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The microlocal Riemann-Hilbert problem on a complex contact manifold

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1 Introduction

On a complex analytic manifold X, let us denote as usual by $\mathcal{O}_X$ the sheaf of holomorphic functions and $\mathcal{D}_X$ the sheaf of linear differential operators. Recall that to any coherent $\mathcal{D}_X$-module $\mathcal{M}$ is associated its characteristic variety char$(\mathcal{M})$ which is an involutif subset of T^*X. In the most simple case where $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \cdot P$ for some $P \in \mathcal{D}_X$ the characteristic variety is just given by the set of zeros of the principal symbol of the operator P. A $\mathcal{D}_X$-module whose characteristic variety is Lagrangian is called a holonomic system. A very basic example is the module associated to an operator of Fuchs type (as for instance the operator $z\frac{d}{dz} - \alpha$ on $X = \mathbb{C}$, $\alpha \in \mathbb{C}$).

The solution functor

$$\mathrm{Sol} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\cdot, \mathcal{O}_X) : \ \mathcal{M}od(\mathcal{D}_X) \longrightarrow \mathrm{D^b}(k_X)$$

sends holonomic systems to complexes with complex constructible cohomology, *i.e.* objects in the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$[1] and more precisely in the category of perverse sheaves[2] $Perv(\mathbb{C}_X)$.

The classical Riemann-Hilbert theorem states that there is an abelian subcategory of the category of holonomic systems $\mathcal{H}ol(\mathcal{D}_X)$, called the category of regular holonomic modules $\mathcal{R}eg\mathcal{H}ol(\mathcal{D}_X)$[3], such that the solution functor induces an equivalence of this category with the category of perverse sheaves. An explicit quasi-inverse has been constructed by Kashiwara in [K1], and the Riemann-Hilbert correspondence can be summarized in the following diagram of quasi-inverse equivalences of categories:

$$Perv(\mathbb{C}_X) \underset{\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\mathcal{O}_X)}{\overset{\mathrm{R}\mathcal{H}om(\,\cdot\,,\mathcal{O}^t)}{\rightleftarrows}} \mathcal{R}eg\mathcal{H}ol(\mathcal{D}_X).$$

Here the "ring" $\mathcal{O}^t \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}(\mathbb{C}_X))$ is the complex of tempered holomorphic functions in the bounded derived category of ind-sheaves on X which has been constructed in [KS2], and $\mathrm{R}\mathcal{H}om(\,\cdot\,,\mathcal{O}^t)$ is isomorphic to Kashiwara's functor of tempered solutions (see [K1] where it is denoted by $\mathrm{T}\mathcal{H}om$). These equivalences can be extended to equivalences of abelian stacks on X.

In [W1],[W2] it is shown that the Riemann-Hilbert correspondence can be microlocalized. Set $\dot{T}^*X = T^*X \setminus T^*X_X$ and let $P^*X = \dot{T}^*X/\mathbb{C}^*$ be the projective cotangent bundle. We fix the notations for the canonical maps by the commutative diagram of spaces

$$\begin{array}{ccc} T^*X & \longleftarrow & \dot{T}^*X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \gamma} \\ X & \underset{\pi}{\longleftarrow} & P^*X. \end{array}$$

Then the microlocal Riemann-Hilbert correspondence (on P^*X) can be summarized by the commutative diagram

$$\begin{array}{ccc} Perv(\mathbb{C}_X) & \underset{\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\mathcal{O}_X)}{\overset{\mathrm{R}\mathcal{H}om(\,\cdot\,,\mathcal{O}^t)}{\rightleftarrows}} & \mathcal{R}eg\mathcal{H}ol(\mathcal{D}_X) \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes}} \\ \mu Perv(\mathbb{C}_X) & \underset{\mathrm{R}\mathcal{I}\mathcal{H}om_{\beta\mathcal{E}_X}(\beta(\,\cdot\,),\mu\mathcal{O}_X)}{\overset{\gamma^{-1}\mathrm{R}\gamma_*\,\mathrm{R}\mathcal{H}om(\,\cdot\,,\mu\mathcal{O}^t)}{\rightleftarrows}} & \mathcal{R}eg\mathcal{H}ol(\mathcal{E}_X). \end{array}$$

Here $\mathcal{E}_X$ denotes the ring of microdifferential operators from [SKK], $\mathcal{R}eg\mathcal{H}ol(\mathcal{E}_X)$ is the category

[1]For any field k, one denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ the full subcategory of the bounded derived category of sheaves of k-vector spaces $\mathrm{D}^{\mathrm{b}}(k_X)$ formed by complexes with complex constructible cohomology, *i.e.* with cohomology sheaves that are locally constant with respect to some complex analytic stratification of X.

