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On the subdissipative Navier-Stokes equations

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Abstract

We derive the various estimates in the scale invariant Besov spaces for the modified 3D Navier-Stokes equations with the dissipation term $(-\Delta)^{\alpha}u$, $0\leq\alpha<\frac{5}{4}$. We also prove the small data unique existence and global stability of a global-in-time solution in $B_{2,1}^{\frac{5}{2}-2\alpha}$.

1 Introduction and Main Results

We are concerned with the subdissipative or hyperdissipative Navier-Stokes equations.

$$(\mathrm{SNS})_{\alpha}\left\{\begin{array}{l}\partial_{t}u+(u\cdot\nabla)u+(-\Delta)^{\alpha}u+\nabla p=f,\mathbb{R}^{3}\times\mathbb{R}_{+},0\leq\alpha<\frac{5}{4},\\ \mathrm{div}\ u=0,\\ u(0,x)=u_{0}(x),\end{array}\right.$$

where u represents the velocity vector field and p is the scalar pressure. J. L. Lions[24] proved the existence of a unique regular solution provided $\alpha\geq\frac{5}{4}$. This modified Navier-Stokes equations are the most studied ones from the numerical point of view. If $\alpha=1$, then above system reduce to the usual Navier-Stokes equations. For the Navier-Stokes equations, Kato[19] proved the local in time existence with initial data $L^{n}(\mathbb{R}^{n})$ and Giga[18] showed that local in time existence with initial data in $L^{p}(\mathbb{R}^{n})$ with $n\leq p<\infty$. Kato and Ponce[20] proved the local in time existence with initial data in some Sobolev space. For the global existence with small data, Kato[19] proved the existence of global solution in $C([0,\infty);L^{3}(\mathbb{R}^{3}))$ if $\|u_{0}\|_{L^{3}}$ is sufficiently small. After Kato's work[19], there were many important improvements using the scaling invariant function spaces. Especially, pioneered by Chemin[11], Cannone-Meyer[6] and Kozono-Yamazaki[23], initial value problem of the Navier-Stokes equations in some Besov spaces were extensively studied (see also [3] and [4]). Especially, Cannone[4] generalized a classical result of Kato on the global existence in $C([0,\infty);L^{3}(\mathbb{R}^{3}))$ to the case that $\|u_{0}\|_{\dot{B}_{q,\infty}^{-\alpha}}$ is sufficiently small with $3<q\leq\infty$ and $\alpha=1-\frac{3}{q}$. Recently, Koch and Tataru[22] showed the global in time existence with initial data in $\mathrm{BMO}^{-1}(\mathbb{R}^{n})$. It is worth of mentioning that there are many recent improvements using the notion of the Besov spaces and Triebel-Lizorkin spaces(see [7], [8] and references therein). Recently, Cannone-Karch[5] proved some existence and uniqueness theorems of global-in-time solutions with external force and small initial conditions in some Besov type spaces in the hyperdissipative cases by using the heat kernel property . We also mention that the authors of the current paper recently proved the small data global existence in the

scaling invariant Besov spaces for the supercritical dissipative quasi-geostrophic equation[9]. This two dimensional supercritical dissipative quasi-geostrophic equation has a similar structure with the three dimensional subdissipative Navier-Stokes equations. Considering scaling analysis, we find that if $u(x,t)$ is a solution of $(\mathrm{SNS})_\alpha$, then $u_\lambda(x,t)=\lambda^{2\alpha-1}u(\lambda x,\lambda^{2\alpha}t)$ is also a solution of $(\mathrm{SNS})_\alpha$. Thus $\dot{B}_{p,q}^{\frac{3}{p}+1-2\alpha}$, $1\le p,q\le\infty$ are scaling invariant function spaces. Our first main result of this paper is the global existence and uniqueness result for the initial value problem $(\mathrm{SNS})_\alpha$ with the initial data small in $\dot{B}_{2,1}^{\frac{5}{2}-2\alpha}$ norm. Precise statement is as follows.

