数理解析研究所講究録 1389 巻 2004 年 27-33

A NEW PROOF OF THE GLOBAL EXISTENCE THEOREM OF KLAINERMAN

三重大学・教育学部 肥田野 久二男 (Kunio Hidano)
Department of Mathematics, Faculty of Education
Mie University

北海道工業大学 横山 和義 (Kazuyoshi Yokoyama)
Hokkaido Institute of Technology

1. Introduction

The commuting vector fields method of John and Klainerman has brought remarkable progress in the theory of large-time existence of small solutions to the Cauchy problem of nonlinear wave equations. The theorem of Klainerman is the most fundamental in this research and it states global existence for $n \geq 4$ and almost global existence for $n = 3$ of small solutions to quadratic, quasi-linear wave equations. The heart of the method of Klainerman is the use of the Killing vector fields and the radial vector field $S = t\partial_t + x \cdot \nabla$ to prove the global Sobolev inequality in the Minkowski space $\mathbb{R}^{n+1}$, known by the name of the Klainerman inequality. Thanks to good commutation relations between these vector fields and the d'Alembertian, the standard energy integral argument together with the Klainerman inequality works efficiently for the proof of the fundamental result.

In this article we go over our recent work [2] on a new proof of the global existence without relying on the Klainerman inequality. In addition to the Klainerman-Sideris inequality, an integrability estimate of the local energy plays a key role in the proof.

The equation we consider is the system of m quasi-linear wave equations

$$(QLW) \quad \begin{cases} \partial_t^2 u^k - c_k^2 \Delta u^k = F^k(\partial u, \partial^2 u), \ t > 0, \ x \in \mathbb{R}^n, \\ u^k(0) = f^k, \ \partial_t u^k(0) = g^k \end{cases}$$

($k = 1, 2, \dots, m$), where $u^k : (0, T) \times \mathbb{R}^n \to \mathbb{R}$, c_k stands for the wave-propagation speed, f^k, $g^k \in C_0^\infty(\mathbb{R}^n)$, and

$$F^k(\partial u, \partial^2 u) = G_{ij}^{k,\alpha\beta\gamma} \partial_\alpha u^i \partial_\beta \partial_\gamma u^j + H_{ij}^{k,\alpha\beta} \partial_\alpha u^i \partial_\beta u^j \ (\partial_0 = \partial_t, \ \partial_i = \partial x^i) \tag{1.1}$$

for real constants satisfying the symmetry condition

$$G_{ij}^{k,\alpha\beta\gamma} = G_{ij}^{k,\alpha\gamma\beta} = G_{ik}^{j,\alpha\beta\gamma}. \tag{1.2}$$

Repeated indices are summed on the right-hand side of (1.1) if lowered and raised. Greek indices have ranged from 0 to n, and Roman from 1 to m. The generators of Euclid rotations are denoted by $\Omega_{ij} = x^i\partial_j - x^j\partial_i$ $(1 \le i < j \le n)$, and the collection of ∂_α and Ω_{ij} is denoted by $Z := \{\partial_0, \partial_1, \dots, \partial_n, \Omega_{12}, \dots, \Omega_{n-1n}\}$. Working mainly with the operators of Z, we shall also use the generator of scaling $S = t\partial_t + x \cdot \nabla$.

Associated with the linear part of (QLW), the energy is defined as

$$E_1(u(t)) = \frac{1}{2}\sum_{k=1}^{m}\int_{\mathbb{R}^n}\left(|\partial_t u^k(t,x)|^2 + c_k^2|\nabla u^k(t,x)|^2\right)dx, \tag{1.3}$$

and the generalized energy as

$$E_l(u(t)) = \sum_{\substack{|a|+b\le l-1\\ b\le 1}} E_1(Z^a S^b u(t)) \quad (l = 2, 3, \dots). \tag{1.4}$$

Here and later on as well, denoting $Z_0 := \partial_0$, $Z_1 := \partial_1$, $\dots$, $Z_n := \partial_n$, $Z_{n+1} := \Omega_{12}$, $\dots$, $Z_\nu := \Omega_{n-1n}$ $(\nu = (n^2+n)/2)$, we mean $Z_0^{a_0}\cdots Z_\nu^{a_\nu}$ by Z^a for a multi-index a. We shall also employ the notation $\partial^a = \partial_0^{a_0}\partial_1^{a_1}\cdots\partial_n^{a_n}$ for a multi-index a. Our main theorem is stated as follows.

