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Asymptotic Analyses for an Exponential Hedging Problem

Jun Sekine
Graduate School of Engineering Science
Osaka University
Toyonaka, Osaka 560-8531, Japan.
E-mail: sekine@sigmath.es.osaka-u.ac.jp

Abstract

Pricing and hedging problems based on the exponential utility maximization are considered in the incomplete market consisting of the derivative security written on the *untradable* asset and the *tradable* asset as the instrument for hedging. In particular, with respect to the correlation ρ of the two asset price processes, two special situations are addressed: (i) $\rho \approx 1$, closely correlated case, (ii) $\rho \approx 0$, almost independent case. Asymptotic expansions of the backward stochastic differential equations for the dual optimization problems with respect to small parameters are studied, and approximations for the prices and the hedging strategies are obtained in explicit forms.

1 Introduction

In Davis (2000), [1], the following special but typical situation in an incomplete market is addressed: let $S^i := (S_t^i)_{t\in[0,T]}$ $(i = 1, 2)$ be the price process of 2-risky assets defined by the stochastic differential equations:

$$\begin{cases} dS_t^1 = S_t^1 \left(\sigma_1 dw_1(t) + \mu_1 dt\right), & S_0^1 > 0, \\ dS_t^2 = S_t^2 \left\{\sigma_2 \left(\sqrt{1-\epsilon^2} dw_1(t) + \epsilon dw_2(t)\right) + \mu_2 dt\right\}, & S_0^2 > 0 \end{cases}$$

on the probability space $(\Omega, \mathcal{F}, P)$ with a 2-dimensional Brownian motion $w := (w_t)_{t\in[0,T]}$, $w_t := (w_1(t), w_2(t))'$ ($(\cdot)'$ denotes the transpose of a vector or a matrix) and the augmented Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$, where $\sigma_1, \sigma_2 > 0$, $\epsilon \in [-1, 1]$ and $\mu_1, \mu_2 \in \mathbf{R}$. Supposing S^1 *untradable* and S^2 *tradable*, and assuming $\epsilon \neq 0$, $\epsilon \ll 1$, i.e., two assets S^1 and S^2 are *closely correlated*:

$$\rho := \frac{d\langle S^1, S^2\rangle}{\sqrt{d\langle S^1\rangle d\langle S^2\rangle}} = \sqrt{1-\epsilon^2} \approx 1,$$

consider the pricing and hedging problem of the derivative security written on the *untradable* asset S^1, whose payoff at the maturity T is given by $F := h(S_T^1)$ with some $h : \mathbf{R}_+ \mapsto \mathbf{R}$.

Let $X^{x,\pi} := (X_t^{x,\pi})_{t\in[0,T]}$ be the value process of the self-financing hedging portfolio, given by

$$X_t^{x,\pi} := e^{rt}\left(x + \int_0^t \pi_u \frac{d\widetilde{S}_u^2}{\widetilde{S}_u^2}\right) \quad \text{for } t \in [0,T],$$

where r is the constant interest rate, $x \in \mathbf{R}$ is the initial capital for hedging, $\pi := (\pi_t)_{t\in[0,T]}$ is the hedging strategy, and $\widetilde{S}_t^2 := e^{-rt}S_t^2$.

In [1], as the *hedging* problem for a seller of the derivative security, the following utility maximization problem, (which we call the exponential hedging problem, following Delbaen et. al; 2002, [2]):

$$\text{(P)} \qquad V^\epsilon(x) := \sup_{\pi\in\mathcal{A}} E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right]$$

with respect to the exponential utility function:

$$U_\gamma(x) := -\frac{e^{-\gamma x}}{\gamma} \quad (\gamma > 0)$$

over an appropriately chosen space $\mathcal{A}$ of admissible strategies is employed. Also, as the *pricing* problem, the quantity called utility indifference price: $p^\epsilon(x,F)$ satisfying

$$\text{(1.1)} \qquad V^\epsilon\left(x + p^\epsilon(x,F)\right) = \sup_{\pi\in\mathcal{A}} E\left[U_\gamma\left(X_T^{x,\pi}\right)\right]$$

is proposed as a coherent price of the derivative security.

To attack the problem (P), a duality method is employed, which is well established for utility maximization problems (cf., Karatzas and Shreve ; 1998, [6], for example). For the value function $v^\epsilon(t,y)$ ($(t,y) \in [0,T]\times\mathbf{R}_+$) of the dual problem (cf., (3.7) for the precise definition), a dynamic-programming equation is derived and the existence of its smooth solution is checked in the setting of [1]. Moreover, the following relations are obtained.

Theorem 1.1 *(Theorem 6.1, 6.4 and 7.3 of Davis, [1])*
1. *For the optimal value of the problem (P) and the utility indifference price defined by (1.1),*

$$\begin{aligned} V^\epsilon(x) &= U_\gamma\left(e^{rT}x - \frac{v^\epsilon(0,S_0^1)}{\gamma}\right), \\ p^\epsilon(x,F) &= \frac{e^{-rT}}{\gamma}\left\{v^\epsilon(0,S_0^1) + \frac{T}{2}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2\right\} \end{aligned}$$

hold for any $x \in \mathbf{R}$, respectively.
2. *An optimal strategy of the problem (P) is given by*

$$\pi_t^* = \frac{e^{-rT}}{\gamma}\left\{\frac{\mu_2 - r}{\sigma_2^2} - \sqrt{1-\epsilon^2}\frac{\sigma_1}{\sigma_2}\partial_x v^\epsilon(t,S_t^1)S_t^1\right\}.$$

3. *As $\epsilon \downarrow 0$, the value function has the expansion*

$$\text{(1.2)} \qquad \begin{aligned} v^\epsilon(t,y) &= \gamma E^0\left[h\left(S_T^1\right)|S_t^1 = y\right] - \frac{1}{2}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2 (T-t) \\ &\quad + \epsilon^2\frac{\gamma^2}{2}\mathrm{Var}^0\left[h\left(S_T^1\right)|S_t^1 = y\right] + O(\epsilon^4). \end{aligned}$$

Here, $E^0[*|\cdot]$ denotes the conditional expectation with respect to the minimal martingale measure P^0, defined by the formula:

$$\frac{dP^0}{dP}\bigg|_{\mathcal{F}_t} := \mathcal{E}_t\left(-\frac{\mu_2 - r}{\sigma_2}\left(\sqrt{1-\epsilon^2}w_1 + \epsilon w_2\right)\right),$$

$\mathrm{Var}^0[*|\cdot] := E^0[(*)^2|\cdot] - \left(E^0[*|\cdot]\right)^2$, *and $O(\epsilon^4)$ depends on the value (t, y).*

In particular, we are interested in the expansion (1.2). From a practical viewpoint, it is an effective and useful expansion: it gives nice, intuitive approximations of the value of the problem (P):

$$\log V^\epsilon(x) - \log U_\gamma\left(e^{rT}x - E^0\left[h\left(S_T^1\right)\right] - \frac{T}{2\gamma}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2 - \epsilon^2\frac{\gamma}{2}\mathrm{Var}^0\left[h(S_T^1)\right]\right) = O(\epsilon^4),$$

and the utility indifference price:

$$p^\epsilon(x, F) = e^{-rT}\left\{E^0\left[h(S_T^1)\right] + \epsilon^2\frac{\gamma}{2}\mathrm{Var}^0\left[h(S_T^1)\right]\right\} + O(\epsilon^4).$$

Also, both quantities $E^0\left[h(S_T^1)|S_t^1 = y\right]$ and $\mathrm{Var}^0\left[h(S_T^1)|S_t^1 = y\right]$ are fairly "computable".

Further, we are interested in the approximation of the optimal strategy, which is not mentioned in [1], and is studied in [10]: under an assumption, the strategy $\widehat{\pi}$ defined by

$$\begin{aligned}\widehat{\pi}_t \quad := \quad & \frac{e^{-rT}}{\gamma}\left[\frac{\mu_2 - r}{\sigma_2^2}\right.\\ & \left. - \sqrt{1-\epsilon^2}\frac{\sigma_1}{\sigma_2}S_t^1\partial_y\left(\gamma E^0\left[h(S_T^1)|S_t^1 = y\right] + \epsilon^2\frac{\gamma^2}{2}\mathrm{Var}^0\left[h(S_T^1)|S_t^1 = y\right]\right)\bigg|_{y=S_t^1}\right]\end{aligned}$$

satisfies the relation

$$\text{(1.3)} \qquad \log V^\epsilon(x) - \log E\left[U_\gamma\left(-F + X_T^{x,\widehat{\pi}}\right)\right] = O(\epsilon^4) \quad \text{as } \epsilon \downarrow 0.$$

In the present paper, we extend the above analysis to **(i)** stochastic mean-return-rate case, and **(ii)** $\epsilon \approx 1$: almost independent case. Instead of treating the dynamic programming equation, we analyze the associated backward stochastic differential equation (abbrev. BSDE, hereafter), which is the approach in Rouge-El Karoui (2000), [9]. Following [10] by the author, we compute the asymptotic expansion of the BSDE with respect to ϵ, which suggests a systematic approach to obtain the expansions such as (1.2-3).