[2]The category $Perv(k_X)$ of perverse sheaves is defined as the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ of complexes $\mathcal{F}$ such that $\dim(\mathrm{supp}(H^i(\mathcal{F}))) \leqslant -i$ and $\dim(\mathrm{supp}(H^i(\mathrm{D}\,\mathcal{F}))) \leqslant -i$ where $\mathrm{D}\,\mathcal{F}$ is the Verdier dual of $\mathcal{F}$, i.e. $\mathrm{D}\,\mathcal{F} = \mathrm{R}\mathcal{H}om(\mathcal{F}, \omega_X)$. See [BBD] for more details.

[3]For a detailed study of regular holonomic modules we refer to [KK]. They are generalizations of modules defined by operators of Fuchs type.

of regular holonomic microdifferential modules[4], $\mu Perv(\mathbb{C}_X)$ the category of microlocal perverse sheaves (cf. [W1] and section 2 below) and μ is the functor of ind-microlocalization (cf. [KSIW] and section 3 below). The horizontal functors in the second line are quasi-inverse equivalences of categories, they define the microlocal Riemann-Hilbert correspondence and can be extended to equivalences of abelian stacks (on P^*X).

Now let Ξ be a complex contact manifold. Then Ξ can be covered by open subsets U which are contact isomorphic to open subsets of the projective bundle P^*X of some complex analytic manifold X. But although the ring $\mathcal{E}_X$ of microdifferential operators is invariant by contact transformations (see section 5) it cannot be glued to a ring $\mathcal{E}_\Xi$. However, in 1996, Kashiwara ([K2]) solved the quantization problem for such manifolds, *i.e.* he constructed the stack of microdifferential modules $\mathcal{M}od(\mathcal{E}, \Xi)$ on Ξ, a canonical stack that is locally equivalent to $\mathcal{M}od(\mathcal{E}_X)$. The stack of microdifferential modules contains the substack of regular holonomic modules, and it was then natural to ask for the stack of microlocal perverse sheaves on Ξ as well as a microlocal Riemann-Hilbert correrspondence on Ξ.

In this note we give a short overview of the microlocal Riemann-Hilbert correspondence on P^*X and a sketch how this construction can be extended to a complex contact manifold Ξ.

2 Microlocal perverse sheaves on P^*X

Let us briefly review the construction of microlocal perverse sheaves on the projective bundle P^*X from [W1]. There have been essentially two earlier attempts to construct these objects, the first by late Andronikof ([An1],[An2]), and a second later by Gelfand-MacPherson-Vilonen [GMV], but to our knowledge, both projects were never completed. The approach we present here is more closely related to Andronikof's work.

Recall that to any $\mathcal{F} \in \mathrm{D^b}(k_X)$ on a real manifold X one can associate its micro-support $\mathrm{SS}(\mathcal{F})$ which is a closed, $\mathbb{R}^+$-conic, involutif subset of the cotangent bundle (cf. [KS1]). Then for any subset $S \subset T^*X$ one defines the microlocalization of $\mathrm{D^b}(k_X)$ on S as the localization of $\mathrm{D^b}(k_X)$ by the triangulated subcategory of sheaves $\mathcal{F}$ such that $\mathrm{SS}(\mathcal{F}) \cap S = \varnothing$. This category will be denoted by $\mathrm{D^b}(k_X, S)$. It is again triangulated, with the same objects as $\mathrm{D^b}(k_X)$, such that any morphism $\mathcal{F} \to \mathcal{G}$ of $\mathrm{D^b}(k_X)$ that can be embedded into an exact triangle

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+}$$

with $\mathrm{SS}(\mathcal{H}) \cap S = \varnothing$ becomes an isomorphism in $\mathrm{D^b}(k_X, S)$. Similarly, we define $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X, S)$ (cf. [KS1]), the microlocalization of the derived category of sheaves with real constructible cohomology, *i.e.* complexes whose cohomology sheaves are locally constant with respect to some subanalytic stratification of X.

In general it is quite hard to compare $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X, S)$ and $\mathrm{D^b}(k_X, S)$ and little is known, but one can prove (cf. [KS1] and [W1]):

Proposition 2.1. *Let $p \in \dot{T}^*X$ and $S = \{p\}$ or $S = \mathbb{C}^*p$. Then the natural functor*

$$\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X, S) \longrightarrow \mathrm{D^b}(k_X, S)$$

[4]See [SKK] and [KK]. A holonomoic $\mathcal{E}_X$-module $\mathcal{M}$ is regular holonomic if locally there exists a coherent $\mathcal{E}(0)$-module $\mathcal{L} \subset \mathcal{M}$ such that $\mathcal{M} = \mathcal{E}_X\mathcal{L}$ and for every homogenous holomorphic function $f \in \mathcal{O}_{\dot{T}^*X}(0)$ with $f|_{\mathrm{supp}(\mathcal{M})} = 0$ we have $f(\mathcal{L}/\mathcal{E}(-1)\mathcal{L}) = 0$. A $\mathcal{D}_X$-module is regular holonomic if and only if its associated $\mathcal{E}_X$-module is regular holonomic

is an embedding.

