数理解析研究所講究録 1391 巻 2004 年 229-242

Stock price process and the long-range percolation

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1 Introduction

We consider a stock market model where traders' analysis of past market data and the long-range percolation communication system between traders bring a single stock price process. In the theory of *econophysics*, they present financial market models with mathematical tools from statistical physics and try to explain the economical phenomena, e.g., using truncated Lévy flight [MS94], herding approach [CB2000] or square lattice percolation [T2002], etc. The balance of microscopic traders' transactions in a financial market reflects fluctuations in a macroscopic stock price.

Usually, the market is somewhat efficient and traders are rather rational, the logarithms of the price change may be expected to distribute Gaussian. Traders, although, could not be always rational, they may follow uncritically the trend in the market and cause a large fluctuation, the logarithms of the price change deviates from Gaussian or has fat tails [M63].

In this article, we suppose there are two types of traders: one type of traders called group A decide their trading strategy by analyzing past market data, another type of traders called group B receive a news of the market from the long-range percolation communication system and obey it unquestioningly. At each discrete time $u = 1, 2, \ldots, n$, a trader gives an order to a broker to purchase or sell the stock, or does not give an order after his market expectation. The stock price will rise when there are much more purchase orders than sell orders, and otherwise it will fall. We obtain the discrete time stock price process up to time n, and as we take a scaling limit of it as n approaches to infinity, we will construct a continuous time stock price process. Moreover, the limit stock price process is a Lévy process, occasionally the price will have a drastic change and exhibit a fat tail.

In group B, there are countably many traders, who are located in a line $\mathbb{Z}$. At each time, a trader in the origin receives a good news with probability γ^+ or bad news with probability γ^-, or does not receive any news with probability $1-\gamma^+-\gamma^-$. We denote the types of news reached to the trader in the origin at time u by a random variable $\bar{\omega}_u$, and we represent $\bar{\omega}_u = +1$, -1 or 0 for good, bad or no news, respectively. Then the trader in the origin announces the news if he receives it to other traders in group B via the long-range percolation communication system between traders in this group. At each time, a pair of trader $\{x, y\}$ $(x, y \in \mathbb{Z})$ make contact with each other with probability p if $|x-y| = 1$, otherwise with probability $\beta|x-y|^{-2}$ where $\beta > 1$ and share the news if at least one of them knew it. We suppose $p = p(u) \in [0, 1)$ depends on time u and we call it the *nearest neighbor percolation probability*. If a pair of trader $\{x, y\}$ $(x, y \in \mathbb{Z})$ make contact with each other at time u, we write $\tilde{\omega}_u(x, y) = +1$; otherwise we write $\tilde{\omega}_u(x, y) = 0$. In this communication system, a pair of traders may communicate frequently when they stay close to each other, although become estranged as getting away. Traders belong to the same communication component with the trader in the origin receive the news, and other traders do not. Let N_n be a positive integer increasing in n and $B_{N_n} = [-N_n, N_n] \cap \mathbb{Z}$ a set of selected traders includes one in the origin. We denote C_{N_n} a random set of the selected traders who receive the news. The trader in the origin representing group B gives an order to a broker to purchase or sell the stock of amount in proportion to a ratio of the number of the selected traders who received the news to the total number of the selected traders, or does not give an order, provided that he receives a good or bad news, or dose not receive any news, respectively. Hence, the order of the stock which the trader in the origin gives at time u is

$$< \hat{\omega}_u >= \bar{\omega}_u \frac{|C_{N_n}|}{|B_{N_n}|}.$$

We think the trader in the origin plays the role of a *advisor* or a *fund manager*.

Let δ be a positive constant with $0 < \delta < \frac{1}{2}$. We define the stopping times T_k and U_k by

$$T_k(\hat{\omega}) = \min\{u \geq 1 : \sum_{\ell=1}^{u} < \hat{\omega}_{U_{k-1}+\ell} > \geq n^{\delta}\},$$
$$U_k(\hat{\omega}) = U_{k-1}(\hat{\omega}) + T_k(\hat{\omega}),$$

for $k \geq 1$, and for $k = 0$ we set $T_0(\hat{\omega}) = 0$ and $U_0(\hat{\omega}) = 0$. Let $q = 1/2 - \delta$. Then we decompose the set of discrete times $\{1, \ldots, n\}$ into random intervals $I_n^{\hat{\omega}}(1), \ldots, I_n^{\hat{\omega}}(n^q + 1)$ where

$$I_n^{\hat{\omega}}(k) = \begin{cases} \{U_{k-1}(\hat{\omega}) + 1, \ldots, U_k(\hat{\omega})\}, & \text{for } k = 1, \ldots, n^q, \\ \{U_{n^q}(\hat{\omega}) + 1, \ldots, n\}, & \text{for } k = n^q + 1. \end{cases}$$

It is known ([NS86, AN86, GKN92]) that there exists a critical value $p_c(\beta) \in (0, 1)$ for the long-range percolation model such that there is a unique infinite cluster almost surely if $p \geq p_c(\beta)$, otherwise there is no infinite cluster. Moreover this critical phenomenon is of the first order, in the sense that the probability that the origin belongs to an infinite component is a discontinuous function of p at critical value $p_c(\beta)$. Let $\{p_k : k = 0, 1, \ldots, n^q + 1\}$ be a increasing sequence in $[0, 1)$. We suppose p_k is subcritical if $k \leq n^q$, and $p_{n^q+1} = p_c(\beta)$. At time u, we take the nearest neighbor percolation probability $p = p_k$, provided that u belongs to $I_n^{\hat{\omega}}(k)$. After $U_{n^q}(\hat{\omega})$, infinitely many traders shall know the news, a rush of orders from traders in group B cause a financial discontinuity.