The organization of this paper is the following. In the next section, the setup is introduced and in Section 3, the relation between the dual problem of the exponential hedging problem and the BSDE having a quadratic growth term in the drift is reviewed. Main results are explained in Section 4, and their proofs is demonstrated in Appendix A. Section 5 is for stating conclusions.

2 Setup

We extend the setup in Introduction in the following way. Let $(\Omega, \mathcal{F}, P^0) := \prod_{i=1}^2(\Omega_i, \mathcal{F}^i, P_i^0)$ be the product of Wiener spaces, i.e., $\Omega_i := C_0([0,T], \mathbf{R})$, $\mathcal{F}^i := \mathcal{B}(\Omega_i)$ and P_i^0 is the Wiener measure, the law of the i-th canonical Brownian motion

$w_i^0 := \left(w_i^0(t)\right)_{t\in[0,T]}$. The filtration $(\mathcal{F}_t)_{t\in[0,T]} := (\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t\in[0,T]}$ is the augmented natural filtration. Sometimes a random variable X on $(\Omega_1, \mathcal{F}^1, P_1)$ is identified with $X \circ j_1$ on $(\Omega, \mathcal{F}, P)$, where $j_1 : \Omega \ni \omega := (\omega_1, \omega_2) \mapsto \omega_1 \in \Omega_1$ is the projection onto the first probability space.

We start with the stochastic differential equation:

$$\begin{cases} dS_t^1 = S_t^1 \left[\sigma_1 dw_1^0(t) + \left\{\mu_1(t) - \sqrt{1-\epsilon^2}\dfrac{\sigma_1(\mu_2(t) - r)}{\sigma_2}\right\} dt\right], & S_0^1 > 0, \\ dS_t^2 = S_t^2 \left[\sigma_2\left(\sqrt{1-\epsilon^2}dw_1^0(t) + \epsilon dw_2^0(t)\right) + rdt\right], & S_0^2 > 0. \end{cases}$$

Here, $\sigma_1, \sigma_2 > 0$, $r \in \mathbf{R}$, and $\epsilon \in (-1,0) \cup (0,1)$ are constant, while μ_1 is a bounded $\mathcal{F}_t^1$-predictable process, i.e., $\mu_1 : [0,T] \times \Omega_1 \ni (t, \omega_1) \mapsto \mu_1(t, \omega_1) \in \mathbf{R}$ is measurable with respect to the predictable σ-algebra. Further, as the condition for μ_2, we impose one of the following

(2.1) μ_2 is a bounded $\mathcal{F}_t^1$-predictable process. Further, $\mu_2(t,\cdot) \in \mathbb{D}_{1;1,2}$ for all $t \in [0,T]$, where $\mathbb{D}_{1;1,2}$ is the completion of the space of Wiener polynomials in the first probability space: $\mathbb{P}_1 := \{F := \phi((f_1 \cdot w_1^0)_T, \ldots, (f_n \cdot w_1^0)_T);\ \phi :$ polynomial in n variables, $f_i \in L^2([0,T])$, $i = 1, \ldots, n\}$ with respect to the norm: $\|F\|_{1;1,2} := \|F\|_{L^2(\Omega_1)} + \left\|\sum_{i=1}^n \partial_{x^i}\phi((f_1 \cdot w_1^0)_T, \ldots, (f_n \cdot w_1^0)_T) f_i(\cdot)\right\|_{L^2([0,T]\times\Omega_1)}$, and it has the bounded Malliavin derivative for $t \in [0,T]$, i.e., $D_{1,s}\mu_2(t,\cdot) \in L^\infty(\Omega_1, \mathbf{R})$ for $s, t \in [0,T]$, where $D_{1,\cdot}(\cdot)$ denotes the Malliavin derivative in the first space.

(2.2) μ_2 is a bounded deterministic process.

Next, let P be the probability measure defined by

$$\left.\frac{dP}{dP^0}\right|_{\mathcal{F}_t} := \mathcal{E}_t\left(\int \lambda\left(\sqrt{1-\epsilon^2}dw_1^0 + \epsilon dw_2^0\right)\right) =: \Lambda_t, \quad \text{where} \quad \lambda := \frac{\mu_2 - r}{\sigma_2}.$$

From the Girsanov theorem, the process $w := (w_1, w_2)'$, given by

$$w_1(t) := w_1^0(t) - \sqrt{1-\epsilon^2}\int_0^t \lambda_u du, \quad w_2(t) := w_2^0(t) - \epsilon\int_0^t \lambda_u du$$

is a $(P, \mathcal{F}_t)$-Brownian motion, and (S^1, S^2) satisfies

$$\begin{cases} dS_t^1 = S_t^1 \{\sigma_1 dw_1(t) + \mu_1(t)dt\}, & S_0^1 > 0, \\ dS_t^2 = S_t^2 \left\{\sigma_2\left(\sqrt{1-\epsilon^2}dw_1(t) + \epsilon dw_2(t)\right) + \mu_2(t)dt\right\}, & S_0^2 > 0. \end{cases}$$

We regard P as the "*real world*" probability measure, S^1, the price process of the *untradable* asset, and S^2, that of the *tradable* asset, respectively, therefore, P^0 is interpreted as the so-called *minimal martingale measure.*

Note that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is not generated by the P-Brownian motion w in general, but that the following martingale representation theorem holds with respect to w.

Lemma 2.1 *Let $G \in L^2(\Omega, \mathcal{F}, P)$. Then, $G = E[G] + \int_0^T \left(\phi_t^G\right)' dw_t$ holds for some 2-dimensional predictable ϕ^G such that $E\left[\int_0^T |\phi_t^G|^2 dt\right] < \infty$.*

Proof. Since $\Lambda_T G$ is P^0-integrable, $\Lambda_T G = E^0[\Lambda_T G] + \int_0^T (\psi_t^G)' dw_t^0 = E[G] + \int_0^T (\psi_t^G)' dw_t^0$ holds for some 2-dimensional predictable ψ^G such that $\int_0^T |\psi_t^G|^2 dt < \infty$.

Let $H_t^G := E^0[\Lambda_T G|\mathcal{F}_t] = E[G] + \int_0^t (\psi_u^G)' dw_u^0$. Then,

$$\begin{aligned} E[G|\mathcal{F}_t] &= \frac{E^0[\Lambda_T G|\mathcal{F}_t]}{\Lambda_t} = E[G] + \int_0^t d\left(\frac{H_u^G}{\Lambda_u}\right) \\ &= E[G] + \int_0^t \frac{\psi_t^G - H_u^G \sigma_2^{-1}(\mu_2(u) - r)}{\Lambda_u} \left(\sqrt{1-\epsilon^2} dw_1(u) + \epsilon dw_2(u)\right) \end{aligned}$$

is observed for $t \in [0, T]$ from the Bayes rule and the Itô formula. By letting $\phi^G := \Lambda^{-1}\left\{\psi^G - H^G \sigma_2^{-1}(\mu_2 - r)\right\}\left(\sqrt{1-\epsilon^2}, \epsilon\right)'$, the lemma follows since the martingale $\int (\phi^G)' dw$ is square integrable: $E\left[\int_0^T |\phi_t^G|^2 dt\right] = \mathrm{Var}[G] < \infty$. ∎

Let F be the payoff of a derivative security maturing at T having the form $F := h(S^1)$ with h, a bounded measurable function on the space $C([0, T], \mathbf{R}_+)$. We assume that the functional $F(\cdot) : \Omega_1 \ni \omega_1 \mapsto F(\omega_1) = h(S^1(\omega_1)) \in \mathbf{R}$ belongs to $\mathbb{D}_{1;1,2}$ and that it has the bounded Malliavin-derivative, i.e.,

$$D_{1,t} F \in L^\infty(\Omega_1, \mathbf{R}) \quad \text{for all } t \in [0, T]. \tag{2.3}$$

We then address the optimization problem (P) over the space of admissible strategies:

$$\mathcal{A} := \left\{\pi : \text{predictable}, E\left[\int_0^T |\pi_t|^2 dt\right] < \infty\right\}.$$

3 Duality and quadratic BSDE

In this section, along the lines in Rouge-El Karoui, [9], we review the duality method to attack the problem (P) and its relation to the BSDE for the dual problem, which has a quadratic growth term in the drift.

