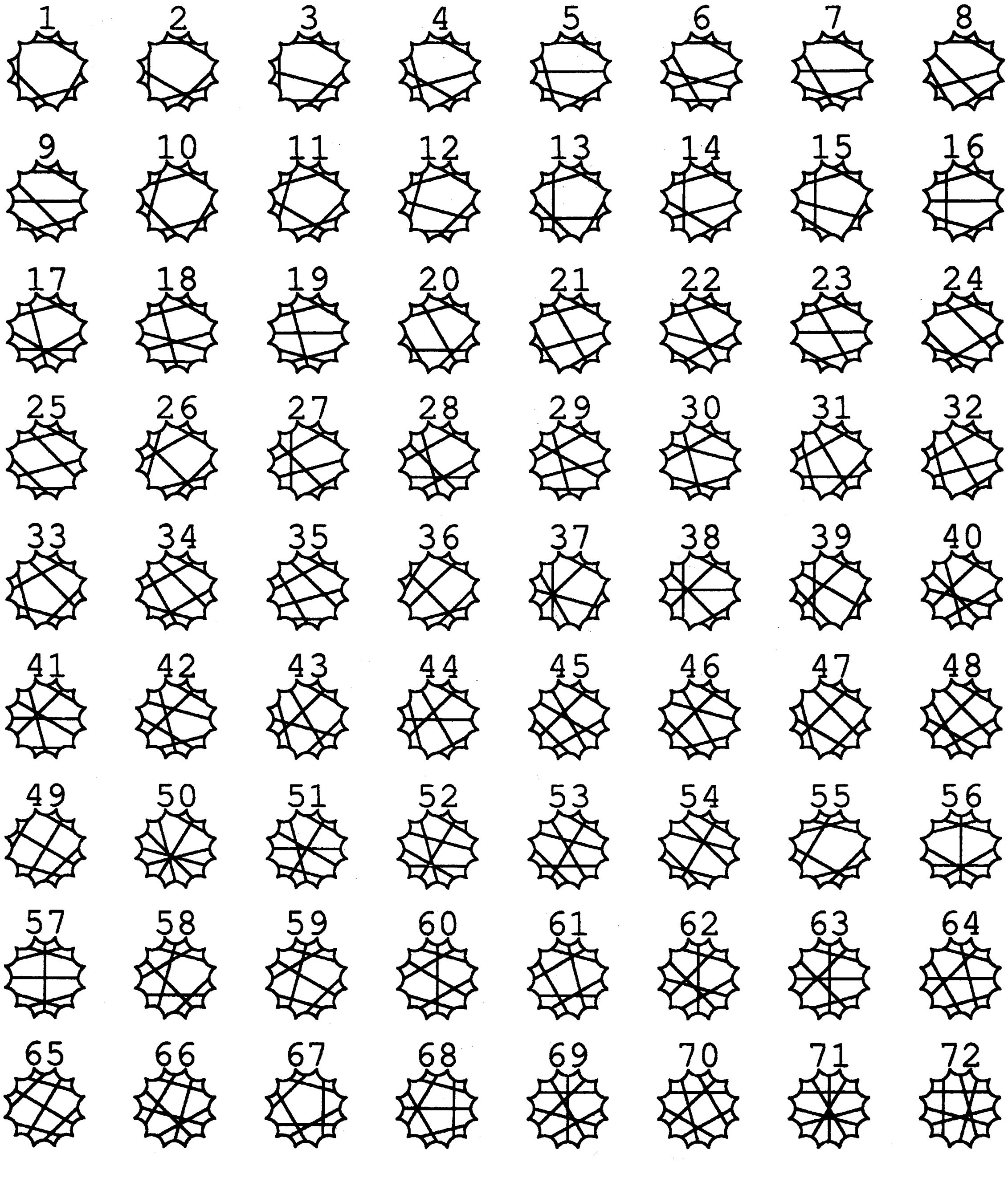

Figure 4: 12-gons with odd side-pairings