Group A consists of N traders. At each time, a trader in this group places an purchasing or selling order with a broker for the unit stock, or does not place any order in accordance with his trading strategy. Traders will act on the mutual interaction, their behavior suppose to be described statistically by a Gibbs distribution. We denote the type of order of the trader $i = 1, \ldots, N$ at time u by a random variable $\omega_u(i)$. We represent

$\omega_u(i) = +1, -1$ or 0 for an order of purchase or sell the unit stock, or no order, respectively. Let ω_u^+ be the number of traders in group A gives an order for purchase the unit stock at time u, and ω_u^- for sell. We denote the *market participants* by $|\omega_u| = (\omega_u^+ + \omega_u^- - d_0) \wedge 0$, where $d_0 > 0$ is the *inefficiency* of the market. We denote the *surplus order of purchase* by

$$< \omega_u >= \begin{cases} \omega_u^+ - \omega_u^- - d_0, & \text{if } \omega_u^+ - \omega_u^- > d_0, \\ -(\omega_u^- - \omega_u^+ - d_0), & \text{if } \omega_u^- - \omega_u^+ > d_0, \\ 0, & \text{otherwise,} \end{cases}$$

Let $f_1(x,t)$ and $f_2(x,t)$ be functions on $\mathbb{R} \times [0,1]$ and continuous in t. At time u, if u belongs to $I_n^{\hat{\omega}}(k)$, then we define a local Hamiltonian by

$$\begin{aligned} H^{u,\hat{\omega}}(\omega) &= \beta_1 |\omega_u|^2 + \beta_2 \Phi(\omega_u | \omega_{u-a}^k, \ldots, \omega_{u-1}^k) \\ &\quad - \beta_3 f_1(|\omega^k|_{u,a}, k/n^q)|\omega_u| - \frac{\beta_4}{\sqrt{n}} f_2(< \omega^k >_{u,a}, k/n^q) < \omega_u >, \end{aligned}$$

where β_1, β_2, β_3, β_4 are positive constants, $\omega_\ell^k = \omega_\ell 1_{\{\ell \in I_n^{\hat{\omega}}(k)\}}$, $|\omega^k|_{u,a} = |\omega_{u-a}^k| + \cdots + |\omega_{u-1}^k|$ and $< \omega^k >_{u,a} = < \omega_{u-a}^k > + \cdots + < \omega_{u-1}^k >$. A *Hamiltonian* is given by

$$H^{\hat{\omega}}(\omega) = \sum_{u=1}^{n} H^{u,\hat{\omega}}(\omega).$$

For a given positive constant τ. Let $f_3(t) > 0$ be a continuous function on $[0,1]$ such that

$$\int_0^1 \frac{1}{f_3(v)} \mathrm{d}v = \tau.$$

The stock price S_u at time u is defined by $S_u = S_{u-1} \exp\{c_0(< \omega_u > + < \hat{\omega}_u >)\}$ where $c_0 > 0$ is the *market depth*. This implies that $S_u = S_0 \exp\{c_0 \sum_{k=1}^{u} (< \omega_u > + < \hat{\omega}_u >)\}$. For $t \in [0,1]$, we denote $k = k(n,t)$ if $[nt] \in I_n^{\hat{\omega}}(k)$. Let $X_t^{(n)} = (W_t^{(n)} + \hat{W}_t^{(n)})/\sqrt{n}$ be a scaled process, where

$$W_t^{(n)} = \begin{cases} \displaystyle\sum_{k=1}^{k(n,t)} \sum_{u \in I_n^{\hat{\omega}}(k)} < \omega_u >, & \text{if } k(n,t) \le n^q, \\ \displaystyle\sum_{u=1}^{U_{n^q} + n^{1-\lambda}} < \omega_u >, & \text{otherwise} \end{cases}$$

and $\hat{W}_t^{(n)}$ is defined in the same way as $W_t^{(n)}$. Then we assume some additional conditions and take the limit as n tends to infinity, using the cluster expansion technique ([P91]), we will obtain a stock price process $S_t = S_0 \exp\{c_0 X_t\}$, here X_t is a Lévy process. We will show the scaled process $X_t^{(n)}$ converges in finite dimensional distribution to the process

$$X_t = \int_0^t (\mu_A(v) + \mu_B(v)) \mathrm{d}v + \int_0^t \sigma_A^2(v) \mathrm{d}B_v + hI_{\{t=\tau\}}$$

where B_t is a standard Brownian motion. Group A contributes toward the trend term $\mu_A(s)$ and the volatility term $\sigma_A^2(s)$ of the price process; Group B contributes toward the trend term $\mu_B(s)$ and the jump term $hI_{\{t=\tau\}}$:

$$\mu_A(t) = \beta_4 \sum_{i(A)=0} < A > f_2(A, s(t))\phi_0(A)e^{\beta_3 f_1(A,s(t))}\frac{\alpha^T(A)}{A!}f_3(s(t)),$$
$$\mu_B(t) = f_3(s(t)),$$
$$\sigma_A^2(t) = \sum_{i(A)=0} < A >^2 \phi_0(A)e^{\beta_3 f_1(A,s(t))}\frac{\alpha^T(A)}{A!}f_3(s(t)).$$

2 Model

In group B, there are countably many traders, who are located in a line $\mathbb{Z}$. Let $\bar{\omega}_u$ be the type of news at time $u = 1, \ldots, n$ in group B. We denote $\bar{\omega}_u = +1$, -1 or 0, if the trader in the origin receives a good news, bad news or no news, respectively. The configuration space of the type of news is $\bar{\Omega}_n = \{+1, -1, 0\}^n$. We suppose that $\{\bar{\omega}_u\}$ are independent, that is each news arrives independently.

At each time $u = 1, \ldots, n$, a pair of traders $\{x, y\}$ $(x, y \in \mathbb{Z})$ is said to be *connected* if they make contact with each other, and we write $\tilde{\omega}_u(x, y) = +1$; otherwise we write $\tilde{\omega}_u(x, y) = 0$. The configuration space of communication system is $\tilde{\Omega}_n = \{0, +1\}^{n\times\mathbb{B}}$, where $\mathbb{B} = \{\{x, y\} | x,\, y \in \mathbb{Z}\}$ is the set of all pairs of traders. We suppose that $\{\tilde{\omega}_u(\{x, y\})\}$ are independent in a pair of traders $\{x, y\}$.