First, prepare a notation

Notation 3.1 *for the process A, $\widetilde{A}$ denotes the process defined by $\widetilde{A}_t := e^{-rt} A_t$,*

and vectors:

$$d_\epsilon := \left(\sqrt{1-\epsilon^2}, \epsilon\right)' \quad \text{and} \quad d_\epsilon^\perp := \left(\epsilon, -\sqrt{1-\epsilon^2}\right)'$$

to recall the expressions

$$\begin{aligned} d\widetilde{S}_t^2 &= \widetilde{S}_t^2 \sigma_2 (d_\epsilon' dw_t + \lambda_t dt) \quad \text{with} \quad \lambda := \frac{\mu_2 - r}{\sigma_2}, \\ \text{and} \quad \widetilde{X}_t^{x,\pi} &= x + \int_0^t \pi_u \sigma_2 (d_\epsilon' dw_u + \lambda_u du). \end{aligned}$$

These imply, for each ν, an element of

$$\mathcal{D} := \left\{\nu := \eta d_\epsilon^\perp;\ \eta : \text{bounded, predictable}\right\},$$

that we can define the equivalent martingale measure P^ν on $(\Omega, \mathcal{F}_T)$ by the formula:

$$\left.\frac{dP^\nu}{dP}\right|_{\mathcal{F}_t} := \mathcal{E}_t\left(-\int (\lambda d_\epsilon - \nu)' dw\right) =: Z_t^\nu,$$

and that the process $Z^\nu \widetilde{X}^{x,\pi}$ is a martingale for all $\pi \in \mathcal{A}$ and $\nu \in \mathcal{D}$, so, in particular, $E\left[\widetilde{Z}_T^\nu X_T^{x,\pi}\right] = x$ holds since $E\left[\sup_{t\in[0,T]} |Z_t^\nu|^2\right] < \infty$ and

$$E\left[\sup_{t\in[0,T]} \left|\widetilde{X}_t^{0,\pi}\right|^2\right] \le C_1 E\left[\int_0^T |\pi_u|^2 du\right] < \infty$$

from Doob's inequality and the boundedness assumptions of σ, λ and ν.

Next, for $f, x \in \mathbf{R}$, and $y > 0$, denote

$$u_\gamma(x; y, f) := U_\gamma(-f + x) - yx \quad \text{and} \quad I_\gamma(y) := \left(U'_\gamma\right)^{-1}(y) = -\frac{1}{\gamma}\log(y)$$

to see the relation

$$\sup_{x\in\mathbf{R}} u_\gamma(x; y, f) = u_\gamma\left(f + I_\gamma(y); y, f\right) = -y\left(f - \frac{1 + \log y}{\gamma}\right).$$

Moreover, for $\pi \in \mathcal{A}$ and $x \in \mathbf{R}, y > 0$, observe the inequalities

$$\begin{aligned}
(3.1) \qquad E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right] - yx &\le \inf_{\nu\in\mathcal{D}} E\left[U_\gamma\left(-F + X_T^{x,\pi}\right) - y\widetilde{Z}_T^\nu X_T^{x,\pi}\right] \\
&\le \inf_{\nu\in\mathcal{D}} \sup_{\pi\in\mathcal{A}} E\left[u_\gamma\left(X_T^{x,\pi}; y\widetilde{Z}_T^\nu, F\right)\right] \\
&\le \inf_{\nu\in\mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}_T^\nu\right); y\widetilde{Z}_T^\nu, F\right)\right]
\end{aligned}$$

to obtain the minimization problem

$$\text{(D)} \qquad \widetilde{V}^\epsilon(y) := \inf_{\nu\in\mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}_T^\nu\right); y\widetilde{Z}_T^\nu, F\right)\right]$$

called the dual problem of the primal problem (P), and to deduce the inequality

$$\text{(3.2)} \qquad V^\epsilon(x) \le \inf_{y>0}\left(\widetilde{V}^\epsilon(y) + yx\right).$$

Indeed, the equality can be established in (3.2) and the following expression is obtained.

Theorem 3.1 *(Theorem 2.1 of Rouge and El Karoui, [9]) It holds that*

$$\text{(3.3)} \qquad V^\epsilon(x) = U_\gamma\left(e^{rT}x - \frac{1}{\gamma}\sup_{\nu\in\mathcal{D}}\{E^\nu[\gamma F] - H(P^\nu|P)\}\right),$$

where $E^\nu[\cdot]$ denotes the expectation with respect to the probability measure P^ν and

$$H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise} \end{cases}$$

is the relative entropy of Q with respect to P.

Remark 3.1. The duality relations similar to (3.3) have been obtained for more general semimartingale S and for other choices of the set of admissible strategies $\mathcal{A}$ by Delbaen et. al. in [2] and by Kabanov and Stricker (2002), [4].

For the computations of the value $V^\epsilon(x)$ and the optimizer, one can solve the BSDE for the value process of the dual problem. Recalling that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is weakly w-Brownian (i.e., Lemma 2.1 holds), we can apply the results in Rouge and El Karoui [9] to obtain the following.

Theorem 3.2 *(Theorem 4.1 and 4.2 of Rouge and El Karoui, [9]) Denote $Z_{t,T}^\nu := Z_T^\nu/Z_t^\nu$, $\widetilde{Z}_{t,T}^\nu := \widetilde{Z}_T^\nu/\widetilde{Z}_t^\nu$, and $\tau := T - t$ for $0 \le t \le T$. Let*

$$\begin{aligned}
&\operatorname*{essinf}_{\nu\in\mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}_{t,T}^\nu\right); y\widetilde{Z}_{t,T}^\nu, F\right) \mid \mathcal{F}_t\right] \\
= \; &\frac{ye^{-r\tau}}{\gamma}\left\{-\operatorname*{esssup}_{\nu\in\mathcal{D}} E^\nu\left[\gamma F - \log Z_{t,T}^\nu|\mathcal{F}_t\right] + (1 + \log y - r\tau)\right\} \\
=: \; &\frac{ye^{-r\tau}}{\gamma}\{-Y_t^\epsilon + (1 + \log y - r\tau)\}.
\end{aligned}$$

There exists $\Xi^\epsilon \in \mathbf{H}_T^{2,2} := \left\{f : \text{2-dim. predictable}; \ E\left[\int_0^T |f_t|^2 dt\right] < \infty\right\}$ *such that* $(Y^\epsilon, \Xi^\epsilon)$ *satisfies*

$$
\text{(3.4)} \qquad \begin{aligned} dY_t^\epsilon &= f(t, \Xi_t^\epsilon, \epsilon)\, dt + (\Xi_t^\epsilon)'\, dw_t, \quad Y_T^\epsilon = \gamma F, \\ \text{where} \quad f(t,\xi,\epsilon) &:= \frac{1}{2}\left\{\lambda_t^2 - \left(\xi, d_\epsilon^\perp\right)^2\right\} + \lambda_t (\xi, d_\epsilon), \end{aligned}
$$

and $(\cdot,\cdot)$ *denotes the standard inner-product in* $\mathbf{R}^2$. *In particular,* $\pi^* \in \mathcal{A}$ *satisfying*

$$
\text{(3.5)} \qquad \pi_t^* := \frac{e^{-rT}}{\gamma}\left\{\frac{\mu_2(t) - r}{\sigma_2^2} + \frac{\sqrt{1-\epsilon^2}}{\sigma_2}\Xi_1^\epsilon(t)\right\} \quad \text{for all } t \in [0,T]
$$

is an optimizer of the primal problem (P), and $\nu^* := (d_\epsilon^\perp, \Xi)\, d_\epsilon^\perp$ *attains the infimum of the dual problem (D). Further,*

$$
\text{(3.6)} \qquad V^\epsilon(x) = U_\gamma\left(e^{rT}x - \frac{Y_0^\epsilon}{\gamma}\right)
$$

holds.

Remark 3.2. The existence and the uniqueness of the solution (Y, Ξ) of the quadratic BSDE (3.4) in the space $\mathbf{H}_T^\infty \times \mathbf{H}_T^{2,2}$, where $\mathbf{H}_T^\infty := \{f \in L^\infty([0,T]\times\Omega);\ \text{predictable}\}$ is ensured by Theorem 2.3 and 2.6 of Kobylanski (2000), [7], (cf., Appendix B of [9], also).

On the other hand, in [1], Davis solves the dynamic programming equation for the value function of the dual problem:

$$
\text{(3.7)} \qquad \begin{aligned} v^\epsilon(t,y) &:= \operatorname*{esssup}_{\nu\in\mathcal{D}} E^\nu\left[\gamma F - \log Z_{t,T}^\nu \;\middle|\; S_t^1 = y\right] \\ &= \operatorname*{esssup}_{\nu\in\mathcal{D}} E^\nu\left[\gamma h\left(S_T^1\right) - \frac{1}{2}\int_t^T \left\{|\lambda_u|^2 + |\nu_u|^2\right\} du \;\middle|\; S_t^1 = y\right], \end{aligned}
$$

recalling the relation

$$
\log Z_{t,T}^\nu = -\int_t^T (\lambda_u d_\epsilon - \nu_u)'\, dw_u^\nu + \frac{1}{2}\int_t^T |\lambda_u d_\epsilon - \nu_u|^2\, du,
$$

where

$$
w^\nu := (w_1^\nu, w_2^\nu)', \quad w_t^\nu := w_t + \int_0^t (\lambda_u d_\epsilon - \nu_u)\, du
$$

is a 2-dimensional P^ν-Brownian motion, and obtains Theorem 1.1, as we explained.

4 Results

We focus on the following two situations:

(i)	$\epsilon \ll 1$	: closely correlated case, with the conditions (2.1,3),
(ii)	$\delta := \sqrt{1-\epsilon^2} \ll 1$	: almost independent case, with the conditions (2.2-3).