Theorem 1 *Let $\alpha\in[0,\frac{5}{4})$ be given. There exists a constant $\epsilon>0$ such that for any $u_0\in B_{2,1}^{\frac{5}{2}-2\alpha}$ and $\|u_0\|_{\dot{B}_{2,1}^{\frac{5}{2}-2\alpha}}+\int_0^\infty\|f(t)\|_{\dot{B}_{2,1}^{\frac{5}{2}-2\alpha}}dt<\epsilon$, the IVP $(\mathrm{SNS})_\alpha$ has a global unique solution u, which belongs to $L^\infty(0,\infty;B_{2,1}^{\frac{5}{2}-2\alpha})\cap L^1(0,\infty;\dot{B}_{2,1}^{\frac{5}{2}})\cap C([0,\infty);B_{2,1}^{\beta})$ with $\beta=\begin{cases}\frac{5}{2}-2\alpha, & \text{if } \frac{1}{2}\le\alpha\le\frac{5}{4}\\ \frac{5}{2}-2\alpha-\delta_1, & \text{if } 0\le\alpha<\frac{1}{2},\end{cases}$ for $\delta_1>0$. Moreover, for any $\sigma>0$, u also belongs to $L^\infty(\sigma,\infty;B_{2,1}^{\frac{5}{2}})\cap L^1(\sigma,\infty;\dot{B}_{2,1}^{\frac{5}{2}+2\alpha})\cap C((0,\infty);B_{2,1}^{\gamma})$, where $\gamma=\begin{cases}\frac{5}{2}-\delta_2, & \text{if } 0\le\alpha<\frac{1}{2},\\ \frac{5}{2}, & \text{if } \frac{1}{2}\le\alpha<\frac{5}{4},\end{cases}$ for any $\delta_2>0$. Furthermore, the solution u satisfies the following estimates*

$$
\begin{aligned}
&\sup_{0\le t<\infty}\|u(t)\|_{B_{2,1}^{\frac{5}{2}-2\alpha}}+C\int_0^\infty\|u(t)\|_{\dot{B}_{2,1}^{\frac{5}{2}}}dt\\
&\le\left(\|u_0\|_{B_{2,1}^{\frac{5}{2}-2\alpha}}+\int_0^\infty\|f\|_{B_{2,1}^{\frac{5}{2}-2\alpha}}dt\right)\exp\left(C\int_0^\infty\|u(t)\|_{\dot{B}_{2,1}^{\frac{5}{2}}}dt\right).
\end{aligned}
$$

Our second main theorem below is concerned with the global stability of the solution of $(\mathrm{SNS})_\alpha$ in the case $\alpha\ge\frac{1}{2}$. For the stability of the usual Navier-Stokes equations, Beirão da Veiga–Secchi[1] and Wiegner[27] obtained L^p-stability with $p>3$ near the $L^\infty(0,\infty;L^{p+2})$-solution. Ponce–Racke–Sideris–Titi[25] proved the H^1-stability of mildly decaying global strong solutions to the Navier-Stokes equations. Recently, Kawanago[21] proved L^3-stability of the solutions near $L^5(0,\infty;L^5)$-solution.

Theorem 2 *Let $\alpha\in[\frac{1}{2},\frac{5}{4})$ be given. Assume that u^1 is a solution of the IVP $(\mathrm{SNS})_\alpha$ with an external force f^1 satisfying $u^1\in C([0,\infty);B_{2,1}^{\frac{5}{2}-2\alpha})\cap L^1(0,\infty;\dot{B}_{2,1}^{\frac{5}{2}})$ and $f^1\in L^1(0,\infty;\dot{B}_{2,1}^{\frac{5}{2}-2\alpha})$. Then there exists a positive constant $\epsilon_0=\epsilon_0(\|u_0^1\|_{B_{2,1}^{\frac{5}{2}-2\alpha}},\|u^1\|_{L^1(0,\infty;\dot{B}_{2,1}^{\frac{5}{2}})})$ such that if $\|u_0^1-u_0^2\|_{\dot{B}_{2,1}^{\frac{5}{2}-2\alpha}}<\epsilon_0$, there exists a unique global solution $u^2\in C([0,\infty);B_{2,1}^{\frac{5}{2}-2\alpha})\cap L^1(0,\infty;\dot{B}_{2,1}^{\frac{5}{2}})\cap C((0,\infty);B_{2,1}^{\frac{5}{2}})$ of $(\mathrm{SNS})_\alpha$ with initial data $u_0^2\in B_{2,1}^{\frac{5}{2}-2\alpha}$.*

Using the similar method originated from Fujita–Kato[17] and Kato[19], we can improve parts of Theorem 1 in the case $\frac{1}{2}<\alpha<\frac{5}{4}$ as follows.