The configuration space of group B is $\hat{\Omega}_n = \bar{\Omega}_n \times \tilde{\Omega}_n$. We denote an element of $\hat{\Omega}_n$ by $\hat{\omega}_u = (\bar{\omega}_u, \tilde{\omega}_u)$. A probability measure $\hat{\mathbf{P}}(\hat{\omega}_u)$ on $\hat{\Omega}_n$ is defined as follows:

$$\hat{\mathbf{P}}(\bar{\omega}_u = +1 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}) = \gamma^+(u),$$
$$\hat{\mathbf{P}}(\bar{\omega}_u = -1 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}) = \gamma^-(u),$$
$$\hat{\mathbf{P}}(\bar{\omega}_u = 0 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}) = 1 - \gamma^+(u) - \gamma^-(u),$$

where $\gamma^+(u) \in [0, 1]$ and $\gamma^-(u) \in [0, 1]$ are the probability a good and bad news arrives at time u, respectively, we call them *news parameters*. The *activity* $\gamma^+(u) + \gamma^-(u)$ of group B is less than 1.

$$\hat{\mathbf{P}}(\tilde{\omega}_u(\{x, y\}) = 0 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}) = 1 - \hat{\mathbf{P}}(\tilde{\omega}_u(\{x, y\}) = +1 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}),$$

and

$$\hat{\mathbf{P}}(\tilde{\omega}_u(\{x, y\}) = +1 | \hat{\omega}_1, \ldots, \hat{\omega}_{u-1}) = \begin{cases} p(u), & (|x - y| = 1), \\ \beta|x - y|^{-2}, & (|x - y| \geq 2), \end{cases}$$

where $\beta > 1$ is a fixed constant, and $p(u)$ $(0 \leq p(u) < 1)$ is a constant, which we call the *nearest neighbor percolation probability*. We sometimes denote $\hat{\mathbf{P}}_p$ when we fix and omit the time u and we focus on the nearest neighbor percolation probability p.

We denote $x \leftrightarrow y$ if x and y belong to the same connected component (at time u), that is, there are $x = x_0, x_1, \ldots, x_k = y \in \mathbb{Z}$ such that $\tilde{\omega}_u(\{x_{\ell-1}, x_\ell\}) = +1$ for all $\ell = 1, \ldots, k$. Only the traders who belong to the same connected component with the trader in the origin receives the news. The set of the all traders who receives the news is

$$C_\infty = C_\infty(0) = \{x \in \mathbb{Z} \,|\, 0 \leftrightarrow x\},$$

where 0 is the origin. We also denote $C_\infty(x)$ the connected component, which includes $x \in \mathbb{Z}$. Let N_n be a positive integer specified later, and let $B_{N_n} = [-N_n, N_n] \cap \mathbb{Z}$ be the selected traders, includes the trader in the origin. The set of the selected traders who receives the news is

$$C_{N_n} = \{x \in B_{N_n} \mid 0 \leftrightarrow x\}.$$

The amount of the order, which the trader in the origin gives, is

$$< \hat{\omega}_u >= \frac{\bar{\omega}_u |C_{N_n}|}{|B_{N_n}|}.$$

We assumed that the news will spread over group B via the specific long-range percolation model. It is known that this kind of model exhibits the first order phase transition. We state some known results on this model as follows.

Theorem 2.1 ([AN86], [GKN92], [NS86]) *For $\beta > 1$, the following holds:*

(1) *There exists a critical value $p_c(\beta) \in (0,1)$ depending on β such that*

$$\hat{\mathbf{P}}_p(|C_\infty| = \infty) \begin{cases} = 0, & (p < p_c(\beta)), \\ \geq \beta^{-1/2}, & (p \geq p_c(\beta)). \end{cases}$$

(2) *For $p \geq p_c(\beta)$, there is a unique infinite cluster almost surely.*

(3) *For $p < p_c(\beta)$, there is a constant $c_2(p,\beta) < \infty$ depending on p and β such that*

$$\tau(x,y) \leq c_2(p,\beta)|x-y|^{-2} \quad \text{for any } x, y \in \mathbb{Z}^2,$$

where $\tau(x,y) = P(x \leftrightarrow y)$ is the connectivity function.

Let $\frac{1}{4} < \lambda < \frac{1}{2}$. We set $c_1(p,\beta) = p + \beta/4$ and

$$\varepsilon(n) = \inf\left\{\varepsilon > 0 \,\middle|\, \frac{16c_2(p_c(\beta)-\varepsilon,\beta)^3}{c_1(p_c(\beta)-\varepsilon,\beta)^2} \leq n^{2\lambda-1/2}\right\} + \frac{1}{n},$$

for each $n \in \mathbb{N}$. Since $16c_2(p_c(\beta)-\varepsilon,\beta)^3/c_1(p_c(\beta)-\varepsilon,\beta)^2 < \infty$ for each $p < p_c(\beta)$ and $\lim_{n\to\infty} n^{2\lambda-1/2} = \infty$, we have

$$\varepsilon(n) \to 0, \qquad \text{as } n \to \infty.$$

We note that

$$\frac{16c_2(p_c(\beta)-\varepsilon(n),\beta)^3}{c_1(p_c(\beta)-\varepsilon(n),\beta)^2}\frac{1}{n^{2\lambda}} \leq \frac{1}{\sqrt{n}} \tag{2.1}$$

holds for each $n \in \mathbb{N}$. Number of traders in group B is given by

$$N_n = \inf\left\{N \in \mathbb{N} : \hat{\mathbf{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_N|}{|B_N|}\right] \leq n^{-\lambda}\right\}.$$

Proposition 2.2 *If n is sufficiently large, then we have*

$$\hat{\mathbf{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] = \frac{1}{n^\lambda} - o\left(\frac{1}{n^\lambda}\right). \tag{2.2}$$

$$\hat{\mathbf{E}}_{p_c(\beta)-\varepsilon(n)}\left[\frac{|C_{N_n}|^2}{|B_{N_n}|^2}\right] \le \frac{1}{\sqrt{n}}. \tag{2.3}$$