Regarding the solution $(Y^\epsilon, \Xi^\epsilon)$ of the BSDE (3.4) as $(Y^{\epsilon,\epsilon}, \Xi^{\epsilon,\epsilon})$, where we define

$$
\text{(4.1)} \qquad \begin{aligned} dY_t^{\epsilon',\epsilon} &= g\left(t, \Xi_t^{\epsilon',\epsilon}, \epsilon'\right) dt + \left(\Xi_t^{\epsilon',\epsilon}\right)' dw_t^0, \quad Y_T^{\epsilon',\epsilon} = \gamma F, \\ \text{where} \quad g(t,\xi,\epsilon') &:= \frac{1}{2}\left\{\lambda_t^2 - \left(\xi, d_{\epsilon'}^\perp\right)^2\right\}, \end{aligned}
$$

(recall that ϵ is contained in $w^0 := w + (\int \lambda du)d_\epsilon$), we compute the asymptotic expansion of $(Y^{\epsilon',\epsilon}, \Xi^{\epsilon',\epsilon})$ with respect to ϵ' at 0, and that of $\left(Y^{\sqrt{1-(\delta')^2},\sqrt{1-\delta^2}}, \Xi^{\sqrt{1-(\delta')^2},\sqrt{1-\delta^2}}\right)$ with respect to δ' at 0, which yield the expansions including (1.2-3).

4.1 Closely correlated case

First, consider the case (i) with the assumptions (2.1) and (2.3). Let $\left(\partial^0_{\epsilon'} Y^{0,\epsilon}, \partial^0_{\epsilon'}\Xi^{0,\epsilon}\right) := \left(Y^{0,\epsilon}, \Xi^{0,\epsilon}\right)$ and introduce the BSDEs:

$$(4.2)\qquad d\left(\partial^i_{\epsilon'} Y^{0,\epsilon}_t\right) = g_i\left(t, \left(\partial^j_{\epsilon'}\Xi^{0,\epsilon}_t\right)_{j=0,\dots,i}, 0\right)dt + \left(\partial^i_{\epsilon'}\Xi^{0,\epsilon}_t\right)' dw^{0,\epsilon}_t, \quad \partial^i_{\epsilon'} Y^{0,\epsilon}_T = 0,$$

using the functions g_i defined inductively

$$\begin{aligned} g_0\left(t, \xi^0, \epsilon'\right) &:= g\left(t, \xi^0, \epsilon'\right) \\ \text{and} \quad g_i\left(t, (\xi^j)_{j=0,\dots,i}, \epsilon'\right) &:= \sum_{j=0}^{i-1}\left(\partial_{\xi^j} g_{i-1}\left(t, (\xi^k)_{k=0,\dots,i-1}, \epsilon'\right), \xi^{j+1}\right) \\ &\quad + \partial_{\epsilon'} g_{i-1}\left(t, (\xi^k)_{k=0,\dots,i-1}, \epsilon'\right). \end{aligned}$$

Formally, it is expected that $\left(\partial^i_{\epsilon'} Y^{0,\epsilon}, \partial^i_{\epsilon'}\Xi^{0,\epsilon}\right)$ is the i-th derivative of the solution of (4.1) with respect to the parameter ϵ' at 0 and that a "Taylor expansion":

$$(4.3)\qquad \widehat{Y}^{\epsilon,n} := \sum_{i=0}^{n} \partial^i_{\epsilon'} Y^{0,\epsilon}\frac{\epsilon^i}{i!}, \quad \widehat{\Xi}^{\epsilon,n} := \sum_{i=0}^{n} \partial^i_{\epsilon'}\Xi^{0,\epsilon}\frac{\epsilon^i}{i!},$$

which satisfies

$$(4.4)\qquad \begin{aligned} d\widehat{Y}^{\epsilon,n}_t &= \left\{g\left(t, \widehat{\Xi}^{\epsilon,n}_t, \epsilon\right) + R^{\epsilon,n}_t\right\}dt + \widehat{\Xi}^{\epsilon,n}_t dw^0_t, \quad \widehat{Y}^{\epsilon,n}_T = \gamma F \\ \text{with} \quad R^{\epsilon,n}_t &:= \sum_{i=0}^{n} g_i\left(t, \left(\partial^j_{\epsilon'}\Xi^{0,\epsilon}_t\right)_{j=0,\dots,i}, 0\right)\frac{\epsilon^i}{i!} - g\left(t, \widehat{\Xi}^{\epsilon,1}_t, \epsilon\right) \end{aligned}$$

gives an "approximation" of the solution of (4.1), if $R^{\epsilon,n}_t(\omega) = o(\epsilon^n)$ is "small" enough. We have not been able to check the differentiability of the solution of the quadratic BSDE (4.1) with respect to ϵ', (note that the standard results on the property, stated in El Karoui et. al. [3], for example, cannot be directly applied), however, an approximation result on the quantities (4.3) can be shown under our assumptions (2.1-3), as we will see.

Define the functional $H \in L^\infty\left(\Omega_1, \mathcal{F}^1\right)$ by

$$H(\omega_1) := \gamma F(\omega_1) - \frac{1}{2}\int_0^T \lambda_u(\omega_1)^2 du$$

to observe the following.

Lemma 4.1 **1.** *The solution of (4.1) at $\epsilon' = 0$ in the space $\mathbf{H}^\infty_T \times \mathbf{H}^{2,2}_T$ is given by*

$$Y^{0,\epsilon}_t = E^0\left[\gamma F - \frac{1}{2}\int_t^T \lambda^2_u du \,\Big|\, \mathcal{F}_t\right], \quad \Xi^{0,\epsilon}_1(t) = E^0\left[D_{1,t}H|\mathcal{F}_t\right],$$

and $\Xi^{0,\epsilon}_2(t) = 0$ *for* $t \in [0,T]$.

2. $\left(\partial^i_{\epsilon'} Y^{0,\epsilon}, \partial^i_{\epsilon'} \Xi^{0,\epsilon}\right) \equiv 0$ *for* $i = 1, 3$.

3. *The solution of (4.2) with* $i = 2$ *in the space* $\mathbf{H}^\infty_T \times \mathbf{H}^{2,2}_T$ *is given by*

$$\begin{aligned}\partial^2_{\epsilon'} Y^{0,\epsilon}_t &= \mathrm{Var}^0\left[\gamma F - \frac{1}{2}\int_t^T \lambda_u^2 du \,\Big|\, \mathcal{F}_t\right],\\ \partial^2_{\epsilon'} \Xi^{0,\epsilon}_1(t) &= 2\left\{E^0\left[HD_{1,t}H|\mathcal{F}_t\right] - E^0\left[H|\mathcal{F}_t\right]E^0\left[D_{1,t}H|\mathcal{F}_t\right]\right\}\end{aligned}$$

and $\partial^2_{\epsilon'} \Xi^{0,\epsilon}_2(t) = 0$ *for* $t \in [0, T]$.

We now extend the expansions (1.2-3) and Theorem 4 in [10], as follows.

Theorem 4.1 *Assume (2.1) and (2.3). Define* $\widetilde{\pi}^{\epsilon,2} := \left(\widetilde{\pi}^{\epsilon,2}_t\right)_{t\in[0,T]} \in \mathcal{A}$ *by the formula*

$$\widetilde{\pi}^{\epsilon,2}_t = \frac{e^{-rT}}{\gamma}\left[\frac{\mu_2(t) - r}{\sigma_2^2} + \frac{\sqrt{1-\epsilon^2}}{\sigma_2}\left\{\Xi^{0,\epsilon}_1(t) + \frac{\epsilon^2}{2}\partial^2_{\epsilon'}\Xi^{0,\epsilon}_1(t)\right\}\right]. \tag{4.5}$$

Then, the relations

$$\begin{aligned}\left\|Y^\epsilon - Y^{0,\epsilon} - \frac{\epsilon^2}{2}\partial^2_{\epsilon'} Y^{0,\epsilon}\right\|_{L^\infty([0,T]\times\Omega)} &= O(\epsilon^4)\\ \text{and}\quad \log V^\epsilon(x) - \log E\left[U_\gamma\left(-F + X^{x,\widetilde{\pi}^{\epsilon,2}}_T\right)\right] &= O(\epsilon^4)\end{aligned}$$

follow as $\epsilon \downarrow 0$.

Corollary 4.1 *Assume (2.2-3). For the utility indifference price,*

$$p^\epsilon(x, F) = e^{-rT}\left\{E^0[F] + \epsilon^2\frac{\gamma}{2}\mathrm{Var}^0[F]\right\} + O(\epsilon^4) \quad \text{as } \epsilon \downarrow 0$$

holds for any $x \in \mathbf{R}$.

It is observed that the price is always higher than that in perfectly correlated ($\epsilon = 0$) case (by neglecting $O(\epsilon^4)$-term), which is intuitively clear.