Recall from [KS1] that the micro-support of an $\mathbb{R}$-constructible sheaf is Lagrangian and moreover that we have the following theorem

Theorem 2.1. *Let* $\mathcal{F} \in D^b_{\mathbb{R}\text{-}c}(k_X)$. *Then*

(i) $\mathcal{F} \in D^b_{\mathbb{C}\text{-}c}(k_X)$ *if and only if* $SS(\mathcal{F})$ *is* $\mathbb{C}^*$*-conic.*

(ii) $\mathcal{F} \in Perv(k_X)$ *if and only if* $\mathcal{F} \in D^b_{\mathbb{C}\text{-}c}(k_X)$ *and for any smooth point* $p \in SS(\mathcal{F})$ *such that* $SS(\mathcal{F}) \to X$ *is of constant rank in a neighborhood of* p *there is a closed submanifold* $Y \subset X$, *and a vector space* M *such that* $\mathcal{F} \simeq M_Y[d_Y]$ *in* $D^b(k_X, \mathbb{C}^*p)$.

Therefore one can define $D^b_{\mathbb{C}\text{-}c}(k_X, S)$ (resp. $D^b_{perv}(k_X, S)$) as the full subcategory of $D^b_{\mathbb{R}\text{-}c}(k_X, S)$ such that $SS(\mathcal{F})$ is $\mathbb{C}^*$-conic on S (resp. $\mathbb{C}^*$-conic on S and for any smooth point $p \in SS(\mathcal{F}) \cap S$ such that $SS(\mathcal{F}) \to X$ is of constant rank in a neighborhood of p there is a closed submanifold $Y \subset X$, and a vector space M such that $\mathcal{F} \simeq M_Y[d_Y]$ in $D^b(k_X, \mathbb{C}^*p)$).

The categories $D^b_{perv}(k_x, U)$ (where $U \subset \dot{T}^*X$ is a conic open subset) clearly define a prestack on the projective bundle P^*X, and we define (see [W1])

Definition 2.1. *The stack* $\mu\mathrm{Perv}(k_X)$ *of microlocal perverse sheaves is the stack associated to the prestack* $U \mapsto D^b_{perv}(k_X, U)$

Using the theory of ind-sheaves of [KS2] and the functor μ of [KSIW], a more explicit description of the stack $\mu\mathrm{Perv}(k_X)$ as a substack of the derived category of ind-sheaves can be obtained (see section 4).

3 Ind-Sheaves and Microlocalization

We need just some very basic definitions and constructions from the theory of analytic ind-sheaves of [KS2]. If $\mathcal{C}$ is a category, one embeds $\mathcal{C}$ into the category of presheaves (of sets) on $\mathcal{C}$ by the fully faithful Yoneda-functor:

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}} \quad ; \quad A \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, A)$$

where $\widehat{\mathcal{C}}$ is the category of contravariant functors $\mathcal{C} \to \mathcal{S}et$. An object in the essential image of the Yoneda-functor is called representable. Note that $\widehat{\mathcal{C}}$ admits all small colimits since the category $\mathcal{S}et$ does but even if $\mathcal{C}$ admits colimits the Yoneda-functor does not commute with them.

One denotes by $\mathrm{Ind}\,\mathcal{C}$ the full subcategory of $\widehat{\mathcal{C}}$ formed by small filtered colimits of representable objects and calls it the category of ind-objects of $\mathcal{C}$. Then $\mathrm{Ind}\,\mathcal{C}$ admits all small filtered colimits. If $\mathcal{C}$ is abelian then $\mathrm{Ind}\,\mathcal{C}$ is abelian and the Yoneda-functor induces an exact fully faithful functor $\mathcal{C} \to \mathrm{Ind}\,\mathcal{C}$.

Now let X be a locally compact topological space with a countable base of open sets and fix a field k. One sets

$$\mathrm{I}(k_X) = \mathrm{Ind}\,\mathcal{M}od^c(k_X)$$

where $\mathcal{M}od^c(k_X)$ denotes the full subcategory of $\mathcal{M}od(k_X)$ formed by sheaves with compact support. We call $\mathrm{I}(k_X)$ the category of ind-sheaves (of k-vector spaces). One can show that the prestack $X \supset U \mapsto \mathrm{I}(k_U)$ is a proper stack (in the sens of [KS2]), in particular it is an abelian stack.

There are three important basic functors for ind-sheaves

$$\begin{array}{rcll} \iota & : & \mathcal{M}od(k_X) \longrightarrow \mathrm{I}(k_X) & ; \quad \mathcal{F} \mapsto \text{“}\varinjlim_{U\subset\subset X}\text{”}\, \mathcal{F}_U \\ \alpha & : & \mathrm{I}(k_X) \longrightarrow \mathcal{M}od(k_X) & ; \quad \text{“}\varinjlim_{i\in I}\text{”}\, \mathcal{F}_i \mapsto \varinjlim_{i\in I} \mathcal{F}_i \\ \beta & : & \mathcal{M}od(k_X) \longrightarrow \mathrm{I}(k_X) & \quad \text{left adjoint to } \alpha \end{array}$$

where we write “$\varinjlim$” for colimits in the category $\mathrm{I}(k_X)$. All three functors induce functors of stacks.