Main Theorem. *Suppose $n \ge 4$. Let l be an integer satisfying $[l/2]+[n/2]+2 \le l-[n/2]-2$. There exists an $\varepsilon > 0$ with the following. If $E_l^{1/2}(u(0)) < \varepsilon$, then (QLW) has a unique smooth global solution. It satisfies*

$$E_\mu^{1/2}(u(t)) < 2\varepsilon,\ E_l^{1/2}(u(t)) \le 2E_l^{1/2}(u(0))(1+t)^{C\varepsilon},\ 0 < t < \infty \tag{1.5}$$

$(\mu = l - [n/2] - 2)$ with a constant C independent of ε. The solution also satisfies for a sufficietly small number $\delta > 0$

$$\begin{aligned}\sum_{\substack{0\le\alpha\le n\\ 1\le i\le m}}\ \sum_{\substack{|a|+b\le\mu\\ b\le 1}} \|\langle x\rangle^{-(1/2)-\delta}\partial_\alpha Z^a S^b u^i\|_{L^2((0,T)\times\mathbb{R}^n)} \\ \le C\varepsilon\left(1+\varepsilon(1+T)^{C\varepsilon}\right),\ 0 < T < \infty\end{aligned} \tag{1.6}$$

with a constant C independent of ε and T.

2. Sketch of the Proof of Main Theorem

In addition to the generalized energy (1.4), we need another type of energy defined as

$$\bar{E}_l(u(t)) = \sum_{|a|\le l-1} E_1(Z^a u(t)) \tag{2.1}$$

for $l = 2, 3, \ldots$. Note that $\bar{E}_l(u(t)) \le E_l(u(t))$.

The auxiliary norm

$$M_l(u(t)) = \sum_{k=1}^{m} \sum_{|a|=2} \sum_{|b|\le l-2} \|\langle c_k t - r\rangle \partial^a Z^b u^k(t)\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

will play an intermediary role in the energy integral argument below. For simplicity we often denote the $L^p(\mathbb{R}^n)$-norm by $\|\cdot\|_{L^p}$.

Our proof consists of the four steps: decay estimate, higher-order energy estimate, space-time L^2-estimate, and lower-order energy estimate. We start with the decay estimate.

Decay Estimate. We use the following Sobolev-type inequalities (see Lemma 4.1 of [1] for the proof).

Lemma 2.1. (1) *Let $n \ge 3$. The inequality*

$$\langle r\rangle^{(n/2)-1}\langle c_j t - r\rangle |\partial u^j(t,x)| \le C\bar{E}^{1/2}_{[n/2]+1}(u(t)) + CM_{[n/2]+2}(u(t)) \tag{2.3}$$

holds.

(2) *Let $n = 4$ and $0 \le d < 1/2$. The inequality*

$$\langle r\rangle^{1+d}|\partial u(t,x)| \le C\bar{E}^{1/2}_4(u(t)) \tag{2.4}$$

holds.

(3) *Suppose $n \ge 5$. The inequality*

$$\langle r\rangle^{(n/2)-1}|\partial u(t,x)| \le C\bar{E}^{1/2}_{[n/2]+2}(u(t)) \tag{2.5}$$

holds.

Since weighted L^2-norms $M_l(u(t))$ appear on the right-hand side of the Sobolev-type inequality (2.3), it is necessary to bound $M_l(u(t))$ by $E_l^{1/2}(u(t))$ for the completion of the energy integral argument. The next crucial inequality, which is due to Klainerman and Sideris [5], is the starting point of the proof.