4.2 Almost independent case

Next, consider the case **(ii)** with the assumptions (2.2) and (2.3). Let $\delta := \sqrt{1-\epsilon^2} \approx 0$, $\delta' := \sqrt{1-(\epsilon')^2} \approx 0$ and denote

$$\overline{Y}^{\delta',\delta} := Y^{\sqrt{1-(\delta')^2},\sqrt{1-\delta^2}}, \quad \overline{\Xi}^{\delta',\delta} := \Xi^{\sqrt{1-(\delta')^2},\sqrt{1-\delta^2}}, \quad \text{and} \quad \overline{d}^\perp_\delta := d^\perp_{\sqrt{1-\delta^2}}.$$

We compute the asymptotic expansion of the BSDE:

$$\begin{aligned}d\overline{Y}^{\delta',\delta}_t &= h\left(t, \overline{\Xi}^{\delta',\delta}_t, \delta'\right)dt + \left(\overline{\Xi}^{\delta',\delta}_t\right)' dw^0_t, \quad \overline{Y}^{\delta',\delta}_T = \gamma F,\\ \text{where}\quad h(t, \xi, \delta') &:= g\left(t, \xi, \sqrt{1-(\delta')^2}\right) = \frac{1}{2}\left\{\lambda_t^2 - \left(\xi, \overline{d}^\perp_{\delta'}\right)^2\right\},\end{aligned} \tag{4.6}$$

with respect to δ' at 0. Let $\left(\partial^0_{\delta'}\overline{Y}^{0,\delta}, \partial^0_{\delta'}\overline{\Xi}^{0,\delta}\right) := \left(\overline{Y}^{0,\delta}, \overline{\Xi}^{0,\delta}\right)$ and introduce the BSDEs:

$$d\partial^i_{\delta'}\overline{Y}^{0,\delta}_t = h_i\left(t, \left(\partial^j_{\delta'}\overline{\Xi}^{0,\delta}_t\right)_{j=0,\dots,i}, 0\right)dt + \left(\partial^i_{\delta'}\overline{\Xi}^{0,\delta}_t\right)' dw^0_t, \quad \partial^i_{\delta'}\overline{Y}^{0,\delta}_T = 0, \tag{4.7}$$

using the functions h_i defined inductively

$$\begin{aligned} h_0\left(t,\xi^0,\delta'\right) &:= h\left(t,\xi^0,\delta'\right) \\ \text{and} \quad h_i\left(t,(\xi^j)_{j=0,\dots,i},\delta'\right) &:= \sum_{j=0}^{i-1}\left(\partial_{\xi^j}h_{i-1}\left(t,(\xi^k)_{k=0,\dots,i-1},\delta'\right),\xi^{j+1}\right) \\ &\quad +\partial_{\delta'}h_{i-1}\left(t,(\xi^k)_{k=0,\dots,i-1},\delta'\right). \end{aligned}$$

We observe the following.

Lemma 4.2 **1.** *The solution of (4.6) at $\delta' = 0$ in the space $\mathbf{H}^\infty_T \times \mathbf{H}^{2,2}_T$ is given by*

$$\overline{Y}^{0,\delta}_t = \log E^0\left[e^{\gamma F}|\mathcal{F}_t\right] - \frac{1}{2}\int_t^T \lambda_u^2 du, \quad \overline{\Xi}^{0,\delta}_1(t) = \gamma\frac{E^0\left[e^{\gamma F}D_{1,t}F|\mathcal{F}_t\right]}{E^0\left[e^{\gamma F}|\mathcal{F}_t\right]}$$

and $\overline{\Xi}^{0,\delta}_2(t) = 0$ for $t \in [0,T]$.

2. $\left(\partial^i_{\delta'}\overline{Y}^{0,\delta}, \partial^i_{\delta'}\overline{\Xi}^{0,\delta}\right) \equiv 0$ *for $i = 1,3$.*

3. *The solution of (4.7) with $i = 2$ in the space $\mathbf{H}^\infty_T \times \mathbf{H}^{2,2}_T$ is given by*

$$\begin{aligned} \partial^2_{\delta'}\overline{Y}^{0,\delta}_t &= -2\left\{\gamma\frac{E^0[e^{\gamma F}F|\mathcal{F}_t]}{E^0\left[e^{\gamma F}|\mathcal{F}_t\right]} - \log E^0\left[e^{\gamma F}|\mathcal{F}_t\right]\right\}, \\ \partial^2_{\delta'}\overline{\Xi}^{0,\delta}_1(t) &= -2\gamma^2\left\{\frac{E^0[e^{\gamma F}FD_{1,t}F|\mathcal{F}_t]}{E^0[e^{\gamma F}|\mathcal{F}_t]} - \frac{E^0[e^{\gamma F}F|\mathcal{F}_t]E^0[e^{\gamma F}D_{1,t}F|\mathcal{F}_t]}{E^0[e^{\gamma F}|\mathcal{F}_t]^2}\right\}, \end{aligned}$$

and $\partial^2_{\delta'}\overline{\Xi}^{0,\delta}_2(t) = 0$ for $t \in [0,T]$.

Using the above lemma, we obtain the following.

Theorem 4.2 *Assume (2.2) and (2.3). Define $\check{\pi}^{\delta,2} := \left(\check{\pi}^{\delta,2}_t\right)_{t\in[0,T]} \in \mathcal{A}$ by the formula*

$$\check{\pi}^{\delta,2}_t = \frac{e^{-rT}}{\gamma}\left[\frac{\mu_2(t)-r}{\sigma_2^2} + \frac{\delta}{\sigma_2}\left\{\overline{\Xi}^{0,\delta}_1(t) + \frac{\delta^2}{2}\partial^2_{\delta'}\overline{\Xi}^{0,\delta}_1(t)\right\}\right]. \tag{4.8}$$

Then, the relations

$$\begin{aligned} \left\|Y^{\sqrt{1-\delta^2}} - \overline{Y}^{0,\delta} - \frac{\delta^2}{2}\partial^2_{\delta'}\overline{Y}^{0,\delta}\right\|_{L^\infty([0,T]\times\Omega)} &= O(\delta^4) \\ \textit{and} \quad \log V^{\sqrt{1-\delta^2}}(x) - \log E\left[U_\gamma\left(-F + X^{x,\check{\pi}^{\delta,2}}_T\right)\right] &= O(\delta^4) \end{aligned}$$

follow as $\delta \downarrow 0$.

Corollary 4.2 *Assume (2.2-3). For the utility indifference price,*

$$p^{\sqrt{1-\delta^2}}(x,F) = \frac{e^{-rT}}{\gamma}\left\{(1+\delta^2)\log E^0[e^{\gamma F}] - \delta^2\gamma\frac{E^0[e^{\gamma F}F]}{E^0[e^{\gamma F}]}\right\} + O(\delta^4) \quad \textit{as } \delta \downarrow 0$$

holds for any $x \in \mathbf{R}$.

From (A.3), $\partial^2_{\delta'}\overline{Y}^{0,\delta} \leq 0$ follows, which implies $p^{\sqrt{1-\delta^2}}(x,F) \leq \frac{e^{-rT}}{\gamma}\log E^0[e^{\gamma F}] + O(\delta^4)$, i.e., the utility indifference price is always lower than that in perfectly independent ($\delta = 0$) case (by neglecting $O(\delta^4)$-term).

4.3 Examples of F

Let (μ_1, μ_2) be deterministic (and bounded). The following are examples of F satisfying (2.3):

(a) European put: $F(\omega_1) := (K - S_T^1(\omega_1))^+$ $(K > 0)$ with $D_{1,t}F(\omega_1) = -\sigma_1 S_T^1(\omega_1) 1_{\{S_T^1(\omega_1) \le K\}}$.

(b) European calls spread: $F(\omega_1) := (S_T^1(\omega_1) - K_1)^+ - (S_T^1(\omega_1) - K_2)^+$, $(K_1 < K_2)$ with $D_{1,t}F(\omega_1) = \sigma_1 S_T^1(\omega_1) 1_{\{K_1 \le S_T^1(\omega_1) \le K_2\}}$.

In these cases, prices and hedging strategies in Theorem 4.1-2 and Corollary 4.1-2 can be computed by using the conditional lognormal distribution function of S_T^1.