Proposition 3.1. *(i) The functor ι is fully faithful and exact.*

(ii) The functor α is exact.

(iii) The functor β is fully faithful and exact.

(iv) The triple (β, α, ι) is a triple of adjoint functors, i.e. β is left adjoint to α and α is left adjoint to ι.

Note that since the functors ι, α, β are exact they are well-defined in the derived categories, guard the adjoint properties and β, ι are still fully faithful. An object $\mathcal{F} \in \mathrm{D^b}(k_X)$ is identified with $\iota\mathcal{F}$ in $\mathrm{D^b}(\mathrm{I}(k_X))$.

There are internal operations on ind-sheaves

$$(\,\cdot\,) \otimes (\,\cdot\,) \qquad \text{and} \qquad \mathcal{IH}om(\,\cdot\,,\,\cdot\,)$$

and an external

$$\mathcal{H}om(\,\cdot\,,\,\cdot\,) :\ \mathrm{I}(k_X) \times \mathrm{I}(k_X) \longrightarrow \mathcal{M}od(k_X).$$

Moreover for any continuous map $f : X \to Y$ between locally compact spaces we get the external operations

$$f^{-1}, f_*, f_{!!},$$

where the notation $f_{!!}$ indicates that $\iota f_! \not\simeq f_{!!}\iota$.

While $\otimes$ and f^{-1} are exact the other functors have a right derived functor and pass to the derived category where we can define Poincaré-Verdier duality, i.e. we have a right adjoint $f^!$ to $\mathrm{R}f_{!!}$ and we get the usual formalism of Grothendieck's six operations. We will not recall here the various natural isomorphisms relating these functors and refer to [KS2] but let us summarize the commutation properties with ι, α, β:

Proposition 3.2. *(i) The functor ι commutes to $\otimes, f^{-1}, f^!, \mathrm{R}f_*$.*

(ii) The functor α commutes to $\otimes, f^{-1}, \mathrm{R}f_*, \mathrm{R}f_{!!}$ and we have $\alpha\, \mathrm{R}\mathcal{IH}om(\,\cdot\,,\,\cdot\,) \simeq \mathrm{R}\mathcal{H}om(\,\cdot\,,\,\cdot\,)$.

(iii) The functor β commutes to $\otimes, f^{-1}$.

In [KSIW], a topological analogon of the microlocalization of a $\mathcal{D}_X$-module is constructed as a functor

$$\mu :\ \mathrm{D^b}(k_X) \longrightarrow \mathrm{D^b}(\mathrm{I}(k_{T^*X}))$$

with values in the derived category of ind-sheaves on T^*X satisfying

$$\mathrm{supp}(\mu(\mathcal{F})) = \mathrm{SS}(\mathcal{F}).$$

Let us briefly put this result into relation with $\mathcal{D}_X$-modules. Recall the microlocalization functor

$$\mathcal{M}od(\mathcal{D}_X) \longrightarrow \mathcal{M}od(\mathcal{E}_X) \quad ; \quad \mathcal{M} \mapsto \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}.$$

Take an operator $P \in \mathcal{D}_X$ and consider $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X \cdot P$. Then $\mathrm{char}(\mathcal{M}) = \sigma(P)^{-1}(0)$ where $\sigma(P)$ is the principal symbol of the operator P. Next recall that an operator $P \in \mathcal{D}_X$ becomes inveritble in $\mathcal{E}_X$ at a point p if and only if its principal symbol does not vanish at P, and then it is easy to see that $\mathrm{supp}(\mathcal{E}_X/\mathcal{E}_X \cdot P) = \sigma(P)^{-1}(0)$. More generally we get $\mathrm{supp}(\mathcal{E}_X \otimes \pi^{-1}\mathcal{M}) = \mathrm{char}(\mathcal{M})$ for any coherent $\mathcal{D}_X$-module. Hence, the microlocalization functor can be seen as a tool to replace a $\mathcal{D}_X$-module by an object supported on the characteristic variety, and the functor μ can be interpreted as the topological analogon of this procedure. The link between these two microlocalization is given by the solution functor which satisfies

$$\mathrm{SS}(\mathrm{Sol}(\mathcal{M})) \doteq \mathrm{char}(\mathcal{M}).$$

One can prove that we get the following nice commutative diagram

$$\begin{array}{ccc} \mathcal{M}od(\mathcal{D}_X) & \xrightarrow{\quad \mathrm{Sol} \quad} & \mathrm{D^b}(k_X) \\ {\scriptstyle \mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes}} \downarrow & & \downarrow {\scriptstyle \mu} \\ \mathcal{M}od(\mathcal{E}_X) & \xrightarrow[\quad \mu\,\mathrm{Sol} \quad]{} & \mathrm{D^b}(\mathrm{I}(k_X)) \end{array}$$

where we set

$$\mu\,\mathrm{Sol}(\mathcal{M}) = R\mathcal{I}\mathcal{H}om_{\beta(\mathcal{E}_X)}(\beta(\mathcal{M}), \mu\mathcal{O}_X).$$