Theorem 3 *Let $\alpha\in(\frac{1}{2},\frac{5}{4})$ be given. Suppose $1\le p<\frac{3}{2\alpha-1}$. There exists a constant $\epsilon>0$ and $\delta>0$ such that for any $u_0\in\dot{B}_{p,\infty}^{\frac{3}{p}+1-2\alpha}$, $\|u_0\|_{\dot{B}_{p,\infty}^{\frac{3}{p}+1-2\alpha}}<\epsilon$ and $\int_0^\infty\|f\|_{\dot{B}_{p,\infty}^{\frac{3}{p}+1-2\alpha}}<\delta$, the IVP $(\mathrm{SNS})_\alpha$ has a global solution $u\in C([0,\infty);\dot{B}_{p,\infty}^{\frac{3}{p}+1-2\alpha})$.*

We remark that if $p\ge\frac{3}{2\alpha-1}$ and $\alpha>\frac{1}{2}$, then we can obtain similar small data global existence results in the Besov type spaces following the idea in [4]. We outline the key steps of proofs of Theorem 1–3 in the section 3. The details of the proofs of Theorem 1–3 are in [10].

2 Function spaces

We first set our notations, and recall definitions of the Besov spaces. We follow [26]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}$, its Fourier transform $\mathcal{F}(f) = \hat{f}$ is defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{-ix\cdot\xi} f(x)dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying $\mathrm{Supp}\,\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \le |\xi| \le 2\}$, and $\hat{\varphi}(\xi) > 0$ if $\frac{1}{2} < |\xi| < 2$. Setting $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$ (In other words, $\varphi_j(x) = 2^{jn}\varphi(2^j x)$.), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) = 1 - \sum_{j\ge k+1} \hat{\varphi}_j(\xi).$$

We observe

$$\mathrm{Supp}\ \hat{\varphi}_j \cap \mathrm{Supp}\ \hat{\varphi}_{j'} = \emptyset \text{ if } |j - j'| \ge 2.$$

Let $s \in \mathbb{R}$, $p, q \in [0, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f = \varphi_j * f$. Then the homogeneous Besov semi-norm $\|f\|_{\dot{B}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left[\sum_{-\infty}^{\infty} 2^{jqs} \|\varphi_j * f\|^q_{L^p}\right]^{\frac{1}{q}} & \text{if } q \in [1, \infty) \\ \sup_j \left[2^{js}\|\varphi_j * f\|_{L^p}\right] & \text{if } q = \infty. \end{cases}$$

The homogeneous Besov space $\dot{B}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\|\cdot\|_{\dot{B}^s_{p,q}}$. For $s > 0$ we define the inhomogeneous Besov space norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'$ as $\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,q}}$. For the simplicity, in the following we denote $\dot{B}^s_{p,\infty}$ and $\dot{B}^{\frac{3}{p}+1-2\alpha}_{p,\infty}$ by $\dot{B}^s_p$ and $\dot{B}_p$, respectively. If $(\rho, p, r) \in [1, \infty]$, we denote

$$\|u\|_{\tilde{L}^\rho_T(\dot{B}^s_{p,r})} = \|(2^{qs}\|\Delta_q u\|_{L^\rho(0,T;L^p)})_{q\in\mathbb{Z}}\|_{l^r(\mathbb{Z})}.$$

We denote briefly $L^\infty(0, \infty; \dot{B}^s_{p,r})$ by $L^\infty(\dot{B}^s_{p,r})$. We denote $(-\Delta)^{\frac{1}{2}}$ by Λ for the notational simplicity. Taking the divergence operation on the first equation of $(\mathrm{SNS})_\alpha$, we have the formula

$$-\Delta p = \sum_{j,k} \partial_j\partial_k(u^j u^k) + \mathrm{div}\ f.$$

This enables us to define the general subdissipative Navier-Stokes type equations

$$\begin{cases} \partial_t u + \Lambda^{2\alpha} u = Q(u, u) + f, & \mathbb{R}^3 \times \mathbb{R}_+, \quad 0 \le \alpha < \frac{5}{4}, \\ u(0, x) = u_0, \end{cases}$$

with $Q(u, u) = -\mathrm{div}(u \otimes u) + \sum_{j,k} \nabla\Delta^{-1}\partial_j\partial_k(u^j u^k)$. This general equations of the usual Navier-Stokes equations was studied by Chemin[14].