Moreover, we can treat path-dependent type options, in principle. For example,

(c) a lookback option: $F(\omega_1) := \left(K - M_T^1(\omega_1)\right)^+$ with $M_t^1 := \min_{s\in[0,t]} S_s^1$

satisfies condition (2.3). In fact, we can observe that $D_{1,t}F(\omega_1) = -\sigma_1 M_T^1(\omega_1) 1_{\{M_T^1(\omega_1)\le K\}} 1_{\{t<\iota(\omega_1+\eta^\epsilon)\}}$. Here, $\iota(\cdot)$ is the time attains the minimum of the P^0-Brownian motion $w_1^0(\cdot)$ on the time interval $[0,T]$, i.e., $\min_{t\in[0,T]} w_1^0(t,\omega_1) = w_1^0(\iota(\omega_1),\omega_1)$, which is uniquely determined for a.e. ω_1 (cf., Remark 2.8.16 of Karatzas and Shreve; 1991, [5]), and $\eta^\epsilon(t) := \int_0^t \left\{\mu_1(u) - \sqrt{1-\epsilon^2}\sigma_1\sigma_2^{-1}(\mu_2(u) - r) - \frac{\sigma_1^2}{2}\right\} du$. The expression follows by letting $G(\omega_1) := S_0^1 \exp\left(\sigma_1 \min_{t\in[0,T]} w_1^0(\omega_1)\right)$, by recalling the relation $M_T^1(\omega_1) = G(\omega_1 + \eta^\epsilon)$, and by observing

$$\lim_{\epsilon\to 0} \frac{G(\omega_1 + \epsilon\phi) - G(\omega_1)}{\epsilon} = \int_0^T \sigma_1 G(\omega_1) 1_{\{t<\iota(\omega_1)\}} \frac{d\phi_t}{dt} dt$$

for all $\phi \in C^1([0,T])$, (cf., Example E.4 in Appendix E of Karatzas and Shreve [6], or Example 41.13 in Chapter IV of Rogers and Williams; 2000, [8]). Further, denoting

$$m_{s,t}^1(\omega_1) := \min_{u\in[s,t]} w_1^0(u, \omega_1 + \eta^\epsilon) = \frac{1}{\sigma_1} \min_{u\in[s,t]} \log\left(\frac{S_u^1(\omega_1)}{S_0^1}\right) \quad \text{and} \quad m_t^1 := m_{0,t}^1,$$

and letting (μ_1, μ_2) constant, we see, for a bounded $I : \mathbf{R} \mapsto \mathbf{R}$ and $J(\cdot) := I(\cdot)\exp(\sigma_1(\cdot)) 1_{\{(\cdot)\le(\sigma^1)^{-1}\log(K/S_0^1)\}}$

$$\begin{aligned}
& E^0\left[I(m_T^1) D_{1,t}F | \mathcal{F}_t\right] \\
= \;& -\sigma_1 S_0^1 E^0\left[I(m_T^1)\exp\left(\sigma_1 m_T^1\right) 1_{\{S_0^1 \exp(\sigma_1 m_T^1)\le K\}} 1_{\{m_t^1 > m_{t,T}^1\}} \;\middle|\; \mathcal{F}_t\right] \\
= \;& -\sigma_1 S_0^1 E^0\left[J\left(b \wedge (a + m_{T-t}^1)\right) 1_{\{m_{T-t}^1 < b-a\}}\right] \Big|_{a=w_1^0(t)+\eta^\epsilon(t), b=m_t^1}
\end{aligned}$$

from the Markov property of the process $(w_1^0(t) + \eta^\epsilon(t), m_t^1)_{t\in[0,T]}$. Therefore, we can compute prices and hedging strategies in Theorem 4.1-2 and Corollary 4.1-2 using the distribution of m_{T-t}^1, whose explicit form is known (cf., Example E.5 of Appendix E in [6], for example).

5 Conclusion

The exponential hedging problem is addressed in the incomplete market consisting of the derivative security written on the *untradable* asset and the *tradable* asset as the instrument for hedging. The correlation ρ of the two asset price processes, or $\sqrt{1-\rho^2}$ is regarded as a small parameter, and the asymptotic expansions of the backward stochastic differential equations for the dual optimization problems with respect to the parameters are studied. Explicit expressions for the expansions are obtained with the help of the Clark-Haussman-Ocone formula, which yield approximations for the utility indifference prices and the optimal hedging strategies.

A Proofs

In this appendix, we give the proofs of Lemma 4.1, Theorem 4.1, and Lemma 4.2. Those of the rest are omitted since Corollary 4.1-2 are deduced from Theorem 1.1 directly, and the proof of Theorem 4.2 is similar that of Theorem 4.1. Actually, Lemma 4.1 and Theorem 4.1 have been obtained in essential forms in [10] (cf., proofs of Lemma 1 and Theorem 4 in [10]), though we show them for our completeness.

A.1 Proof of Lemma 4.1.

1. Suppose $\Xi_2^{0,\epsilon} \equiv 0$, then

$$dY_t^{0,\epsilon} = \frac{1}{2}\lambda_t^2 dt + \Xi_1^{0,\epsilon}(t)dw_1^0(t), \quad Y_T^{0,\epsilon} = \gamma F$$

is observed. The expression for $Y^{0,\epsilon}$ and the relation

$$E^0[H|\mathcal{F}_t] = Y_0^{0,\epsilon} + \int_0^t \Xi_1^{0,\epsilon}(u)dw_1^0(u) \quad \text{for } t \in [0,T]$$

follows from a standard result of linear BSDE (cf., El Karoui et. al; 1997, [3]) and the result on the uniqueness of the quadratic BSDE studied in Kobylanski (2000), [7]. The expression for $\Xi_1^{0,\epsilon}$ is obtained from the Clark-Haussman-Ocone formula.

2-3. Observe that

$$\begin{aligned} d_\epsilon^\perp &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \epsilon\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{\epsilon^2}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\epsilon^3}{3!}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + O(\epsilon^4) \\ &=: d_0^\perp + \sum_{i=1}^3 \frac{\epsilon^i}{i!}\partial_{\epsilon'}^i d_0^\perp + O(\epsilon^4), \end{aligned}$$

where $O(\epsilon^4) \in \mathbf{R}^2$ is a vector with the norm $|O(\epsilon^4)| \sim \epsilon^4$.

(i) Noting that

$$g_1\left(t, (\xi^j)_{j=0,1}, 0\right) = -\left(\xi^0, d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\}$$

and that $\Xi_2^0 \equiv 0$, we can deduce

$$d\left(\partial_{\epsilon'} Y_t^{0,\epsilon}\right) = \partial_{\epsilon'}\Xi_t^{0,\epsilon} dw_t^0, \quad \partial_{\epsilon'} Y_T^{0,\epsilon} \equiv 0$$

and $\left(\partial_{\epsilon'} Y^{0,\epsilon}, \partial_{\epsilon'}\Xi^{0,\epsilon}\right) \equiv 0$.

(ii) Observing that

$$\begin{aligned} & g_2\left(t, (\xi^j)_{j=0,1,2}, 0\right) \\ = & -\left(\xi^1, d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\} - \left(\xi^0, d_0^\perp\right)\left\{\left(\xi^2, d_0^\perp\right) + \left(\xi^1, \partial_{\epsilon'} d_0^\perp\right)\right\} \\ & -\left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\} - \left(\xi^0, d_0^\perp\right)\left\{\left(\xi^1, \partial_{\epsilon'} d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'}^2 d_0^\perp\right)\right\}, \end{aligned}$$

we rewrite the BSDE for $\left(\partial^2_{\epsilon'} Y^{0,\epsilon}, \partial^2_{\epsilon'} \Xi^{0,\epsilon}\right)$ as

$$d\left(\partial^2_{\epsilon'} Y^{0,\epsilon}_t\right) = -\left(\Xi^{0,\epsilon}_1(t)\right)^2 dt + \left(\partial^2_{\epsilon'} \Xi^{0,\epsilon}_t\right)' dw^0_t, \quad \partial^2_{\epsilon'} Y^{0,\epsilon}_T \equiv 0$$

since $\Xi^{0,\epsilon}_2 \equiv 0$ and $\partial_{\epsilon'} \Xi^{0,\epsilon} \equiv 0$. This standard linear BSDE on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$, (or $\left(\Omega, \mathcal{F}, P^0, (\mathcal{F}_t)_{t\in[0,T]}\right)$) has the unique solution satisfying

$$\begin{aligned} \partial^2_{\epsilon'} Y^{0,\epsilon}_t &= E^0\left[\int_t^T \left(\Xi^{0,\epsilon}_1(u)\right)^2 du \,\middle|\, \mathcal{F}_t\right], \\ \partial^2_{\epsilon'} Y^{0,\epsilon}_0 + \int_0^t \partial^2_{\epsilon'} \Xi^{0,\epsilon}_1(u) dw^0_1(u) &= E^0\left[\int_0^T \left(\Xi^{0,\epsilon}_1(u)\right)^2 du \,\middle|\, \mathcal{F}_t\right], \end{aligned}$$

and $\partial^2_{\epsilon'} \Xi^{0,\epsilon}_2 \equiv 0$. The expression for $\partial^2_{\epsilon'} \Xi^{0,\epsilon}_1$ is deduced from the relation

$$\begin{aligned} &\int_0^T \left(\Xi^{0,\epsilon}_1(t)\right)^2 dt \\ &= \left(\int_0^T \Xi^{0,\epsilon}_1(t) dw^0_1(t)\right)^2 - 2\int_0^T \left(\int_0^t \Xi^{0,\epsilon}_1(u) dw^0_1(u)\right) \Xi^{0,\epsilon}_1(t) dw^0_1(t) \\ &= \left(H - E^0[H]\right)^2 - 2\int_0^T \left(E^0[H|\mathcal{F}_t] - E^0[H]\right) \Xi^{0,\epsilon}_1(t) dw^0_1(t) \\ &= H^2 - \left(E^0[H]\right)^2 - 2\int_0^T E^0[H|\mathcal{F}_t] E^0[D_{1,t}H|\mathcal{F}_t] dw^0_1(t), \end{aligned}$$

the Clark-Haussman-Ocone formula, and the chain rule for differentiation.
(iii) For $\left(\xi^j\right)_{j=0,1,2,3}$ such that $\xi^0_2 = \xi^2_2 = 0$ and $\xi^1 = 0$, we can check that

$$g_3\left(t, (\xi^j)_{j=0,1,2,3}, 0\right) = 0,$$

so the equation

$$d\left(\partial^3_{\epsilon'} Y^{0,\epsilon}_t\right) = \partial^3_{\epsilon'} \Xi^{0,\epsilon}_t dw^0_t, \quad \partial^3_{\epsilon'} Y^{0,\epsilon}_T \equiv 0$$

and $\left(\partial^3_{\epsilon'} Y^{0,\epsilon}, \partial^3_{\epsilon'} \Xi^{0,\epsilon}\right) \equiv 0$ are deduced. ∎

A.2 Proof of Theorem 4.1.

