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Remark on homotopy types of twisted complex projective spaces

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1 Introduction.

The main purpose of this note is to announce the recent results given in the preprints ([5], [11]) and is to consider the remaining several related unsolved problems. Let $m \geq 0$ and $n \geq 2$ be integers and let M be a simply-connected $2n$ dimensional finite Poincaré complex. Then it is called an m-twisted $\mathbb{C}\mathrm{P}^n$ if there is an isomorphism $H_*(M,\mathbb{Z}) \cong H_*(\mathbb{C}\mathrm{P}^n,\mathbb{Z})$ and $x_1 \cdot x_1 = mx_2$, where $x_k \in H^{2k}(M,\mathbb{Z}) \cong \mathbb{Z}$ $(k = 1,2)$ denotes the generator. If M is an m-twisted $\mathbb{C}\mathrm{P}^n$, it has the homotopy type of the form

$$(1) \qquad M \simeq S^2 \cup_{m\eta_2} e^4 \cup e^6 \cup \cdots \cup e^{2n-2} \cup e^{2n}.$$

We denote by $\mathcal{M}_m^n$ the set consisting of all homotopy equivalence classes of m-twisted $\mathbb{C}\mathrm{P}^n$'s. If $n = 2$, it is easy to see that $\mathcal{M}_1^2 = \{[\mathbb{C}\mathrm{P}^2]\}$ and $\mathcal{M}_m^2 = \emptyset$ if $m \neq 1$. If $n = 3$, it is known in [9] that $\mathrm{card}(\mathcal{M}_m^3) = 2 + (-1)^m$, where $\mathrm{card}(V)$ denotes the number of a finite set V. For example, if $m = 0$ or 1, then $\mathcal{M}_1^3 = \{[\mathbb{C}\mathrm{P}^3]\}$ and $\mathcal{M}_0^3 = \{[M_0],[M_1],[M_2]\}$, where $i_k : S^k \to S^2 \vee S^4$ denotes the inclusion $(k = 2,4)$ and we take $M_0 = S^2 \times S^4 = S^2 \vee S^4 \cup_{[i_2.i_4]} e^6$, $M_1 = S^2 \vee S^4 \cup_{i_4 \circ \eta_4 + [i_2,i_4]} e^6$ and $m_2 = S^2 \vee S^4 \cup_{i_2 \circ \eta_2^3 + [i_2,i_4]} e^6$.

In general, we can show that $\mathcal{M}_m^n \neq \emptyset$ for any $m \geq 0$ when $n \geq 5$ is an odd integer, which is shown by using a technique of the theory of transformation groups (cf. [1]). So it seems interesting to study the set $\mathcal{M}_m^n$ when $n \geq 4$ is an even integer. More precisely, we consider the following:

Problem. *Let $n \geq 4$ and $m \geq 0$ be integers.*

(a) *Then is the set $\mathcal{M}_m^n$ an emptyset or not? Moreover, if $\mathcal{M}_m^n \neq \emptyset$, can we determine the number* $\mathrm{card}(\mathcal{M}_m^n)$ *and representatives of $\mathcal{M}_m^4$?*

(b) *Let M be an m-twisted $\mathbb{C}P^n$. Then does it has the homotopy type of closed smooth manifolds of dimension $2n$?*

The precise statement of this paper is as follows.

Theorem 1.1. *Let $m \geq 0$ be an integer and let (a, b) denote the greatest common divisor of positive integers a, b.*

(i) *If $m \equiv 1 \pmod 2$, $\mathrm{card}(\mathcal{M}_m^4) = (m, 3)$.*

(ii) *If $m \equiv 0 \pmod 2$ and and it is not divisible by 8, $\mathcal{M}_m^4 = \emptyset$.*

(iii) *If $m \equiv 0 \pmod 8$ and $m \neq 0$, $\mathcal{M}_m^4 \neq \emptyset$ and its number is estimated as $3 \leq \mathrm{card}(\mathcal{M}_m^4) \leq 2^5 \cdot 3 \cdot m(m, 3)$.*

(iv) *In particular, if $m = 0$, $\mathcal{M}_0^4 \neq \emptyset$ and its number is estimated as $3 \leq \mathrm{card}(\mathcal{M}_m^4) \leq 2^7 \cdot 3^2$.*

Theorem 1.2. *Let $m \geq 0$, $n \geq 2$ be integers and let M be an m-twisted $\mathbb{C}P^n$. Then it has the homotopy type of topological manifolds of dimension $2n$. In particular, if $n = 4$, then it also has the homotopy type of* PL-*manifolds of dimension* 8.