For a $\mathcal{D}_X$-module $\mathcal{M}$ we obtain the important formula

$$\mu(\mathrm{Sol}(\mathcal{M})) = \mu\,\mathrm{Sol}(\mathcal{E}_X \otimes \pi^{-1}\mathcal{M}). \tag{1}$$

4 The microlocal Riemann-Hilbert correspondence on P^*X

Note that the functor μ indues

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, U) \longrightarrow \mathrm{D^b}(\mathrm{I}(k_X))|_U.$$

For a conic open subset $U \subset \dot{T}^*X$ set

$$\mu Perv(k_U) = \{\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(k_U)) \mid \forall p \in U\ \exists \tilde{\mathcal{F}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, V), p \in V \text{ such that } \mathcal{F} \mathrel{\dot{\simeq}} \mu\tilde{\mathcal{F}} \text{ on } V\}.$$

To see that this definition of microlocal perverse sheaves coincides with the definition as the stack associated to the prestack $U \mapsto \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, U)$, first note that by definition the functor μ induces

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, U) \longrightarrow \mu Perv(k_U).$$

Then one starts by showing that $U \mapsto \mu Perv(k_U)$ is a stack on X. Therefore μ induces a functor from the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}$ to $\mu Perv$. This functor induces equivalences of categories in the stalks and therefore is an equivalence of stacks (cf. [W1] for a detailed proof).

If $\mathcal{M}$ is a regular holonomic $\mathcal{E}_X$-module then one can show that $\mu\,\mathrm{Sol}(\mathcal{M})$ is a microlocal perverse sheaf. Note that this result is a direct consequence of the classical Riemann-Hilbert correspondence together with isomorphism (1) if $\mathcal{M}$ is coming from a $\mathcal{D}_X$-module. In general one can use the fact that a regular holonomic system in generic position always comes from a $\mathcal{D}_X$-module (see [KK]), and that by a quantized contact transformation (see section 5 below), we can reduce the problem to the generic case.

Finally one gets
Theorem 4.1. *The microlocal solution functor induces an equivalence of abelian stacks*

$$\mu\,\mathrm{Sol}:\ \mathcal{R}eg\mathcal{H}ol(\mathcal{E}_X) \longrightarrow \mu\mathcal{P}erv(\mathbb{C}_X).$$

The proof can be done be either looking at this functor in the stalks or by showing directly that the functor $\gamma^{-1}\,\mathrm{R}\gamma_*\,\mathrm{R}\mathcal{H}om(\,\cdot\,,\mu\mathcal{O}^t_X)$ is well-defined and quasi-inverse to $\mu\,\mathrm{Sol}$ (for details, see [W2]).

5 Invariant by quantized contact transformation

Microdifferential modules (as well as regular holonomic systems) are invariant under quantized contact transformation. Consider $U_X \subset \dot{T}^*X$, $U_Y \subset \dot{T}^*Y$. A $\mathbb{C}^*$-homogeneous symplectic isomorphism

$$\chi:\ U_X \xrightarrow{\sim} U_Y$$

is often called a contact transformation (although strictly speaking, the contact structures are defined on the projective bundles). Invariance under "quantized contact transformations" means that locally we can construct an equivalence of categories

$$\chi_*\mathcal{M}od(\mathcal{E}_X)|_{U_X} \longrightarrow \mathcal{M}od(\mathcal{E}_Y)|_{U_Y}.$$

This equivalence is actually defined by a ring isomorphism $\chi_*\mathcal{E}_X|_{U_X} \simeq \mathcal{E}_Y|_{U_Y}$, and it preserves regular holonomic modules.

Similarly, invariance under "quantized contact transformations" means for microlocal perverse sheaves that locally we can construct from χ an equivalence of categories

$$\Phi_{\mathcal{K}}:\ \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, U_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_Y, U_Y).$$

The equivalence $\Phi_{\mathcal{K}}$ is explicitly given by an integral transform and depends on the choice of a kernel $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(k_{Y\times X})$:
Theorem 5.1. *Let X, Y be two real manifolds, $U_X \subset \dot{T}^*X$, $U_Y \subset \dot{T}^*Y$ open subsets and*

$$\chi:\ U_X \longrightarrow U_Y$$

a complex contact transformation. Set

$$\Lambda = \Big\{((y;\eta),(x;\xi)) \in U_Y \times U_X^a \mid (y,\eta) = \chi(x,-\xi)\Big\}.$$