Lemma 2.2 (Klainerman-Sideris inequality). *Assume $\kappa \geq 2$ and $n \geq 2$. The inequality*

$$M_\kappa(u(t)) \leq CE_\kappa^{1/2}(u(t)) + C \sum_{|a| \leq \kappa - 2} \|(t+r)\Box Z^a u(t)\|_{L^2} \tag{2.6}$$

holds for any smooth function $u : \mathbb{R}_+^{1+n} \to \mathbb{R}^m$ when the right-hand side is finite.

In what follows we assume that u is a smooth local (in time) solution to (QLW). Using the Sobolev-type inequalities of Lemma 2.1, we obtain with the help of a bootstrap-type argument

Lemma 2.3. *Let $n \geq 4$ and let l be large so that*

$$\left[\frac{l}{2}\right] + \left[\frac{n}{2}\right] + 2 \leq l - \left[\frac{n}{2}\right] - 2. \tag{2.7}$$

Set $\mu = l - [n/2] - 2$. There exists a small, positive constant ε_0 with the following property: Suppose that, for a local smooth solution u of (QLW), the supremum of $E_\mu^{1/2}(u(t))$ on an interval $(0,T)$ is sufficiently small so that

$$\sup_{0<t<T} E_\mu^{1/2}(u(t)) \leq \varepsilon_0. \tag{2.8}$$

Then

$$M_\mu(u(t)) \leq CE_\mu^{1/2}(u(t)),\ 0 < t < T \tag{2.9}$$

and

$$M_l(u(t)) \leq CE_l^{1/2}(u(t)),\ 0 < t < T \tag{2.10}$$

hold with a constant C independent of T.

Under the smallness of the lower-order energy of local solutions, we have therefore obtained the decay estimate such as

$$\langle r\rangle^{(n/2)-1}\langle c_j t - r\rangle |\partial u^j(t,x)| \leq CE_{[n/2]+2}^{1/2}(u(t)). \tag{2.11}$$

Higher-order Energy. Allowing the higher-order energy to grow polynomially in time and bounding the lower-order energy uniformly in time, we shall accomplish the energy-integral argument. For the higher-order energy, we have the following result (see Section 6 of [2]).

Lemma 2.4. *Let $n \geq 4$ and suppose that l is large so that (2.7) holds. Set $\mu \equiv l - [n/2] - 2$. Suppose that initial data of a local solution to (QLW) satisfy $E_l^{1/2}(u(0)) < \varepsilon$ for a sufficiently small ε such that $2\varepsilon \leq \varepsilon_0$ (see (2.8) for ε_0). Let T_0 be the* *supremum of all $T > 0$ for which the unique local solution satisfies*

$$E_\mu^{1/2}(u(t)) < 2\varepsilon, \quad 0 < t < T. \tag{2.12}$$

Then the solution has the bound

$$E_l^{1/2}(u(t)) \leq 2E_l^{1/2}(u(0))(1+t)^{C\varepsilon}, \quad 0 < t < T_0. \tag{2.13}$$

Since the estimate (2.11) for $n \geq 4$ only implies the rather weak time decay such as $(1+t)^{-1}$ at best, we have no hope of bounding the higher-order energy $E_l(u(t))$ uniformly in time. It will be shown that the weak bound (2.13) combined with a certain space-time L^2-estimate is actually sufficient to bound the *lower-order* energy uniformly in time. Recall that we have assumed in Lemma 2.3 the smallness only of the lower-order energy.

Space-time L^2-estimate. A key to our proof is that a space-time L^2-estimate can compensate for the weak bound (2.13).

Lemma 2.5. *Let $\delta > 0$. Under the same assumptions as in Lemma 2.4 the local solution satisfies*

$$\begin{aligned} &\sum_{\substack{\alpha=0,\dots,3\\ i=1,\dots,m}} \sum_{\substack{|a|+b\leq\mu\\ b\leq 1}} \|\langle r\rangle^{-(1/2)-\delta}\partial_\alpha Z^a S^b u^i\|_{L^2((0,t)\times\mathbb{R}^n)} \\ &\leq CE_{\mu+1}^{1/2}(0) + C\varepsilon^2(1+t)^{C\varepsilon}, \quad 0 < t < T_0 \end{aligned} \tag{2.14}$$

for constants C independent of ε.