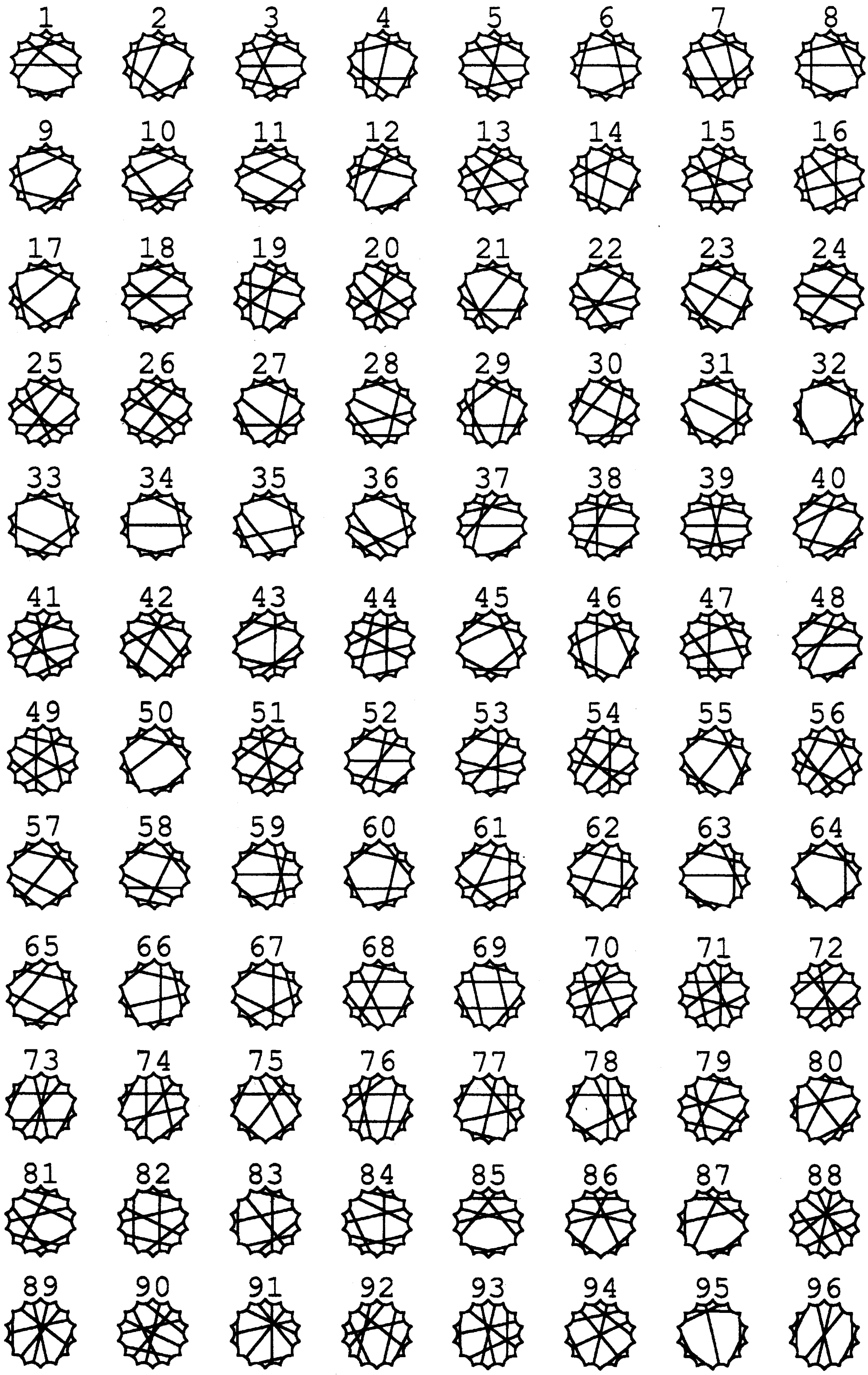

Figure 5: 14-gons with odd side-pairings