Let $p_X \in U_X$ and $p_Y = \chi(p_X)$.
There exist open neighborhoods X' of $\pi(p_X)$, Y' of $\pi(p_Y)$, U'_X of p_X, U'_Y of p_Y with $U'_X \subset T^*X' \cap U_X$, $U'_Y \subset T^*Y' \cap U_Y$ and a kernel $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(k_{Y'\times X'})$ such that:

(1) χ *induces a contact transformation* $U'_X \xrightarrow{\sim} U'_Y$,

(2)

$$\Big((U''_Y \times T^*X') \cup (T^*Y' \times U''^a_X)\Big) \cap \mathrm{SS}(\mathcal{K}) \subset \Lambda \cap (U''_Y \times U''^a_X)$$

for every open subsets $U''_X \subset U'_X$ *and* $U''_Y = \chi(U''_X)$,

(3) composition with $\mathcal{K}$ *induces an equivalence of prestacks*

$$\Phi_{\mathcal{K}} = \mathcal{K}\circ : \chi_* \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_{X'}, *)|_{U'_X} \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_{Y'}, *)|_{U'_Y},$$

a quasi-inverse being given by $\Phi_{\mathcal{K}^*}$ *with* $\mathcal{K}^* = r_* \mathrm{R}\mathcal{H}om(\mathcal{K}, \omega_{Y\times X|X})$ *where* $r : Y \times X \to X \times Y$ *switches the factors.*

(4) $\mathrm{SS}(\Phi_{\mathcal{K}}(\mathcal{F})) \cap U''_Y = \chi(\mathrm{SS}(\mathcal{F}) \cap U''_X)$,

This equivalence passes of course to the associated stacks of microlocal perverse sheaves. Note that the kernel $\mathcal{K}$ of the theorem is not uniquely determined by χ, in fact any microlocal perverse sheaf of rank 1 supported on Λ will work. Therefore it is important to classify such kernels.

6 Microlocal perverse sheaves with smooth support

Recall that by the microlocal Riemann Hilbert theorem microlocal perverse sheaves correspond to regular holonomic microdifferential modules. The classification theorem for regular holonomic systems with smooth support has been known to the specialists for a long time and can be found in [K1]:

Theorem 6.1. *Let* $\Lambda \subset \dot{T}^*X$ *be a smooth homogenous Lagrangian variety. Then there is a canonical equivalence of stacks on* Λ

$$\mathcal{R}eg\mathcal{H}ol(\mathcal{E}_X, \Lambda) \xrightarrow{\sim} \mathrm{Loc}(\Lambda, [\Omega^{\otimes 1/2}_{\Lambda|X}])$$

This equivalence furthermore identifies simple systems with local systems of rank one.

Here $\mathrm{Loc}(\Lambda, [\Omega^{\otimes 1/2}_{\Lambda|X}])$ denotes the stack of twisted local systems by the bundle of relative half-densities. There are several ways to define stacks of twisted sheaves and we refer to [D'AP] for a survey on the subject. Categories of sheaves on a complex manifold that are twisted by a complex line bundle (or a locally free holomorphic bundle of rank one) are easily described in the following intuitive way:

Let X be a complex manifold and $\pi : \Lambda \to X$ a complex line bundle on X. Denote by

$$\mu : \mathbb{C}^* \times \Lambda \longrightarrow \Lambda .$$

the natural action of $\mathbb{C}^*$ on the fibers of Λ and by

$$p_2 : \mathbb{C}^* \times \Lambda \longrightarrow \Lambda$$

the projection.

Then we can define a twisted sheaf (with respect to Λ) as follows.

$$\mathrm{Mod}(X, L^{\otimes 1/2}) = \{(\mathcal{F}, \alpha) \mid \mathcal{F} \in \mathrm{Mod}(\dot{L}) \; ; \; \alpha : \mu^{-1}\mathcal{F} \simeq p_2^{-1}\mathcal{F} \otimes p_1^{-1}\mathbb{Z}_{1/2}\}$$

Let $t : U \longrightarrow \dot{L}$ be a (local) section. Then we have (locally) an equivalence

$$\mathrm{Mod}(X, L^{\otimes 1/2}) \simeq \mathrm{Mod}(X)$$

defined by $(\mathcal{F}, \theta) \mapsto t^{-1}\mathcal{F}$. Hence a stack of twisted modules is locally isomorphic to a stack of modules.

By the microlocal Riemann-Hilbert theorem, an analogue to Theorem 6.1 holds for microlocal perverse sheaves (over the complex numbers). The aim of this section is to prove the topological analogue for any field, hence without the Riemann-Hilbert theorem.