2 Homotopy groups

In this section we shall give the rough idea of the proof of Theorem 1.1.

For each integer $m \geq 0$, we denote by L_m the CW complex defined by $L_m = S^2 \cup_{m\eta_2} e^4$. Then we recall the following:

Lemma 2.1. *Let $m \geq 0$ be an integer.*

(i) $$\pi_5(L_m) = \begin{cases} \mathbb{Z} \cdot b_m & \text{if } m \equiv 1 \pmod 2, \\ \mathbb{Z} \cdot b_m \oplus \mathbb{Z}/4 \cdot \gamma_m & \text{if } m \equiv 2 \pmod 4, \\ \mathbb{Z} \cdot b_m \oplus \mathbb{Z}/2 \cdot \gamma_m \oplus \mathbb{Z}/2 \cdot i_*(\eta_2^3) & \text{if } m \equiv 0 \pmod 4, \end{cases}$$

where we take $b_m = [i, i_4]$ and $\gamma_m = i_4 \circ \eta_4$ if $m = 0$, and $2\gamma_m = i_*(\eta_2^3)$ if $m \equiv 2 \pmod 4$.

(ii) Let M be an m-twisted $\mathbb{C}P^4$ and $M^{(6)}$ denote its 6-skelton. Then there is a homotopy equivalence

$$M^{(6)} \simeq \begin{cases} X_m & \text{if } m \equiv 1 \pmod 2 \\ V_m & \text{if } m \equiv 0 \pmod 2, \ V \in \{X, Y\} \end{cases}$$

where we take $X_m = L_m \cup_{mb_m} e^6$ and $Y_m = L_m \cup_{mb_m + i_*(\eta_2)} e^6$.

Proof. This can be proved using standard computation of homotopy groups and the method given in [9]. □

Lemma 2.2. *Let $j_{1*} : \pi_7(X_m) \to \pi_7(X_m, L_m)$ denote the induced homomorphism.*

(i) If $m \equiv 1 \pmod 2$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_{1*}(\varphi_m) = [\beta_m, i]_r + \epsilon_m \cdot \beta_m \circ \eta_5'$, and there is an isomorphism

$$\pi_7(X_m) = \mathbb{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \oplus \mathbb{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbb{Z} \cdot \varphi_m.$$

(ii) If $m \equiv 0 \pmod 8$ and $m \neq 0$, there exists some element $\varphi_m \in \pi_7(X_m)$ such that, $j_{1*}(\varphi_m) = [\beta_m, i]_r$, and there is an isomorphism

$$\begin{aligned} \pi_7(X_m) &= \mathbb{Z} \cdot \varphi_m \oplus \mathbb{Z}/4 \cdot j_*(f_m \circ \widetilde{\nu'}) \oplus \mathbb{Z}/2 \cdot j_*(f_m \circ \sigma \circ \eta_6) \\ &\oplus \mathbb{Z}/2 \cdot (j \circ i)_*(\eta_2 \circ \omega \circ \eta_6) \oplus \mathbb{Z}/(m,3) \cdot j_*(f_m \circ \omega_m) \\ &\oplus \mathbb{Z}/2 \cdot j_*(b_m \circ \eta_5^2) \oplus \mathbb{Z}/m \cdot j_*([b_m, i_*(\eta_2)]) \oplus \mathbb{Z}/2 \cdot \tilde{\eta}_5. \end{aligned}$$