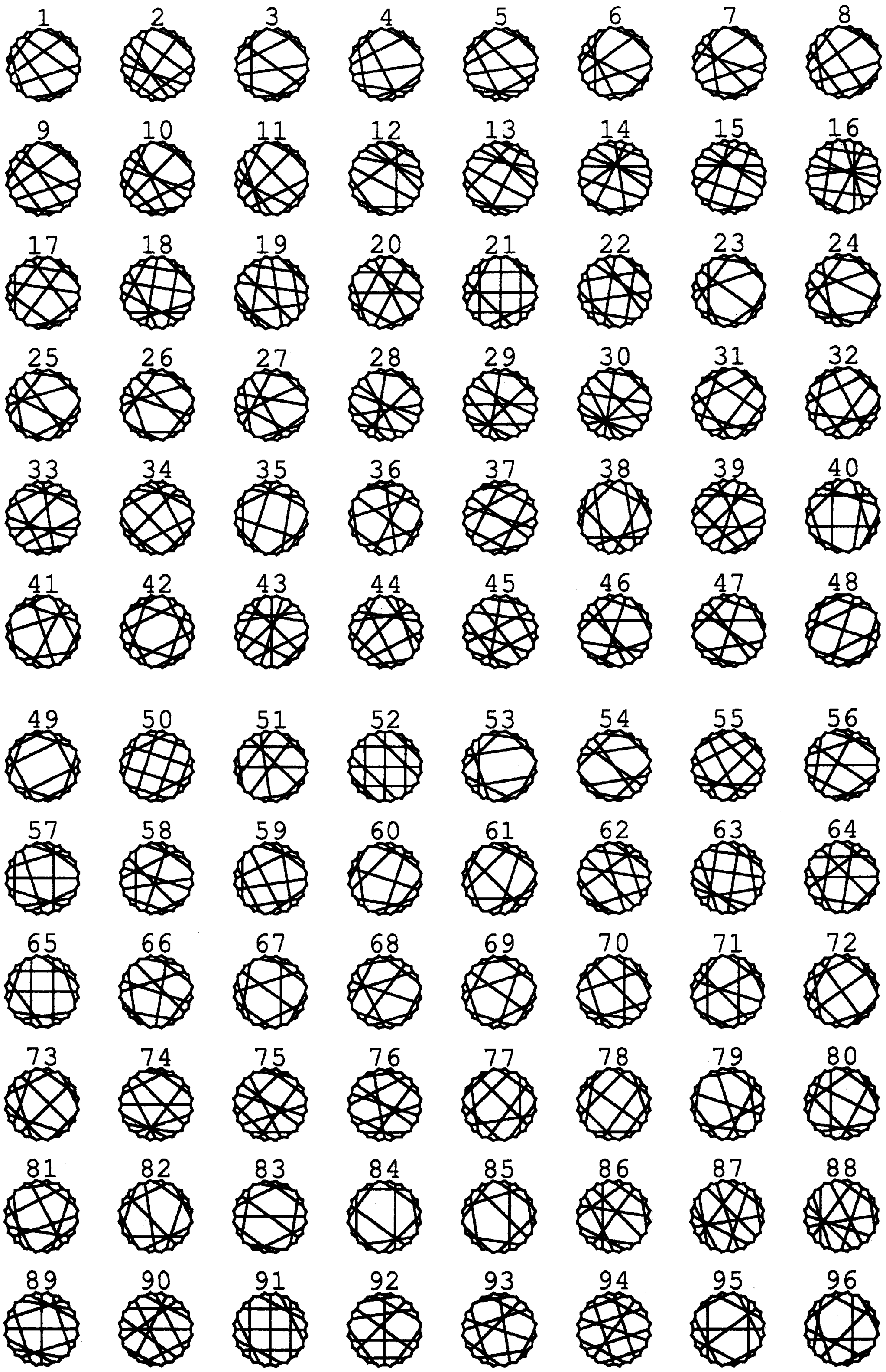

Figure 6: 20-gons with odd side-pairings (1)