Proposition 2.3 *For every $\beta > 1$, if $p \ge p_c(\beta)$ then we have*

$$\hat{\mathbf{E}}_p\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \ge \beta^{-1} \tag{2.4}$$

$$\hat{\mathbf{E}}_p\left[\frac{|C_{N_n}|^2}{|B_{N_n}|^2}\right] \ge \beta^{-3/2} \tag{2.5}$$

Let δ be a positive constant with $\frac{3}{8} < \delta < \frac{1}{2}$. We define the stopping times T_k and U_k by

$$T_k(\hat{\omega}) = \min\{u \ge 1 : \sum_{u=1}^{u} < \hat{\omega}_{U_{k-1}+u} > \ge n^\delta\},$$
$$U_k(\hat{\omega}) = U_{k-1}(\hat{\omega}) + T_k(\hat{\omega}),$$

for $k \ge 1$, and for $k = 0$ we set $T_0(\hat{\omega}) = 0$ and $U_0(\hat{\omega}) = 0$. Let $q = 1/2 - \delta$ $(< \delta)$. We decompose the set of discrete times $\{1, \dots, n\}$ into random intervals $I_n^{\hat{\omega}}(1), \dots, I_n^{\hat{\omega}}(n^q+1)$ where

$$I_n^{\hat{\omega}}(k) = \begin{cases} \{U_{k-1}(\hat{\omega})+1, \dots, U_k(\hat{\omega})\}, & \text{for } k = 1, \dots, n^q, \\ \{U_{n^q}(\hat{\omega})+1, \dots, n\}, & \text{for } k = n^q+1. \end{cases}$$

Let $\{p_k : k = 0, 1, \dots, n^q+1\}$ be a increasing sequence defined by

$$p_0 = \inf\left\{p > 0;\ \hat{\mathbf{E}}_p\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \ge \frac{\sup\{f_3(t) : t \in [0,1]\}}{\sqrt{n}}\right\},$$

and

$$p_k = \begin{cases} p_0 + \dfrac{p_c(\beta) - \varepsilon(n) - p_0}{n^q} k, & \text{if } k \le n^q, \\ p_c(\beta), & \text{if } k = n^q + 1. \end{cases}$$

At time u, we take the nearest neighbor percolation probability $p(u) = p_k$, provided that u belongs to $I_n^{\hat{\omega}}(k)$. Note that $p_{n^q} = p_c(\beta) - \varepsilon(n)$ approaches to $p_c(\beta)$ as n tends to infinity.

By noticing that from Proposition 2.2 we have

$$\frac{\sup\{f_3(t) : t \in [0,1]\}}{\sqrt{n}} \le \hat{\mathbf{E}}_{p_k}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] \le \frac{1}{n^\lambda} \quad \text{for } k \le n^q,$$

we take the news parameters $\gamma^+(u) = \gamma^+{}_k$ and $\gamma^-(u) = \gamma^-{}_k$ at time u satisfying

$$\hat{\mathbf{E}}[< \hat{\omega}_u >] = (\gamma^+{}_k - \gamma^-{}_k)\hat{\mathbf{E}}_{p_k}\left[\frac{|C_{N_n}|}{|B_{N_n}|}\right] = \begin{cases} \dfrac{1}{\sqrt{n}} f_3\left(\dfrac{k}{n^q}\right), & \text{if } k \le n^q, \\ \dfrac{h}{n^{1/2-\lambda}}, & \text{if } k = n^q + 1. \end{cases}$$

provided that u belongs to $I_n^{\hat{\omega}}(k)$, where $h \in \mathbb{R}$ is a constant. We note that $\gamma^+{}_k > \gamma^-{}_k$ whenever $k \le n^q$.

Lemma 2.4 *If*

$$0 < \varepsilon < \delta/2, \tag{2.6}$$

then for any $k = 1, \ldots, n^q$,

$$\hat{\mathbf{P}}_{p_k}\left(\left|T_k - \frac{n^{\delta+1/2}}{f_3(k/n^q)}\right| > n^{\delta/2+1/2+\varepsilon}\right) \le \frac{c}{n^{2\varepsilon}} \tag{2.7}$$

for a sufficiently large n, *where* c *is a positive constant.*

We call $\mathbf{r} = \{r_k\}_{k=1}^{n^q}$ an *admissible sequence* if

$$\left|r_k - \frac{n^{\delta+1/2}}{f_3(k/n^q)}\right| \le n^{\delta/2+1/2+\varepsilon} \qquad \text{for all } k = 1, \ldots, n^q.$$

For $k = 1, \ldots, n^q$, we set $s_k = r_1 + r_2 + \cdots + r_k$ and let $I(k) = I_n(k, r)$ be a set of consecutive numbers $\{s_{k-1}+1, s_{k-1}+2, \ldots, s_k\}$, and we write $I(n^q+1) = \{s_{n^q}+1, \ldots, n\}$. For an admissible sequence $\mathbf{r} = \{r_k\}_{k=1}^{n^q}$, we have

$$\begin{aligned} r_1 + \cdots + r_{n^q} &= \sum_{k=1}^{n^q} \frac{n^{\delta+1/2}}{f_3(k/n^q)} + o(n) \\ &= n^{\delta+1/2} n^q \sum_{k=1}^{n^q} \frac{1}{n^q} \frac{1}{f_3(k/n^q)} + o(n) \\ &= n \int_0^1 \frac{\mathrm{d}x}{f_3(x)} + o(n) = n\tau + o(n). \end{aligned}$$

Corollary 2.5 *If*

$$0 < q/2 < \varepsilon < \delta/2, \tag{2.8}$$

then we have

$$\hat{\mathbf{P}}(\{T_k\}_{k=1}^{n^q} \textit{ is an admissible sequence}) \to 1 \quad (n \to \infty). \tag{2.9}$$