(iii) If $m = 0$, then $X_0 = S^2 \vee S^4 \vee S^6$ and there is an isomorphism

$$\begin{aligned} \pi_7(X_0) &= \mathbb{Z} \cdot j_4 \circ \nu_4 \oplus \mathbb{Z} \cdot [j_2, j_6] \oplus \mathbb{Z}/2 \cdot j_2 \circ \eta_2 \circ \omega \circ \eta_6 \oplus \mathbb{Z}/2 \cdot j_6 \circ \eta_6 \\ &\oplus \mathbb{Z}/12 \cdot j_4 \circ E\omega \oplus \mathbb{Z}/2 \cdot [j_2, j_4 \circ \eta_4^2] \oplus \mathbb{Z}/2 \cdot [j_2 \circ \eta_2, j_4 \circ \eta_4] \\ &\oplus \mathbb{Z}/2 \cdot [j_2 \circ \eta_2^2, j_4], \end{aligned}$$

where $j_k : S^k \to S^2 \vee S^4 \vee S^6$ $(k = 2, 4, 6)$ denote the corresponding inclusions.

Proof. The proof is given using standard computations of homotopy groups. □

Similarly we obtain:

Lemma 2.3. *Let $m \geq 0$ be an integer with $m \equiv 0 \pmod 8$, and let $j_2 : \pi_7(Y_m) \longrightarrow \pi_7(Y_m, L_m)$ be the induced homomorphism. Then there exists some element $\varphi'_m \in \pi_7(Y_m)$ such that, $j_{2*}(\varphi'_m) = [\beta'_m, i]_r$, and there is an isomorphism*

$$\begin{aligned}\pi_7(Y_m) &= \mathbb{Z}\cdot\varphi'_m \oplus \mathbb{Z}/4\cdot j'_*(f_m\circ\widetilde{\nu'}) \oplus \mathbb{Z}/2\cdot j'_*(f_m\circ\sigma\circ\eta_6)\\ &\oplus\mathbb{Z}/2\cdot j'_*(i_*(\eta_2\circ\omega\circ\eta_6)) \oplus \mathbb{Z}/(m,3)\cdot j'_*(f_m\circ\omega_m)\\ &\oplus\mathbb{Z}/2\cdot j'_*(b_m\circ\eta_5^2) \oplus \mathbb{Z}/m\cdot j'_*([b_m, i_*(\eta_2)]) \qquad \text{if } m\neq 0,\end{aligned}$$

$$\begin{aligned}\pi_7(Y_0) &= \mathbb{Z}\cdot j'_*(i_4\circ\nu_4) \oplus \mathbb{Z}\cdot\varphi'_0 \oplus \mathbb{Z}/2\cdot j'_*([i, i_4\circ\eta_4^2]) \oplus \mathbb{Z}/12\cdot j'_*(i_4\circ E\omega)\\ &\oplus\mathbb{Z}/2\cdot j'_*([i_*(\eta_2), i_4\circ\eta_4]) \oplus \mathbb{Z}/2\cdot j'_*([i_*(\eta_2^2), i_4])\\ &\oplus\mathbb{Z}/2\cdot j'_*(\eta_2\circ\omega\circ\eta_6) \qquad \text{if } m=0.\end{aligned}$$

Sketch proof of Theorem 1.1. If we use some lemmas given in [10] concerning the relation between cup-products and relative Whitehead products, we can show the desired assertions. □

3 Surgery obstructions

First, we shall give rough idea of the proof of Theorem 1.2.

Sketch proof of Theorem 1.2. Since M is a finite Poicaré complex, it follows from Theorem of Spivak that there is a spherical fiber space over M with fber S^N (N: suuficiently large). Then by using the result of Stasheff, it is classified by the map $f_M : M \longrightarrow BSG$. Let us consider whether it lifts to $B\mathrm{STop}$ or not. Its obstructions lie in $H^k(M, \pi_{k-1}(SG/\mathrm{STop}))$ for all $k \geq 1$. However, since $\pi_j(SG/\mathrm{STop}) = 0$ and $H^j(M) = 0$ if $j \equiv 1 \pmod 2$, all obstructions vanish. Hence, the map f_m lifts to $B\mathrm{STop}$. If we recall Theorem of the type of Browder-Novikov [6], we can show that M has the homotopy type of topological manifolds of dimension $2n$.

Because $\pi_{2k-1}(G/O) = 0$ for $1 \leq k \leq 4$, if $n = 4$ the map f_m lifts to BSO and it follows from the Browder-Novokov type Theorem ([4], Corollary 2.17) that M has the homotopy type of PL-manifolds of dimension 8. □

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