For the proof of Lemma 2.5 we need the integrability (in time) estimate of finite-energy solutions.

Lemma 2.6. *Let $n \geq 1$, $(f,g) \in \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ and $\delta > 0$. Suppose that u solves the Cauchy problem $\Box u = 0$ with the data $(u(0), \partial_t u(0)) = (f,g)$. Then this solution satisfies*

$$\sum_{\alpha=0}^{n} \|\langle x\rangle^{-(1/2)-\delta}\partial_\alpha u\|_{L^2((0,\infty)\times\mathbb{R}^n)} \leq C\big(\|\nabla f\|_{L^2} + \|g\|_{L^2}\big). \tag{2.15}$$

Remark. Let $F \in L^1((0,T); L^2(\mathbb{R}^n))$. Suppose that u solves the Cauchy problem $\Box u = F$ with zero data at $t = 0$. Because the estimate (2.15) obviously remains true for the data $(0, g)$ with $g \in L^2(\mathbb{R}^n)$, we can obtain by the Duhamel principle

$$\sum_{\alpha=0}^{n} \|\langle x\rangle^{-(1/2)-\delta}\partial_\alpha u\|_{L^2((0,T)\times\mathbb{R}^n)} \leq C\int_0^T \|F(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} d\tau. \tag{2.16}$$

We draw attention to the fact that, essentially in line with §27 of Mochizuki [7], one can prove Lemma 2.6 for $n = 1$ or $n \geq 3$ by the multiplier method. In Proposition 2.8 of [6] Metcalfe has discussed global (in time) integrability of the local energy by making use of the space-time Fourier transform, and he has proved

$$||\langle x\rangle^{-(n-1)/4}\partial u||_{L^2((0,\infty)\times\mathbb{R}^n)} \leq C(||\nabla f||_{L^2} + ||g||_{L^2})$$

for $n \geq 4$. The present authors have verified that, without any essential modifications, the argument of Metcalfe is actually valid for the proof of (2.15) for all $n \geq 1$. Therefore the estimate (2.16) is also true for all $n \geq 1$ by the Duhamel principle.

Lower-order Energy. Using the decay estimate (2.11) and rather weak bounds (2.13)–(2.14), we can obtain the uniform (in time) bound of the lower-order energy. The key is the use of the dyadic decomposition of the time interval $(0, T_0)$. This technique has been devised by Sogge (see [8] on page 363). Following Section 8 of [2], we describe how to obtain the uniform (in time) bound of the lower-order energy $E_\mu(u(t))$ by Sogge's dyadic decomposition technique .

Our consideration starts with the standard inequality

$$E_\mu^{1/2}(u(t)) \leq E_\mu^{1/2}(u(0)) + C \sum_{\substack{|a|+b\leq\mu-1 \\ b\leq 1}} ||Z^a S^b F(\partial u, \partial^2 u)||_{L^1(0,t;L^2(\mathbb{R}^n))} \tag{2.17}$$

for $\mu \equiv l - [n/2] - 2$ as before. It is necessary to estimate the contributions from quasi-linear terms

$$\sum_{\substack{|a|+|b|+|a'|+|b'|\leq\mu-1 \\ b+b'\leq 1}} \int_0^t ||\partial Z^a S^b u^i(\tau)\partial^2 Z^{a'} S^{b'} u^j(\tau)||_{L^2} d\tau \tag{2.18}$$

and from semi-linear terms

$$\sum_{\substack{|a|+|b|+|a'|+|b'|\leq\mu-1 \\ b+b'\leq 1}} \int_0^t ||\partial Z^a S^b u^i(\tau)\partial Z^{a'} S^{b'} u^j(\tau)||_{L^2} d\tau. \tag{2.19}$$