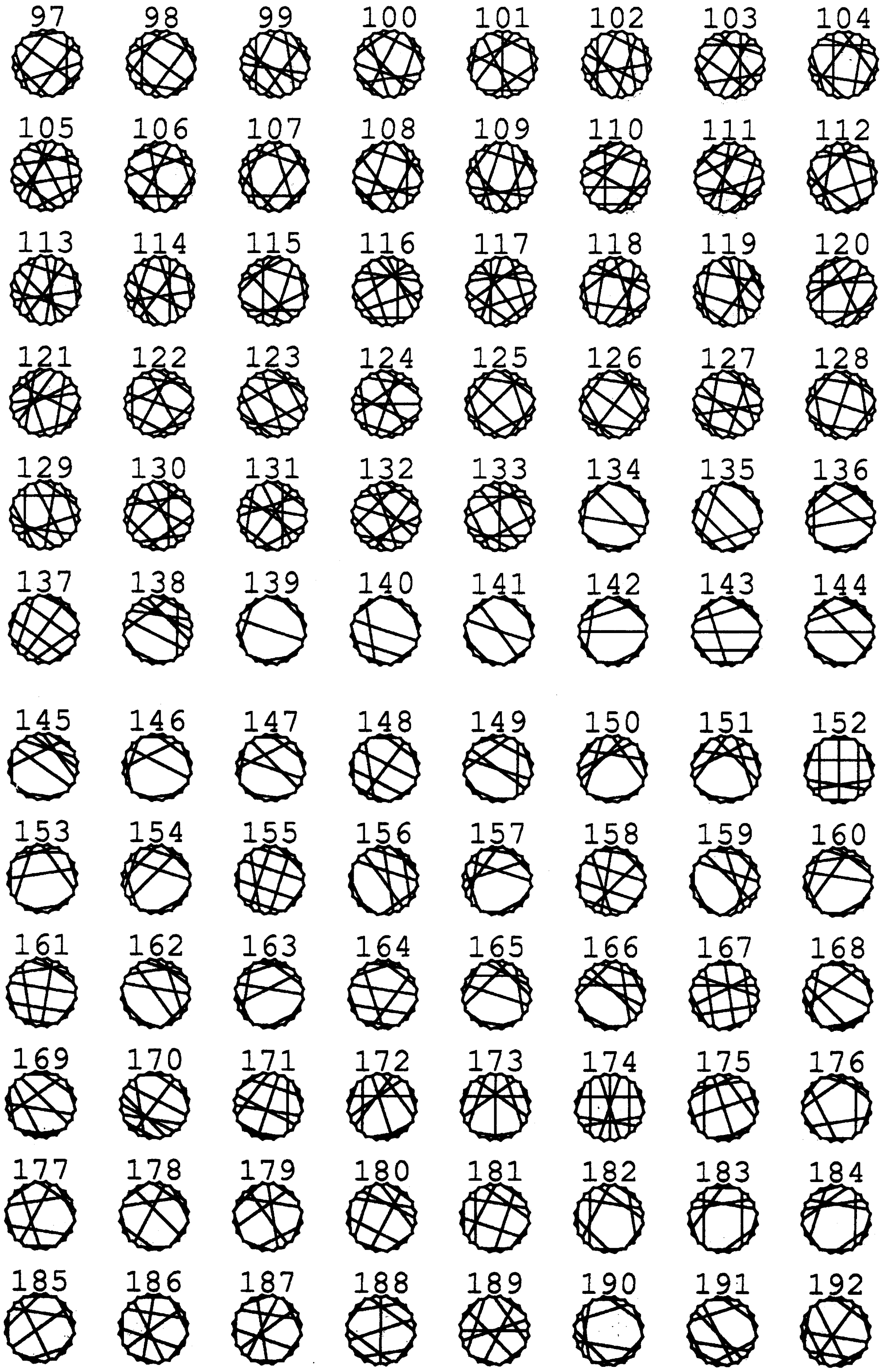

Figure 7: 20-gons with odd side-pairings (2)