Theorem 6.2. *Let $\Lambda \subset \dot{T}^*X$ be a smooth homogenous Lagrangian variety. Then there is a canonical equivalence*

$$\mu\mathrm{Perv}(k_\Lambda) \xrightarrow{\sim} \mathrm{Loc}(k_\Lambda, [\Omega_{\Lambda|X}^{\otimes 1/2}]).$$

Let us briefly explain how the theorem can be proved. First consider a point $p \in \dot{T}^*X$. It is well known (cf. [KS1]) that by fixing a function φ on X such that $\Lambda_\varphi = \{(x, \mathrm{d}\,\varphi_x)\}$ is transversal to Λ at p, then $\mu hom(\mathbb{C}_{\varphi=0}, \mathcal{F})$ is a local system on Λ in a neighborhood of p and we get an equivalence of categories

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, p) \longrightarrow \mathrm{Loc}(k_\Lambda)_p \quad ; \quad \mathcal{F} \mapsto \mu hom(\mathbb{C}_{\varphi=0}, \mathcal{F})_p$$

However this equivalence, although it could be extended to some neighborhood of p depends on the choice of the function φ, and it could not be used to patch the equivalence of the theorem. Therefore the idea is to consider simultaneously all possible φ. We will not give a fully detailed description here but sketch a rough approach in local coordinates.

For $p \in \dot{T}^*X$ set

(1) $\lambda_0(p) = T_p\pi^{-1}\pi(p)$

(2) $\lambda_\Lambda(p) = T_p\Lambda$

and consider the following bundles on Λ

$$\mathcal{L} = \Big\{(p.\lambda) \mid p \in \Lambda, \lambda \subset T_p(T^*X) \text{ Lagrangian} \ ; \ \lambda \cap \lambda_\Lambda(p) = \{0\}\Big\}$$

$$\mathcal{L}_0 = \Big\{(p.\lambda) \mid \lambda \cap \lambda_\Lambda(p) = \{0\}, \lambda \cap \lambda_0(p) = \{0\}\Big\} \subset \mathcal{L}$$

Locally on $\mathcal{L}_0$ we can chose coordinates as follows. Each Lagrangian subspace λ is uniquely determined by a symmetric matrix A and the relation $\xi = Ax$. We will fix local coordinates $(x, (x_0, \xi_0, A))$ on $X \times \mathcal{L}_0$ and define $\varphi : X \times \mathcal{L}_0 \longrightarrow \mathbb{R}$ by

$$\varphi(x, (x_0, \xi_0, A)) = \mathrm{Re}(\langle x - x_0, \xi_0\rangle + \frac{1}{2}\langle A(x - x_0), x - x_0\rangle)$$

Let $p_1 : X \times \mathcal{L}_0 \to X$ be the projection and $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, U_X)$. Using results from [KS1] it can be shown that the complex

$$\mathrm{L}(\mathcal{F}) = \mu hom(k_{\varphi \geqslant 0}, p_1^{-1}\mathcal{F})$$

defines a local system on $X \times \mathcal{L}_0$.

We now want to extend this local system to $\mathfrak{L}$. Set $Z = \mathfrak{L} \setminus \mathfrak{L}_0$ and

$$W = \{p \in Z \mid \text{the projection } \Lambda \to X \text{ is not of constant rank in a neighborhood of } p\}$$

Since W is of codimension $\geqslant 2$, in order to extend the local system, it is enough to calculate the monodromy at a generic point of Z. Hence we may assume that $\Lambda = T^*_Y X$ and $\mathcal{F} = E \otimes \mathbb{C}_Y$ for some vector space E. A direct calulation with the tools from [KS2] and the language of [D'AP] then shows that $\mathrm{L}(\mathcal{F})$ extends as a twisted sheaf with twist $\Omega_{\Lambda|X}^{\otimes 1/2}$ to $\mathfrak{L}$ and we can descend it by a local section to Λ.

7 The contact case

Now let $\chi : U_X \xrightarrow{\sim} U_Y$ be a contact isomorphism and

$$\Lambda = \Big\{((y;\eta),(x;\xi)) \in U_Y \times U_X^a \mid (y,\eta) = \chi(x,-\xi)\Big\}.$$

Then Λ is a smooth Lagrangian variety. Morover since it is the graph of a contact transformation the bundle Ω_Λ is trivial and we get by Theorem 6.2 the canonical equivalence of stacks

$$\mu\mathrm{Perv}(k_\Lambda) \xrightarrow{\sim} \mathrm{Loc}(k_\Lambda, [\Omega_{Y \otimes X}^{-\otimes 1/2}]),$$

or equivalently the equivalence

$$\mu\mathrm{Perv}(k_\Lambda, [\Omega_{Y \otimes X}^{\otimes 1/2}]) \xrightarrow{\sim} \mathrm{Loc}(k_\Lambda).$$

Then the constant sheaf $\mathbb{C}_\Lambda$ defines a canonical twisted simple microlocal perverse sheaf $\mathcal{K}_\chi$ with twist $\Omega_Y^{\otimes 1/2} \otimes \Omega_X^{\otimes -1/2}$ on $Y \times X$, where we used the fact that twisting by the globally defined bundle Ω_X is trivial. Hence, as in Theorem 5.1, $\mathcal{K}_\chi$ induces (cf. [D'AP]) the functor

$$\Phi_{\mathcal{K}_\chi} : \chi_* \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *, [\Omega_X^{\otimes 1/2}])|_{U_X} \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_Y, *, [\Omega_Y^{\otimes 1/2}])|_{U_Y},$$

which locally can be identified with the functor of Theorem 5.1 and therefore induces an equivalence of the associated stacks of (twisted) microlocal perverse sheaves.