3 Outline of the Proofs

The main ingredients of the proofs of Theorem 1–3 are the followings.
(i) Commutator type of estimates
(ii) Moser type of inequalities in the Besov spaces
(iii) Heat kernel type estimates

(i) Commutator type of estimates

Proposition 1 *If s satisfies $s \in (-\frac{N}{p}-1, \frac{N}{p}]$, then we have*

$$\|[u,\Delta_q]w\|_{L^p} \le c_q 2^{-q(s+1)} \|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|w\|_{\dot{B}^s_{p,1}}$$

with $\sum_{q\in\mathbb{Z}} c_q \le 1$. In the above, we denote

$$[u,\Delta_q]w = u\Delta_q w - \Delta_q(uw).$$

(ii) Moser type of inequalities in the Besov spaces

Proposition 2 *Let $s>0$, $q\in[1,\infty]$, then there exists a constant C such that the following inequality holds :*

$$\|fg\|_{\dot{B}^s_{p,q}} \le C\left(\|f\|_{L^{p_1}}\|g\|_{\dot{B}^s_{p_2,q}} + \|g\|_{L^{r_1}}\|f\|_{\dot{B}^s_{r_2,q}}\right),$$

for homogeneous Besov spaces, where p_1, $r_1 \in [1,\infty]$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{r_1}+\frac{1}{r_2}$. Let s_1, $s_2 \le \frac{N}{p}$ such that $s_1+s_2>0$, $f\in \dot{B}^{s_1}_{p,1}$ and $g\in\dot{B}^{s_2}_{p,1}$. Then $fg \in \dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}$ and

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{N}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}.$$

Proposition 3 *Let $\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\rho}$. Set $s_{1,2}=s_1+s_2-\frac{3}{p}$. If $s_i<\frac{3}{p}$ and $s_1+s_2>0$, then we have*

$$\|Q(u,v)\|_{\tilde{L}^{\rho}_T(\dot{B}^{s_{1,2}-1}_p)} \le C\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s_1}_p)}\|v\|_{\tilde{L}^{\rho_2}_T(\dot{B}^{s_2}_p)}. \tag{1}$$

(iii) Heat kernel type estimates

Proposition 4 *Let $\alpha \ge 0$ be given. There exists a constant $C>0$ such that*

$$\|e^{-t\Lambda^{2\alpha}}u_0\|_{\tilde{L}^{\rho}_T(\dot{B}^{s+\frac{2\alpha}{\rho}}_{p,r})} \le C\|u_0\|_{\dot{B}^s_{p,r}}. \tag{2}$$

If u is a solution of

$$\begin{cases} \partial_t u + \Lambda^{2\alpha}u = f, & \mathbb{R}^3\times\mathbb{R}_+, \\ u(0,x)=0, \end{cases}$$

then we have

$$\|u\|_{\tilde{L}^{\rho}_T(\dot{B}^{s+2\alpha}_p)} \le C\|f\|_{\tilde{L}^{\rho}_T(\dot{B}^s_p)}, \tag{3}$$

and

$$\|u\|_{\tilde{L}^{\rho_1}_T(\dot{B}^{s+2\alpha(1+\frac{1}{\rho_1}-\frac{1}{\rho_2})}_p)} \le C\|f\|_{\tilde{L}^{\rho_2}_T(\dot{B}^s_p)}, \tag{4}$$

where $\rho_1 \ge \rho_2$.

For the proofs of Theorem 1–2, we define the following two iterating sequences.

$$
\text{(I)}\begin{cases} \partial_t u^{n+1} + (u^n\cdot\nabla)u^{n+1} + \Lambda^{2\alpha}u^{n+1} + \nabla p^{n+1} = f^{n+1}, \quad \mathbb{R}^3\times\mathbb{R}_+, \quad 0\le 2\alpha<\frac{5}{2},\\ \text{div } u^{n+1}=0,\\ u^{n+1}(x,0)=u_0^{n+1}(x)=\sum_{q\le n+1}\Delta_q u_0, \quad f^{n+1}=\sum_{q\le n+1}\Delta_q f, \end{cases}
$$

and

$$
\text{(II)}\begin{cases} \partial_t U^{n+1} + (U^{n+1}\cdot\nabla)u^1 - (U^n\cdot\nabla)U^{n+1} + (u^1\cdot\nabla)U^{n+1}\\ \qquad +\Lambda^{2\alpha}U^{n+1} + \nabla P^{n+1} = f^{n+1}, \quad \mathbb{R}^3\times\mathbb{R}_+, \quad \frac{1}{2}\le\alpha<\frac{5}{4},\\ \text{div } U^{n+1}=0,\\ U(0,x)=\sum_{q\le n+1}(\Delta_q u_0^1-\Delta_q u_0^2), \quad f^{n+1}=\sum_{q\le n+1}\Delta_q(f^1-f^2). \end{cases}
$$