Group A consists of N traders. We represent $\omega_u(i) = +1, -1$ or 0 for an order of purchase or sell the unit stock, or no order, respectively. So the configuration space of positions for group A is $\Omega_n = \{+1, -1, 0\}^{n \times N}$. Let ω_u^+ and ω_u^- be the number of traders in group A gives an order for purchase and sell the unit stock at time u, respectively. Let us fix a positive constant d_0. We denote the *market participants* or *activity* of group A by

$$|\omega_u| = \begin{cases} \omega_u^+ + \omega_u^- - d_0, & \text{if } \omega_u^+ + \omega_u^- > d_0, \\ 0, & \text{otherwise,} \end{cases}$$

and the *surplus order of purchase* by

$$<\omega_u>= \begin{cases} \omega_u^+ - \omega_u^- - d_0, & \text{if } \omega_u^+ - \omega_u^- > d_0, \\ -(\omega_u^- - \omega_u^+ - d_0), & \text{if } \omega_u^- - \omega_u^+ > d_0, \\ 0, & \text{otherwise.} \end{cases}$$

We say ω_u is *active* or group A is active at time u if $|\omega_u| \neq 0$, otherwise, we say it is *static.*

A *Gibbs measure* on Ω_n with respect to $\hat{\omega} \in \hat{\Omega}$, which reflects the trading strategy of the traders in group A, is defined by

$$\mathbf{P}^{\hat{\omega}}(\omega) = \frac{1}{Z_n^{\hat{\omega}}} \exp\left[-H^{\hat{\omega}}(\omega)\right],$$

where $Z_n^{\hat{\omega}}$ is a normalized constant. A *Hamiltonian* is given by

$$H^{\hat{\omega}}(\omega) = \sum_{u=1}^{n} H^{u,\hat{\omega}}(\omega).$$

A local Hamiltonian which describes the traders' behavior in group A at time u in random interval $I_n^{\hat{\omega}}(k)$ is given by

$$\begin{aligned} H^{u,\hat{\omega}}(\omega) &= \beta_1|\omega_u|^2 + \beta_2\Phi(\omega_u|\omega_{u-a}^k, \ldots, \omega_{u-1}^k) \\ &\quad - \beta_3 f_1(|\omega^k|_{u,a}, k/n^q)|\omega_u| - \frac{\beta_4}{\sqrt{n}} f_2(<\omega^k>_{u,a}, k/n^q) <\omega_u>, \end{aligned}$$

where β_1, β_2, β_3, β_4 are positive constants. The first term keeps control over the activity of group A. The second term represents the trading strategies, traders analyze past a time steps' data within the random interval $I_n^{\hat{\omega}}(k)$, where $\omega_\ell^k = \omega_\ell 1_{\{\ell \in I_n^{\hat{\omega}}(k)\}}$ is the restriction of ω_ℓ to $I(k)$ and a is a positive integer. The third term plays a key role to generate the volatility of the stock price, where $f_1(x,t)$ is a real valued function on $[0,\infty) \times [0,1]$, which is continuous in t, and $|\omega^k|_{u,a} = |\omega_{u-a}^k| + \cdots + |\omega_{u-1}^k|$. If $f_1 > 0$ then the activity is increasing and cause a large volatility, otherwise the activity is decreasing and cause a small volatility. The fourth term plays a key role to generate the trend of the stock price, where $f_2(x,t)$ is a real valued function on $\mathbb{R} \times [0,1]$, which is continuous in t, and $<\omega^k>_{u,a} = <\omega_{u-a}^k> + \cdots + <\omega_{u-1}^k>$. If $f_2 > 0$ then the stock price process will be in an up trend, otherwise in a down trend. We assume that the local Hamiltonian satisfies the following conditions (A.1)-(A.5),

(A.1) If ω_u is static, that is $|\omega_u| = 0$, then

$$\Phi(\omega_u|\omega_{u-a}^k, \ldots, \omega_{u-1}^k) = 0.$$

(A.2) For any $\omega \in \Omega_n$,

$$\Phi(-\omega_u| - \omega_{u-a}^k, \ldots, -\omega_{u-1}^k) = \Phi(\omega_u|\omega_{u-a}^k, \ldots, \omega_{u-1}^k),$$

where $-\omega_u = (-\omega_u(1), \ldots, -\omega_u(N))$.

(A.3) There is a positive constant c such that

$$|\Phi(\omega_u|\omega_{u-a}^k, \ldots, \omega_{u-1}^k)| \leq c|\omega_u|^2.$$

(A.4) There is a positive constant c such that

$$|f_1(x,t)| \leq cx \quad \text{for any } x > 0.$$

(A.5) There is a positive constant c such that

$$|f_2(x,t)| \leq c|x| \quad \text{for any } x.$$

And $f_2(x,t) + f_2(-x,t) > 0$ for any x.

The coupling measure $\mathbf{P}$ on $\Omega \times \hat{\Omega}$, which describe the behavior of all traders both in group A and in group B, is defined by

$$\mathbf{P}(\omega, \hat{\omega}) = \mathbf{P}^{\hat{\omega}}(\omega)\hat{\mathbf{P}}(\hat{\omega}).$$

When the total amount of the stock order $< \omega_u > + < \hat{\omega}_u >$ is positively large, it is expected that there is a strong driving activity on the part of buyer and the stock price is going to move in upper direction. On the other hand, market is going to fall when $< \omega_u > + < \hat{\omega}_u >$ is negative. We define the stock price S_u at time u by

$$S_u = e^{c_0(<\omega_u>+<\hat{\omega}_u>)} S_{u-1}, \tag{2.10}$$

where $c_0 > 0$ is the *market depth.* Note that we think S_u is the *closing price* other than opening price. This recurrence formula implies for any $u = 1, \ldots, n$

$$S_u = S_0 \exp\{c_0 \sum_{l=1}^{u} (< \omega_l > + < \hat{\omega}_l >)\}, \tag{2.11}$$

where S_0 is initial stock price at time 0.