Moreover, since $\Phi_{\mathcal{K}_\chi}$ is now given by a canonical kernel $\mathcal{K}_\chi$ one can prove that if we consider a composition of contact transformations $\chi_3 = \chi_2 \circ \chi_1$ we get a canonical isomorphism $\Phi_{\mathcal{K}_{\chi_3}} \simeq \Phi_{\mathcal{K}_{\chi_2}} \circ \Phi_{\mathcal{K}_{\chi_1}}$, that will satisfy the usual cocycle condition for iterated compositions. In other words, we can use the equivalences $\Phi_{\mathcal{K}_\chi}$ to patch the stacks $\mu Perv(k_X, [\Omega_X^{\otimes 1/2}])$ to a globally defined stack of microlocal perverse sheaves on Ξ.

Finally let us recall from [W2] that the microlocal solution functor is also invariant by quantized contact transformation in a compatible way with the classic case. This means that we have the following commutative diagram (up to isomorphism of functors) of equivalences of categories (or stacks)

$$\begin{array}{ccc}
\chi_* \mathcal{R}eg\mathcal{H}ol(\mathcal{E}_X)|_{U_X} & \xrightarrow{\mu\,\mathrm{Sol}} & \chi_* \mu Perv(k_X)|_{U_X} \\
\sim\downarrow & & \downarrow\sim \\
\mathcal{R}eg\mathcal{H}ol(\mathcal{E}_Y)|_{U_Y} & \xrightarrow[\mu\,\mathrm{Sol}]{} & \mu Perv(k_Y)|_{U_Y}
\end{array}$$

This diagram still exists in the twisted case:

$$\begin{array}{ccc}\chi_*\mathcal{M}od(\mathcal{E}_X,[\Omega_X^{\otimes 1/2}])|_{U_X} & \xrightarrow{\mu\mathrm{Sol}} & \chi_*\mu Perv(k_X,[\Omega_X^{\otimes 1/2}])|_{U_X} \\ \downarrow & & \downarrow{\scriptstyle \Phi_{\mathcal{K}_X}} \\ \mathcal{M}od(\mathcal{E}_Y,[\Omega_Y^{\otimes 1/2}])|_{U_Y} & \xrightarrow[\mu\mathrm{Sol}]{} & \mu Perv(k_Y,[\Omega_Y^{\otimes 1/2}])|_{U_Y}\end{array}$$

The equivalences on the left hand side were used by Kashiwara to patch the stack of microdifferential modules [5], those on the right hand side define the stack of microlocal perverse sheaves, hence the microlocal solution functor can be patched to Ξ. It is an equivalence on Ξ because it is a functor of stacks that is locally an equivalence, and we can prove the microlocal Riemann-Hilbert theorem on Ξ:

Theorem 7.1. *Let Ξ be a contact manifold. Then for any field k there exists a canonical abelian stack $\mu Perv(k_\Xi)$ of microlocal perverse sheaves on Ξ which, over the complex numbers, is equipped with an equivalence of stacks*

$$\mathcal{R}eg\mathcal{H}ol(\mathcal{E},\Xi) \longrightarrow \mu Perv(\mathbb{C}_\Xi)$$

such that if $U \subset \Xi$ is an open subset that is contact isomorphic to an open subset $V \subset P^*X$ of the projective bundle to some complex analytic manifold X, we can identify it with the equivalence

$$\mu\mathrm{Sol}:\ \mathcal{R}eg\mathcal{H}ol(\mathcal{E},[\Omega_X^{\otimes 1/2}]) \longrightarrow \mu Perv(\mathbb{C}_X,[\Omega_X^{\otimes 1/2}]).$$

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[5] Strictly speaking, in [K1], Kashiwara replaces $\mathcal{E}_X$ by the globally defined ring $\tilde{\mathcal{E}}_X = \Omega_X^{\otimes -1/2} \otimes \mathcal{E}_X \otimes \Omega_X^{\otimes 1/2}$ of twisted microdifferential operators, and then he works with $\mathcal{M}od(\tilde{\mathcal{E}}_X)$. Note however that there is a canonical equivalence $\mathcal{M}od(\tilde{\mathcal{E}}_X) \simeq \mathcal{M}od(\mathcal{E}_X,[\Omega_X^{\otimes 1/2}])$.

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