First, observe, in the BSDE (4.4) with $n = 2$, that $\|R^{\epsilon,2}\|_{L^\infty([0,T],\Omega)} = O(\epsilon^4)$ holds because of the boundedness of λ^ϵ, $\partial^i_{\epsilon'} d^\perp_0$, and $\partial^i_{\epsilon'} \Xi^{0,\epsilon}$ $(i = 0, \dots, 3)$, which is a consequence of Lemma 4.1.

Next, introduce the linear BSDE for $\left(\Delta Y^{\epsilon,2}, \Delta \Xi^{\epsilon,2}\right) := (Y^\epsilon - \widehat{Y}^{\epsilon,2}, \Xi^\epsilon - \widehat{\Xi}^{\epsilon,2})$, described as

$$\begin{cases} d\Delta Y^{\epsilon,2}_t = \left\{-\frac{1}{2}\left(\Xi^\epsilon_t + \widehat{\Xi}^{\epsilon,2}_t, d^\perp_\epsilon\right)\left(\Delta \Xi^{\epsilon,2}_t, d^\perp_\epsilon\right) - R^{\epsilon,2}_t\right\} dt + \Delta \Xi^{\epsilon,2}_t dw^0_t, \\ \Delta Y^{\epsilon,2}_T \equiv 0 \end{cases}$$

to observe the relation:

$$-\Gamma_s \Delta Y^{\epsilon,2}_s = -\Gamma_t \Delta Y^{\epsilon,2}_t - \int_s^t \Gamma_u R^{\epsilon,2}_u du + M_t - M_s \tag{A.1}$$

for $0 \le s \le t \le T$, where $\Gamma := (\Gamma_t)_{t\in[0,T]}$ is the solution of the SDE:

$$d\Gamma_t = \Gamma_t \left\{\frac{1}{2}\left(\Xi^\epsilon_t + \widehat{\Xi}^{\epsilon,2}_t, d^\perp_\epsilon\right)\left(d^\perp_\epsilon\right)' dw^0_t\right\}, \quad \Gamma_0 = 1$$

and $M := (M_t)_{t\in[0,T]}$ is the P^0-local-martingale defined by

$$M_t := \int_0^t \Gamma_u \left\{ \Delta\Xi_u^{\epsilon,2} + \frac{1}{2}\Delta Y_u^{\epsilon,2}\left(\Xi_u^{\epsilon} + \widehat{\Xi}_u^{\epsilon,2}, d_\epsilon^{\perp}\right) d_\epsilon^{\perp} \right\}' dw_u^0.$$

For a sequence of increasing stopping times $(\tau_m)_{m\in\mathbf{N}}$, which localizes the local martingale M, we deduce the relation

$$\Gamma_{t\wedge\tau_m} \left|\Delta Y_{t\wedge\tau_m}^{\epsilon,2}\right| \le E^0 \left[\Gamma_{T\wedge\tau_m} \left|\Delta Y_{T\wedge\tau_m}^{\epsilon,2}\right| + \epsilon^4 C_1 \int_{t\wedge\tau_m}^{T\wedge\tau_m} \Gamma_u du \,\Big|\, \mathcal{F}_{t\wedge\tau_m} \right].$$

with some constant $C_1 > 0$ from (A.1). The first term of the right-hand-side is

$$\le E^0\left[\Gamma_{T\wedge\tau_m} \,\big|\, \mathcal{F}_{t\wedge\tau_m}\right] \left\|\Delta Y_{T\wedge\tau_m}^{\epsilon,2}\right\|_{L^\infty(\Omega)} \le \Gamma_{t\wedge\tau_m} \left\|\Delta Y_{T\wedge\tau_m}^{\epsilon,2}\right\|_{L^\infty(\Omega)} \to 0$$

as $m \to \infty$ by using the optional stopping theorem, and the second term of the right-hand-side is

$$= \epsilon^4 C_1 E^0 \left[\int_{t\wedge\tau_m}^{T\wedge\tau_m} \Gamma_u du \,\Big|\, \mathcal{F}_{t\wedge\tau_m} \right] \to \epsilon^4 C_1 E^0 \left[\int_t^T \Gamma_u du \,\Big|\, \mathcal{F}_t \right] \le \epsilon^4 C_1 T \Gamma_t$$

as $m \to \infty$ for a continuous version of $E^0[\int_{\cdot}^T \Gamma_u du | \mathcal{F}_{\cdot}]$ by using the monotone convergence theorem. Therefore, $\|\Delta Y_t^{\epsilon,2}\|_{L^\infty([0,T]\times\Omega)} = O(\epsilon^4)$ follows.

Finally, define the process $\widehat{\nu}^{\epsilon,2} := \left(\widehat{\nu}_t^{\epsilon,2}\right)_{t\in[0,T]}$ by

$$\widehat{\nu}_t^{\epsilon,2} := \left(\widehat{\Xi}_t^{\epsilon,2}, d_\epsilon^{\perp}\right) d_\epsilon^{\perp}, \tag{A.2}$$

to deduce the relation $\widehat{\Xi}_t^{\epsilon,2} = \left\{\gamma e^{rT}\sigma_2 \widehat{\pi}_t^{\epsilon,2} - \lambda_t\right\} d_\epsilon + \widehat{\nu}_t^{\epsilon,2}$ and

$$\begin{aligned} \gamma F &= \widehat{Y}_0^{\epsilon,2} + \int_0^T \left(\gamma e^{rT}\sigma_2 \widehat{\pi}_t^{\epsilon,2} d_\epsilon - \lambda_t d_\epsilon + \widehat{\nu}_t^{\epsilon,2}\right)' dw_t^0 \\ &\quad + \int_0^T \left(\frac{|\lambda_t^\epsilon|^2 - |\widehat{\nu}_t^{\epsilon,2}|^2}{2} + R_t^{\epsilon,2} \right) dt \end{aligned}$$

from (4.4-5) and (A.2). Therefore, for $x \in \mathbf{R}$, we obtain that

$$\begin{aligned} F + I_\gamma\left(\widehat{\mathcal{Y}}^{\epsilon,2}(x) Z_T^{\widehat{\nu}^{\epsilon,2}}\right) &= X_T^{x,\widehat{\pi}^{\epsilon,2}} + \int_0^T R_t^{\epsilon,2} dt, \\ \text{where} \quad \widehat{\mathcal{Y}}^{\epsilon,2}(x) &= \exp\left(\widehat{Y}_0^{\epsilon,2} - \gamma e^{rT} x\right), \end{aligned}$$

which implies

$$\begin{aligned} &\log E\left[U_\gamma\left(-F + X_T^{x,\widehat{\pi}^{\epsilon,2}}\right)\right] \\ =\ &\log E\left[U_\gamma\left(I_\gamma\left(\widehat{\mathcal{Y}}^{\epsilon,2}(x) Z_T^{\widehat{\nu}^{\epsilon,2}}\right) - \int_0^T R_t^{\epsilon,2} dt\right)\right] \\ =\ &-\frac{1}{\gamma}\widehat{\mathcal{Y}}^{\epsilon,2}(x) + O(\epsilon^4) \\ =\ &\log U_\gamma\left(e^{rT}x - \frac{\widehat{Y}_0^{\epsilon,2}}{\gamma}\right) + O(\epsilon^4) \\ =\ &\log U_\gamma\left(e^{rT}x - \frac{Y_0^{\epsilon}}{\gamma}\right) + O(\epsilon^4) \quad \text{as } \epsilon \downarrow 0. \blacksquare \end{aligned}$$

A.3 Proof of Lemma 4.2.