A *polymer* ξ with respect to $\hat{\omega} \in \hat{\Omega}$ is a collection $(\eta, b(\xi), k(\xi))$ with the following five conditions:

(P.1) $k(\xi) \in \{1, 2, \ldots, n^q + 1\}$ and $b(\xi)$ is a set of consecutive numbers in $I_n^{\hat{\omega}}(k(\xi))$.

(P.2) $\eta = \{\eta_u\}_{u \in b(\xi)}$ with $\eta_u \in \{+1, -1, 0\}^N$ for all $u \in b(\xi)$.

(P.3) For each $u \in b(\xi)$, there is $\ell = 0, \ldots, a$ with $u + \ell \in b(\xi)$ such that $\omega_{u+\ell}$ is active, that is $|\omega_{u+\ell}| \neq 0$.

(P.4) Let u_0 be the left end point of $b(\xi)$. If $u_0 \neq U_{k(\xi)-1} + 1$, then η_{u_0+a} is active and η_{u_0+l} is static for $l = 0, \ldots, a-1$.

We denote $\mathcal{D}$ the set of all polymers.

A pair of polymers ξ and ξ' is said to be *compatible* if $b(\xi) \cap b(\xi') = \emptyset$. A family of polymers $\{\xi^i\}$ is said to be compatible if each pair of polymers ξ^i and ξ^j $(i \neq j)$ is compatible. A pair of polymers ξ and ξ' is said to be *incompatible* if it is not compatible.

Let m_0 be the number of static elements in $\{+1, -1, 0\}^N$, which can be expressed by

$$m_0 = \sum_{k_1+k_2 \leq d_0} \frac{N!}{k_1!k_2!(N-k_1-k_2)!}.$$

We denote the statistical weight of a polymer $\xi = (\eta, b(\xi), k(\xi))$ by

$$\mathcal{W}(\xi) = \exp\{-|b(\xi)| \log m_0 - \sum_{u \in b(\xi)} H^{u,\hat{\omega}}(\eta)\}, \tag{2.12}$$

here, we regard η as an element in Ω_n with $\eta_u = 0$ for $u \notin b(\xi)$, so that the expression $H^{u,\hat{\omega}}(\eta)$ makes sense.

For given $\hat{\omega} \in \hat{\Omega}$ and for $u = 1, 2, \dots, n$, we denote $k(u) = 1, \dots, n^q+1$ if $u \in I_n^{\hat{\omega}}(k(u))$. Take $\omega \in \Omega_n$, a set

$$\bigcup_{\substack{u=1,\dots,n \\ \omega_u \text{ is active}}} \{u-a, \dots, u-1, u\} \cap I_n^{\hat{\omega}}(k(u))$$

is decomposed into the sets $W_1, \dots, W_m$ of consecutive numbers and there is a increase sequence $\{k_i\}_{i=1}^m$ of positive integers such that $I_n^{\hat{\omega}}(k_i)$ is a unique random interval includes W_i. Then for any $\omega \in \Omega$, a collection of triples $\{\xi^i\}_{i=1}^m = \{(\{\omega_u\}_{u\in W_i}, W_i, k_i)\}_{i=1}^m$ is a compatible family of polymers, we say ω *forms* $\{\xi^i\}_{i=1}^m$. For any compatible family of polymers $\{\xi^i\}_{i=1}^m$, we denote

$$\mathbf{P}^{\hat{\omega}}(\xi^1, \dots, \xi^m) = \mathbf{P}^{\hat{\omega}}(\omega \text{ forms } \{\xi^i\}_{i=1}^m) = \frac{1}{Z_n^{\hat{\omega}}} \Xi_n^{\hat{\omega}}(\xi^1, \dots, \xi^m),$$

where

$$\Xi_n^{\hat{\omega}}(\xi^1, \dots, \xi^m) = \sum_{\substack{\omega\in\Omega_n \\ \omega \text{ forms } \{\xi^i\}_{i=1}^m}} \exp\{-H^{\hat{\omega}}(\omega)\}. \tag{2.13}$$

For any compatible family of polymers $\{\xi^i = (\eta^i, b(\xi^i), k(\xi^i)\}$ and any ω which forms $\{\xi^i\}_{i=1}^m$, if $u \in \{1, \dots, n\} \backslash \cup_{i=1}^m b(\xi^i)$, then ω_u is static, so that $H^{u,\hat{\omega}}(\omega) = 0$, and if $u \in b(\xi^i)$, then ω_u coincides with η_u^i for $i = 1, \dots, m$. Hence we have

$$\Xi(\xi^1, \dots, \xi^m) = \Big(\prod_{i=1}^m \exp\Big\{-\sum_{u\in b(\xi^i)} H^{u,\hat{\omega}}(\eta^i)\Big\}\Big) \times m_0^{n-\sum_{i=1}^m |b(\xi^i)|}.$$

This implies the polymer representation

$$\mathbf{P}_n^{\hat{\omega}}(\xi^1, \dots, \xi^m) = \frac{1}{\tilde{Z}_n^{\hat{\omega}}} \prod_{i=1}^m \mathcal{W}(\xi^i), \tag{2.14}$$

where $\tilde{Z}_n^{\hat{\omega}} = Z_n^{\hat{\omega}}/m_0^n$.

We denote $\mathcal{X}$ the space of mappings A from $\mathcal{D}$ to $\mathbb{N} \cup \{0\}$ satisfying

$$|A| := \sum_{\xi\in\mathcal{D}} A(\xi) < \infty.$$

We denote $suppA = \{\xi \in \mathcal{D} \mid A(\xi) \neq 0\}$, $A! = \prod_{\xi\in suppA} A(\xi)!$, $b(A) = \sum_{\xi\in suppA} b(\xi)$ and

$$\alpha(A) = \begin{cases} 1, & \text{if } A! = 1 \text{ and any pair of } \xi^i, \xi^j \in suppA \text{ are compatible,} \\ 0, & \text{otherwise.} \end{cases}$$