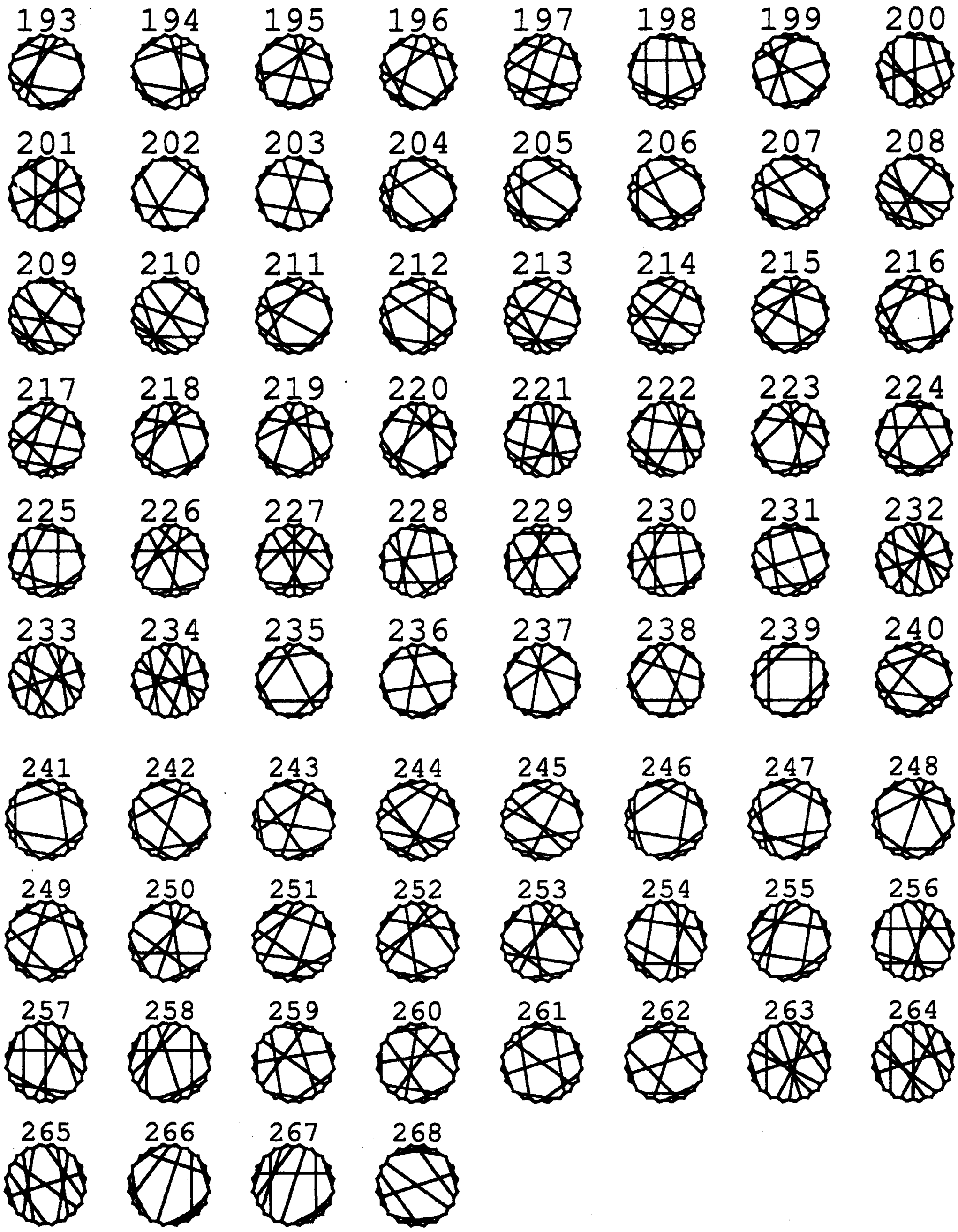

Figure 8: 20-gons with odd side-pairings (3)