The first equation of (II) is an iterating linearized equation of the differences u^1-u^2. By using the commutator type of estimates, the Moser type of inequalities in the Besov spaces and Gronwall's inequality, we have the following inequalities for (I)

$$
\sup_{0\le t<\infty}\|u^{n+1}(t)\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}} + C_1\int_0^\infty \|u^{n+1}(t)\|_{\dot B^{\frac{5}{2}}_{2,1}}\,dt
$$
$$
\le\left(\|u_0^{n+1}\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}}+\int_0^\infty\|f^{n+1}\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}}\right)\exp\left(C_2\int_0^\infty\|u^n(t)\|_{\dot B^{\frac{5}{2}}_{2,1}}\,dt\right). \tag{5}
$$

By using the induction, we have

$$
\sup_{0\le t<\infty}\|u^{n+1}(t)\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}} + C_1\int_0^\infty \|u^{n+1}(t)\|_{\dot B^{\frac{5}{2}}_{2,1}}\,dt\le M\epsilon, \tag{6}
$$

for some $M>0$.
For the estimates of the solution of (II), we have

$$
\sup_{0\le t<\infty}\|U^{n+1}(t)\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}} + \frac{C_3}{2}\int_0^\infty \|U^{n+1}(t)\|_{\dot B^{\frac{5}{2}}_{2,1}}\,dt
$$
$$
\le\left(\|U_0^{n+1}\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}}+\int_0^\infty\|f^{n+1}\|_{\dot B^{\frac{5}{2}-2\alpha}_{2,1}}\right)\exp\left(C_4\int_0^\infty\|u^1\|_{\dot B^{\frac{5}{2}}_{2,1}}\,dt\right). \tag{7}
$$

By the induction we have the similar results for (II). Thus we have the uniform estimates of the solutions of (I) and (II). To show the existence, we consider the equations of the differences of the solutions of (I) and (II), i.e. $\delta u^{n+1}=u^{n+1}-u^n$ and $\delta U^{n+1}=U^{n+1}-U^n$, respectively. We obtain the following equations of the differences

$$
\text{(I}'\text{)}\begin{cases} \partial_t\delta u^{n+1} + (u^n\cdot\nabla)\delta u^{n+1} + (\delta u^n\cdot\nabla)u^n\\ \qquad +\Lambda^{2\alpha}\delta u^{n+1} + \nabla\delta p^{n+1} = \delta f^{n+1}, \quad \mathbb{R}^3\times\mathbb{R}_+, \quad 0\le 2\alpha<\frac{5}{2},\\ \text{div } \delta u^{n+1}=0,\\ \delta u^{n+1}(x,0)=\Delta_{n+1}u_0, \end{cases}
$$

and

$$
\text{(II}'\text{)}\begin{cases} \partial_t\delta U^{n+1} - (U^n\cdot\nabla)\delta U^{n+1} - (\delta U^n\cdot\nabla)U^n\\ \qquad +\Lambda^{2\alpha}\delta U^{n+1} + (\delta U^{n+1}\cdot\nabla)u^1 + (u^1\cdot\nabla)\delta U^{n+1} + \nabla\delta P^{n+1} = \delta f^{n+1},\\ \text{div } \delta U^{n+1}=0,\\ \delta U(0,x)=(\Delta_{n+1}u_0^1-\Delta_{n+1}u_0^2). \end{cases}
$$

Similarly to a priori estimates, we have for η satisfying $\eta = \max\{0, 1-2\alpha\}$

$$\sup_{0\le t<\infty} \|\delta u^{n+1}(t)\|_{\dot B^{\frac52-2\alpha-\eta}_{2,1}} + C_5\int_0^\infty \|\delta u^{n+1}(t)\|_{\dot B^{\frac52-\eta}_{2,1}} dt$$
$$\le \left(\|\delta u_0^{n+1}\|_{\dot B^{\frac52-2\alpha-\eta}_{2,1}} + \int_0^\infty \|\delta f^{n+1}\|_{\dot B^{\frac52-2\alpha-\eta}_{2,1}} dt\right)\exp\left(C_6\int_0^\infty \|u^n(t)\|_{\dot B^{\frac52}_{2,1}} dt\right)$$
$$+C_7 \sup_{0\le t<\infty}\|\delta u^n(t)\|_{\dot B^{\frac52-2\alpha-\eta}_{2,1}} \int_0^\infty \|u^n\|_{\dot B^{\frac52}_{2,1}} dt \exp\left(C_8\int_0^\infty \|u^n(\tau)\|_{\dot B^{\frac52}_{2,1}} d\tau\right), \tag{8}$$