Let $G(A)$ be a graph whose vertex set is $suppA$ and edge set is all incompatible pairs in $suppA$. For any graph G, we denote $|G|$ the number of its edges. The *Ursell function* $\alpha^T(A)$ is given by

$$\alpha^T(A) = \sum_{G'\subset G(A)} (-1)^{|G'|},$$

where the summation is over all connected subgraph G' of G whose vertex set is also *suppA*. If $\alpha^T(A) \neq 0$ then there is a unique $k(A) = 1, 2, ..., n^q + 1$ such that $b(A)$ is included in $I_n^{\hat{\omega}}(k(A))$, since if such $k(A)$ dose not exist then from the definition of the polymer, $G(A)$ is disconnected. Hence, for any $A \in \mathcal{X}$ with $\alpha^T(A) \neq 0$ we define

$$f_1(A, k(A)/n^q) = \sum_{\xi \in \mathcal{D}} \sum_{u \in b(\xi)} f_1(|\eta^{k(\xi)}|_{u,a}, k(A)/n^q)|\eta_u| A(\xi),$$

$$f_2(A, k(A)/n^q) = \sum_{\xi \in \mathcal{D}} \sum_{u \in b(\xi)} f_2(< \eta^{k(\xi)} >_{u,a}, k(A)/n^q) < \eta_u > A(\xi).$$

For any $A \in \mathcal{X}$, we write

$$< A >= \sum_{\xi \in \mathcal{D}} \sum_{u \in b(\xi)} < \eta_u > A(\xi), \quad < A >^2= \sum_{\xi \in \mathcal{D}} \sum_{u \in b(\xi)} < \eta_u >^2 A(\xi).$$

A function space $\mathcal{L}$ is given by

$$\mathcal{L} = \{\varphi \ : \ \mathcal{X} \to \mathbb{C} \ ; \ \sup_{|A|=n} |\varphi(A)| < \infty \text{ for any } n\}.$$

An element $\varphi \in \mathcal{L}$ is said to be *multiplicative* if

$$\varphi(A_1 + A_2) = \varphi(A_1)\varphi(A_2) \text{ for all } A_1, A_2 \in \mathcal{X}.$$

For any $\xi = (\eta, b(\xi), k(\xi)) \in \mathcal{D}$, put

$$\phi_0(\xi) = \exp\big[- |b(\xi)| \log m_0 - \sum_{u \in b(\xi)} \{\beta_1 |\eta_u|^2 + \beta_2 \Phi(\eta_u \mid \eta_{u-a}^{k(\xi)}, \dots, \eta_{u-1}^{k(\xi)})\}\big],$$

$$\phi_1(\xi) = \exp\big[\beta_3 \sum_{u \in b(\xi)} f_1(|\eta^{k(\xi)}|_{u,a}, k(\xi)/n^q)|\eta_u|\big],$$

$$\phi_2(\xi) = \exp\big[\frac{\beta_4}{\sqrt{n}} \sum_{u \in b(\xi)} f_2(< \eta^{k(\xi)} >_{u,a}, k(\xi)/n^q) < \eta_u > \big].$$

For any $A \in \mathcal{X}$, we set

$$\phi_0(A) = \prod_{\xi \in \mathcal{D}} \phi_0(\xi)^{A(\xi)}, \quad \phi_1(A) = \prod_{\xi \in \mathcal{D}} \phi_1(\xi)^{A(\xi)}, \quad \phi_2(A) = \prod_{\xi \in \mathcal{D}} \phi_2(\xi)^{A(\xi)}.$$

Note that ϕ_0, ϕ_1 and ϕ_2 are multiplicative functions. If $\alpha^T(A) \neq 0$, then

$$\phi_1(A) = \exp\{\beta_3 f_1(A, k(A)/n^q)\}, \quad \phi_2(A) = \exp\left\{\frac{\beta_4}{\sqrt{n}} f_2(A, k(A)/n^q)\right\},$$

Proposition 2.6 (1) *There exists $\beta_0 > 0$ such that for any $\beta_1 > \beta_0$, we have*

$$\sum_{\substack{A \in \mathcal{X} \\ 0 \in b(A)}} |\alpha^T(A)| \frac{\phi_0(A)\phi_1(A)\phi_2(A)}{A!} \leq \frac{\theta(\beta_1)}{1 - \theta(\beta_1)}. \tag{2.15}$$

(2) *For any* $0 < c < 1$ *and any* $\beta_1 > \beta_0/(1-c)$, *we have*

$$\sum_{\substack{A\in\mathcal{X} \\ 0\in b(A),|A|_2\geq k}} |\alpha^T(A)|\frac{\phi_0(A)\phi_1(A)\phi_2(A)}{A!} \leq \frac{\theta((1-c)\beta_1)}{1-\theta((1-c)\beta_1)}\mathrm{e}^{-c\beta_1 k}. \tag{2.16}$$

where

$$|A|_2 = \sum_{\xi=(\eta,b(\xi))\in\mathcal{D}} \sum_{u\in b(\xi)} |\eta_u|^2 A(\xi).$$

(3) *For any multiplicative function* $\varphi \in \mathcal{L}$ *and any* $\beta_1 > \beta_0$,

(2.17)

$$\sum_{A\in\mathcal{X}} \alpha(A)\frac{\phi_0(A)\phi_1(A)\phi_2(A)\varphi(A)}{A!} = \exp\Big[\sum_{A\in\mathcal{X}} \alpha^T(A)\frac{\phi_0(A)\phi_1(A)\phi_2(A)\varphi(A)}{A!}\Big]$$
$$= \exp\Big[\sum_{A\in\mathcal{X}} \alpha^T(A)\frac{\phi_0(A)\varphi(A)}{A!}\mathrm{e}^{\beta_3 f_1(A,k(A)/n^q)}\mathrm{e}^{\beta_4 f_2(A,k(A)/n^q)/\sqrt{n}}\Big].$$

Especially,

$$\tilde{Z}_n^{\hat{\omega}} = \exp\Big[\sum_{A\in\mathcal{X}} \alpha^T(A)\frac{\phi_0(A)\phi_1(A)\phi_2(A)}{A!}\Big]. \tag{2.18}$$

3 Statement of Result

Let us denote that

$$W_u = \sum_{l=1}^{u} < \omega_l >, \qquad \hat{W}_u = \sum_{l=1}^{u} < \hat{\omega}_l > .$$

Then by (2.11), the stock price process is describe as $S_u = S_0\mathrm{e}^{c_0(W_u+\hat{W}_u)}$.