1. Suppose $\overline{\Xi}_2^{0,\delta} \equiv 0$ and observe the BSDE:

$$d\overline{Y}_t^{0,\delta} = \frac{1}{2}\left\{\lambda_t^2 - \left(\overline{\Xi}_1^{0,\delta}(t)\right)^2\right\}dt + \overline{\Xi}_1^{0,\delta}(t)dw_1^0(t), \quad \overline{Y}_T^{0,\delta} = \gamma F.$$

Let $W_t := \exp\left(\overline{Y}_t^{0,\delta} - \frac{1}{2}\int_t^T \lambda_u^2 du\right)$. We can deduce the equation

$$dW_t = W_t\overline{\Xi}_1^{0,\delta}(t)dw_1^0(t), \quad W_T = e^{\gamma F}$$

and its solution $W_t := E^0\left[e^{\gamma F}|\mathcal{F}_t\right]$, $\Xi_1^{0,\delta}(t) := \gamma W_t^{-1}E^0\left[e^{\gamma F}D_{1,t}F \mid \mathcal{F}_t\right]$. The uniqueness follows from the result of Kobylanski, [7].

2-3. Observe that

$$\begin{aligned}
\overline{d}_\delta^\perp := d_{\sqrt{1-\delta^2}}^\perp &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \delta\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{\delta^2}{2}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \frac{\delta^3}{3!}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + O(\delta^4) \\
&=: \overline{d}_0^\perp + \sum_{i=1}^3 \frac{\delta^i}{i!}\partial_{\delta'}^i\overline{d}_0^\perp + O(\delta^4),
\end{aligned}$$

where $O(\delta^4) \in \mathbf{R}^2$ is a vector with the norm $|O(\delta^4)| \sim \delta^4$.

(i) Noting that

$$h_1\left(t, (\xi^j)_{j=0,1}, 0\right) = -\left(\xi^0, \overline{d}_0^\perp\right)\left\{\left(\xi^1, \overline{d}_0^\perp\right) + \left(\xi^0, \partial_{\delta'}\overline{d}_0^\perp\right)\right\},$$

$\overline{\Xi}_1^{0,\delta} \in \mathbf{H}_T^\infty$, and $\overline{\Xi}_2^{0,\delta} \equiv 0$, we have a standard linear BSDE:

$$d\left(\partial_{\delta'}\overline{Y}_t^{0,\delta}\right) = -\partial_{\delta'}\overline{\Xi}_1^{0,\delta}(t)\overline{\Xi}_1^{0,\delta}(t)dt + \partial_{\delta'}\overline{\Xi}_t^{0,\delta}dw_t^0, \quad \partial_{\delta'}\overline{Y}_T^{0,\delta} \equiv 0$$

with the solution 0.

(ii) Observing that

$$\begin{aligned}
& h_2\left(t, (\xi^j)_{j=0,1,2}, 0\right) \\
= \; & -\left(\xi^1, \overline{d}_0^\perp\right)\left\{\left(\xi^1, \overline{d}_0^\perp\right) + \left(\xi^0, \partial_{\delta'}\overline{d}_0^\perp\right)\right\} - \left(\xi^0, \overline{d}_0^\perp\right)\left\{\left(\xi^2, \overline{d}_0^\perp\right) + \left(\xi^1, \partial_{\delta'}\overline{d}_0^\perp\right)\right\} \\
& -\left(\xi^0, \partial_{\delta'}\overline{d}_0^\perp\right)\left\{\left(\xi^1, \overline{d}_0^\perp\right) + \left(\xi^0, \partial_{\delta'}\overline{d}_0^\perp\right)\right\} - \left(\xi^0, \overline{d}_0^\perp\right)\left\{\left(\xi^1, \partial_{\delta'}\overline{d}_0^\perp\right) + \left(\xi^0, \partial_{\delta'}^2\overline{d}_0^\perp\right)\right\},
\end{aligned}$$

we rewrite the BSDE for $\left(\partial_{\delta'}^2\overline{Y}^{0,\delta}, \partial_{\delta'}^2\overline{\Xi}^{0,\delta}\right)$ as

$$d\left(\partial_{\delta'}^2\overline{Y}_t^{0,\delta}\right) = \overline{\Xi}_1^{0,\delta}(t)\left(\overline{\Xi}_1^{0,\delta}(t) - \partial_{\delta'}^2\overline{\Xi}_1^{0,\delta}(t)\right)dt + \left(\partial_{\delta'}^2\overline{\Xi}_t^{0,\delta}\right)' dw_t^0, \quad \partial_{\delta'}^2\overline{Y}_T^{0,\delta} \equiv 0$$

since $\overline{\Xi}_2^{0,\delta} \equiv 0$ and $\partial_{\delta'}\overline{\Xi}^{0,\delta} \equiv 0$. This standard linear BSDE has the solution satisfying

$$\text{(A.3)} \qquad \begin{aligned}
\partial_{\delta'}^2\overline{Y}_t^{0,\delta} &= -\overline{E}^{0,\gamma}\left[\int_t^T \left(\overline{\Xi}_1^{0,\delta}(u)\right)^2 du \,\middle|\, \mathcal{F}_t\right], \\
\partial_{\delta'}^2\overline{Y}_0^{0,\delta} + \int_0^t \partial_{\delta'}^2\overline{\Xi}_1^{0,\delta}(u)d\overline{w}_1^{0,\gamma}(u) &= -\overline{E}^{0,\gamma}\left[\int_0^T \left(\overline{\Xi}_1^{0,\delta}(u)\right)^2 du \,\middle|\, \mathcal{F}_t\right],
\end{aligned}$$

and $\partial^2_{\delta'}\overline{\Xi}_2^{0,\delta} \equiv 0$ for $t \in [0,T]$, where $\overline{E}^{0,\gamma}[\cdot]$ is the expectation with respect to the probability measure $\overline{P}^{0,\gamma}$ defined by

$$\frac{d\overline{P}^{0,\gamma}}{dP^0}\Big|_{\mathcal{F}_t} := \frac{E^0\left[e^{\gamma F}|\mathcal{F}_t\right]}{E^0\left[e^{\gamma F}\right]} = \mathcal{E}_t\left(\int \overline{\Xi}_1^{0,\delta} dw_1^0\right)$$

and $\overline{w}_1^{0,\gamma}(t) := w_1^0(t) - \int_0^t \overline{\Xi}_1^{0,\delta}(u)du$. Noting the relation

$$\text{(A.4)} \qquad -\int_t^T \left(\overline{\Xi}_1^{0,\delta}(u)\right)^2 du = -2\left\{\gamma F - \log E^0\left[e^{\gamma F}|\mathcal{F}_t\right] - \int_t^T \overline{\Xi}_1^{0,\delta}(u)d\overline{w}_1^{0,\gamma}(u)\right\},$$

we obtain the expression for $\partial^2_{\delta'}\overline{Y}^{0,\delta}$ from the Bayes rule. Further, recalling the relation

$$e^{\gamma F}F = E^0\left[e^{\gamma F}F\right] + \int_0^T E^0\left[e^{\gamma F}(\gamma F + 1)D_{1,t}F|\mathcal{F}_t\right]dw_1^0(t)$$

from the Clark-Haussman-Ocone formula and the chain rule for differentiation, we observe that

$$\begin{aligned}
F = \frac{e^{\gamma F}F}{e^{\gamma F}} &= \overline{E}^{0,\gamma}[F] + \int_0^T d\left(\frac{E^0[e^{\gamma F}F|\mathcal{F}_t]}{E^0[e^{\gamma F}|\mathcal{F}_t]}\right) \\
&= \overline{E}^{0,\gamma}[F] + \int_0^T \frac{1}{E^0[e^{\gamma F}|\mathcal{F}_t]}\left\{E^0\left[e^{\gamma F}(\gamma F + 1)D_{1,t}F|\mathcal{F}_t\right]\right. \\
&\qquad \left. -\gamma\frac{E^0[e^{\gamma F}F|\mathcal{F}_t]E^0[e^{\gamma F}D_{1,t}F|\mathcal{F}_t]}{E^0[e^{\gamma F}|\mathcal{F}_t]}\right\}d\overline{w}_1^{0,\gamma}(t).
\end{aligned}$$

This, together with (A.4) for $t = 0$, yields the expression for $\partial^2_{\delta'}\overline{\Xi}^{0,\delta}$.
(iii) For $\left(\xi^j\right)_{j=0,1,2,3}$ such that $\xi_2^0 = \xi_2^2 = 0$ and $\xi^1 = 0$, we can check that

$$h_3\left(t, (\xi^j)_{j=0,1,2,3}, 0\right) = -\xi_1^3\xi_1^0,$$

so the equation

$$d\left(\partial^3_{\delta'}\overline{Y}_t^{0,\delta}\right) = -\partial^3_{\delta'}\overline{\Xi}_1^{0,\delta}(t)\overline{\Xi}_1^{0,\delta}(t)dt + \partial^3_{\delta'}\overline{\Xi}_t^{0,\delta}dw_t^0, \quad \partial^3_{\delta'}\overline{Y}_T^{0,\delta} \equiv 0$$

and $\left(\partial^3_{\delta'}\overline{Y}^{0,\delta}, \partial^3_{\delta'}\overline{\Xi}^{0,\delta}\right) \equiv 0$ are deduced. ∎

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