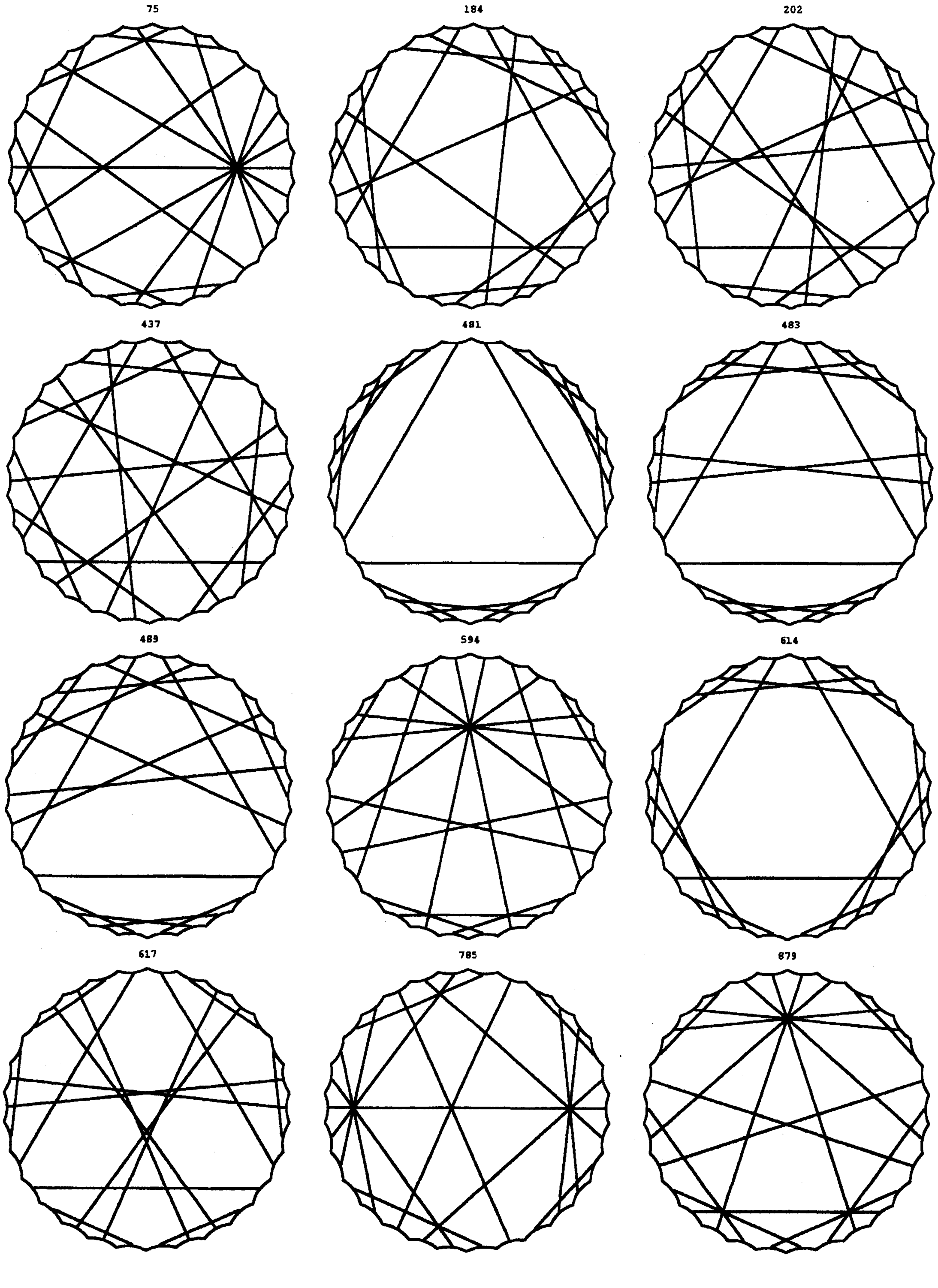

Figure 9: Side-pairing patterns which induce extremal surfaces admitting two extremal disks