For every $t\in(0,1]$ there exists a unique $k = k(n,t) = 1,\dots,n^q+1$ such that $[nt] \in I_n^{\hat{\omega}}(k(n,t))$. A scaled process $\{W_t^{(n)}\}_{t\in[0,1]}$ of $\{W_u\}_{u=1}^n$ is defined as follows: Set $W_0^{(n)} = 0$. For $t\in(0,1]$

$$W_t^{(n)} = \begin{cases} W_{U_{k(n,t)}}, & \text{if } k(n,t)\leq n^q, \\ W_{U_{n^q}+n^{1-\lambda}}, & \text{otherwise.} \end{cases}$$

A scaled process $\{\hat{W}_t^{(n)}\}_{t\in[0,1]}$ of $\{\hat{W}_u\}_{u=1}^n$ is defined in a similar way. And let

$$X_t^{(n)} = \frac{1}{\sqrt{n}}(W_t^{(n)} + \hat{W}_t^{(n)}).$$

For any $t\in[0,\tau]$, we define $s(t)\in[0,1]$ as

$$\int_0^{s(t)} \frac{1}{f_3(v)}\mathrm{d}v = t.$$

Since $f_3(t)$ is a positive function, $s(t)$ is well-defined. Then we have $s'(t)/f_3(s(t)) = 1$, thus we obtain

$$s(t) = \int_0^t f_3(s(x))\mathrm{d}x + C.$$

Since $s(0) = 0$, we see that C is zero. Hence

$$s(t) = \int_0^t f_3(s(x))\mathrm{d}x. \tag{3.1}$$

From the definition of $s(t)$,

$$\sum_{u=1}^{s(t)n^q} \frac{1}{f_3(u/n^q)} \frac{1}{n^q} = t + o(1), \quad \text{as } n \to \infty.$$

On the other hand, for any $t \leq \tau$,

$$\sum_{u=1}^{k(n,t)} \frac{n^{\delta+1/2}}{f_3(u/n^q)} - k(n,t)n^{\delta/2+1/2+\varepsilon} \leq [nt] \leq \sum_{u=1}^{k(n,t)} \frac{n^{\delta+1/2}}{f_3(u/n^q)} + k(n,t)n^{\delta/2+1/2+\varepsilon}$$

Divide by n, since $k(n,t) \leq n^q$ and $\varepsilon < \delta/2$,

$$\sum_{u=1}^{k(n,t)} \frac{1}{f_3(u/n^q)} \frac{1}{n^q} = t + o(1).$$

Since $f_3(t) > 0$, there is a constant $c > 0$ such that

$$o(1) = \frac{1}{n^q} \left| \sum_{u=1}^{k(n,t)} \frac{1}{f_3(u/n^q)} - \sum_{u=1}^{s(t)n^q} \frac{1}{f_3(u/n^q)} \right| \geq \frac{c}{n^q} |k(n,t) - s(t)n^q|.$$

Hence we have

$$k(n,t) = s(t)n^q + o(n^q). \tag{3.2}$$

We denote the trend terms and volatility term of limit price process as follow:

$$\mu_A(t) = \beta_4 \sum_{i(A)=0} < A > f_2(A, s(t))\phi_0(A)\mathrm{e}^{\beta_3 f_1(A,s(t))} \frac{\alpha^T(A)}{A!} f_3(s(t)), \tag{3.3}$$

$$\mu_B(t) = f_3(s(t)), \tag{3.4}$$

$$\sigma_A^2(t) = \sum_{i(A)=0} < A >^2 \phi_0(A)\mathrm{e}^{\beta_3 f_1(A,s(t))} \frac{\alpha^T(A)}{A!} f_3(s(t)). \tag{3.5}$$

Theorem 3.1 *For $\frac{3}{8} < \delta < \frac{1}{2}$, $\frac{1}{4} < \lambda < \frac{1}{2}$ and $q = \frac{1}{2} - \delta$, the process $X_t^{(n)}$ converges in finite dimensional distribution to the process*

$$X_t = \int_0^t (\mu_A(v) + \mu_B(v))\mathrm{d}v + \int_0^t \sigma_A^2(v)\mathrm{d}B_v + h1_{\{t=\tau\}}, \quad \textit{for all } t \in [0, \tau], \tag{3.6}$$

where B_t is a standard Brownian motion.

Let $\{\tau_i : i = 1, 2, \ldots\}$ be i.i.d. sequence of exponential holding times with mean $1/c$, and we write $\tau_0 = 0$. When $\tau_i \leq 1$, the stock price is continuous on each random interval (τ_{i-1}, τ_i), and it jumps at each random time τ_i, and jumps are i.i.d. with distribution ρ. The stock price process on $(\tau_{i-1}, \tau_i]$ behaves just like on $[0, \tau_1]$. Then by using the same argument in the proof of Theorem 3.1 repeatedly, we will obtain the following.

Theorem 3.2 *The scaled process $X_t^{(n)}$ converges in finite dimensional distribution to the process*

$$X_t = \int_0^t (\mu_A(v) + \mu_B(v))\mathrm{d}v + \int_0^t \sigma_A^2(v)\mathrm{d}B_v + Y_t, \quad \text{for all } t \in [0, 1],$$

where the jump term Y_t is a compound Poisson process, that is

$$Y_t = \int_{[0,t]} \int_{(-\infty,\infty)\setminus\{0\}} x N_p(\mathrm{d}s\mathrm{d}x),$$

where $N_p(\mathrm{d}s\mathrm{d}x)$ is a Poisson random measure. The Lévy measure of Y_t is $\mu(\mathrm{d}x) = c\rho(\mathrm{d}x)$ with $c > 0$ and $\rho((-\infty, \infty)) = 1$.

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