Set $c_0 \equiv \min\{\, c_j \,:\, j = 1, 2, \ldots, m\}$ for the propagation speeds. Here we only deal with the model case (2.18) with $|a| + |b| + |a'| + |b'| = 0$. We divide (2.18) into three pieces

$$\begin{aligned} &\sum \int_0^1 ||\partial u^i(\tau)\partial^2 u^j(\tau)||_{L^2} d\tau \\ &+ \sum \int_1^t ||\cdots||_{L^2(\{(\tau,x):r<c_0\tau/2\})} d\tau + \sum \int_1^t ||\cdots||_{L^2(\{(\tau,x):r>c_0\tau/2\})} d\tau \\ &\equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.20}$$

We focus our attention on I_2 only. By (2.11), (2.13) and (2.14) we see for small $\delta > 0$

$$
\begin{aligned}
(2.21) \qquad & ||\partial u^i(\tau)\partial^2 u^j(\tau)||_{L^2(\{(\tau,x):r<c_0\tau/2\})} \\
&\le \langle\tau\rangle^{-1}||\langle r\rangle^{-(1/2)-\delta}\partial u^i(\tau)||_{L^2}||\langle r\rangle^{(1/2)+\delta}\langle c_j\tau - r\rangle\partial^2 u^j(\tau)||_{L^\infty} \\
&\le C\langle\tau\rangle^{-1+C\varepsilon}E_l^{1/2}(u(0))||\langle r\rangle^{-(1/2)-\delta}\partial u^i(\tau)||_{L^2}.
\end{aligned}
$$

Abusing the notation a bit to mean T_0 by 2^{N+1}, we have

$$
\begin{aligned}
(2.22) \qquad I_2 &\le C\varepsilon\sum_{j=0}^{N}\int_{2^j}^{2^{j+1}}\langle\tau\rangle^{-1+C\varepsilon}||\langle r\rangle^{-(1/2)-\delta}\partial u^i(\tau)||_{L^2}d\tau \\
&\le C\varepsilon\sum_{j=0}^{N}\Big(\int_{2^j}^{2^{j+1}}\langle\tau\rangle^{-2+2C\varepsilon}d\tau\Big)^{1/2}||\langle r\rangle^{-(1/2)-\delta}\partial u^i||_{L^2((0,2^{j+1})\times\mathbb{R}^n)} \\
&\le C\varepsilon^2\sum_{j=0}^{\infty}(2^j)^{(-1/2)+C\varepsilon} \le C\varepsilon^2.
\end{aligned}
$$

Acknowledgments. We are grateful to Professor M. Yamazaki for inviting us to the conference "Harmonic Analysis and Nonlinear Partial Differential Equations" held in July 2003 at RIMS.

References

[1] K. Hidano, *An elementary proof of global or almost global existence for quasi-linear wave equations.* To appear in Tohoku Math. J.

[2] K. Hidano and K. Yokoyama, *A new proof of the global existence theorem of Klainerman.* Submitted for publication.

[3] M. Keel, F. Smith and C. D. Sogge, *Almost global existence for some semilinear wave equations.* J. Anal. Math. **87** (2002), 265–279.

[4] S. Klainerman, *Remarks on the global Sobolev inequalities in the Minkowski space $\mathbb{R}^{n+1}$.* Comm. Pure Appl. Math. **40** (1987), 111–117.

[5] S. Klainerman and T. C. Sideris, *On almost global existence for nonrelativistic wave equations in 3D.* Comm. Pure Appl. Math. **49** (1996), 307–321.

[6] J. L. Metcalfe, *Global existence for semilinear wave equations exterior to nontrapping obstacles.* arXiv:math.AP/0210246. To appear in Houston J. Math.

[7] K. Mochizuki, *Scattering theory for wave equations.* Kinokuniya, Tokyo, 1984 (In Japanese).

[8] C. D. Sogge, *Global existence for nonlinear wave equations with multiple speeds.* "Harmonic Analysis at Mount Holyoke", Contemp. Math., No. 320, Amer. Math. Soc., Providence, RI, 2003.