and

$$\sup_{0\le t<\infty} \|\delta U^{n+1}(t)\|_{\dot B^{\frac52-2\alpha}_{2,1}} + C_9\int_0^\infty \|\delta U^{n+1}(t)\|_{\dot B^{\frac52}_{2,1}} dt$$
$$\le \left(\|\delta U_0^{n+1}\|_{\dot B^{\frac52-2\alpha}_{2,1}} + \int_0^\infty \|\delta f^{n+1}\|_{\dot B^{\frac52-2\alpha}_{2,1}} dt + C_{10}\sup_{0\le t<\infty}\|\delta U^n(t)\|_{\dot B^{\frac52-2\alpha}_{2,1}}\int_0^\infty \|U^n\|_{\dot B^{\frac52-2\alpha}_{2,1}} dt\right)\times$$
$$\exp\left(\frac{2C_{12}\epsilon_0}{C_{11}}\exp\left(C_{13}\|u^1\|_{L^1(0,\infty;\dot B^{\frac52}_{2,1})}\right) + C_{14}\|u^1\|_{L^1(0,\infty;\dot B^{\frac52}_{2,1})}\right).$$

Choosing ϵ sufficiently small and using the iteration argument, we conclude that u^n and U^n converge to u and U, respectively in $L^\infty(0,\infty; B^{\frac52-2\alpha-\eta}_{2,1}) \cap L^1(0,\infty;\dot B^{\frac52-\eta}_{2,1})$. This is the end of the sketch of the proofs of Theorem 1-2.

To prove Theorem 3, we consider following iterating sequences

$$\begin{cases} \partial_t w_{n+1} + \Lambda^{2\alpha} w_{n+1} = Q(e^{-t\Lambda^{2\alpha}}u_0, e^{-t\Lambda^{2\alpha}}u_0) + 2Q(e^{-t\Lambda^{2\alpha}}u_0, w_n) + Q(w_n,w_n) + f_{n+1}, \\ \quad \mathbb{R}^3\times\mathbb{R}_+, \frac12 < \alpha < \frac54, \\ w_{n+1}(0,x) = 0. \end{cases}$$

Using Proposition 4, we have for $\rho > \max\{\frac{2\alpha}{2\alpha-1}, 2\}$,

$$\begin{aligned} \|w_{n+1}\|_{L^\infty(\dot B_p)} \le\ & C\|Q(w_n,w_n)\|_{L^\infty(\dot B_p^{\frac3p+1-4\alpha})} \\ & + C\|Q(e^{-t\Lambda^{2\alpha}}u_0, w_n)\|_{\tilde L^\rho(\dot B_p^{\frac3p+1-2\alpha(\frac{2\rho-1}{\rho})})} \\ & + C\|Q(e^{-t\Lambda^{2\alpha}}u_0, e^{-t\Lambda^{2\alpha}}u_0)\|_{\tilde L^{\frac\rho2}(\dot B_p^{\frac3p+1-4\alpha\frac{\rho-1}{\rho}})} + C\|f_{n+1}\|_{L^1(\dot B_p)}. \end{aligned}$$

By using Proposition 3, we obtain

$$\|w_{n+1}\|_{L^\infty(\dot B_p)} \le C_{15}(\|w_n\|_{L^\infty(\dot B_p)} + \|u_0\|_{\dot B_p})^2 + \|f_{n+1}\|_{L^1(\dot B_p)}.$$

Choosing appropriately small ϵ and δ such that $\|u_0\|_{\dot B_p} < \epsilon$, $\|w_1\|_{L^\infty(\dot B_p)} < \epsilon$, and $4C_{16}\epsilon^2+\delta \le \epsilon$. Then we have $\|w_n\|_{L^\infty(\dot B_p)} \le \epsilon$, for all n. Using the similar argument as in the proof of Theorem 1 and 2, we have $w_n \to w$ in $C([0,\infty);\dot B_p)$. This is the end of the sketch of the proof of Theorem 3. □

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