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SPECTRAL GEOMETRY AND A CHARACTERIZATION OF HYPERSURFACES IN A COMPLEX GRASSMANN MANIFOLD

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ABSTRACT. In this article, the author will study compact Kähler hypersurfaces M in a complex Grassmann manifold $G_r(\mathbb{C}^n)$ of r-planes, and give an upper bound for the first eigenvalue of the Laplacian (Theorem A). In the case that $r=2$, $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. When the tangent bundle TM and the normal bundle $T^{\perp}M$ of M satisfy the property that $\mathfrak{J}T^{\perp}M \subset TM$, the author obtain sharper estimate (Theorem B). It is an interesting problem that *"What is M satisfying $\mathfrak{J}T^{\perp}M \subset TM$?"*. If M is Einstein, without the assumption of homogeneity, we shall show that M is congruent to a certain Kähler C-space (Theorem C). Theorems A, B and C are showed in the section 2, and are proved in later sections.

1. COMPLEX GRASSMANN MANIFOLDS OF r-PLANES

In this section, we discuss geometries of complex Grassmann manifolds of r-planes and their first standard imbeddings. For details, see [9] and [2].

$M_{r,s}(\mathbb{C})$ denotes the set of all $r\times s$ matrices with entries in $\mathbb{C}$, and $M_r(\mathbb{C})$ stands for $M_{r,r}(\mathbb{C})$. I_r and O_r denote the identity r-matrix and the zero r-matrix.

Let $M_r(\mathbb{C}^n)$ be the complex Stiefel manifold which is the set of all unitary r-systems of $\mathbb{C}^n$, i.e.,

$$M_r(\mathbb{C}^n)=\{Z\in M_{n,r}(\mathbb{C})\mid Z^*Z=I_r\}.$$

The complex r-plane Grassmann manifold $G_r(\mathbb{C}^n)$ is defined by

$$G_r(\mathbb{C}^n)=M_r(\mathbb{C}^n)/U(r).$$

The origin o of $G_r(\mathbb{C}^n)$ is defined by $\pi(Z_0)$, where $Z_0=\begin{pmatrix} I_r\\ 0\end{pmatrix}$ is an element of $M_r(\mathbb{C}^n)$, and $\pi: M_r(\mathbb{C}^n)\longrightarrow G_r(\mathbb{C}^n)$ is the natural projection.

The left action of the unitary group $\tilde{G}=SU(n)$ on $G_r(\mathbb{C}^n)$ is transitive, and the isotropy subgroup at the origin o is

$$\begin{aligned}\tilde{K}&=S(U(r)\cdot U(n-r))\\ &=\left\{\begin{pmatrix} U_1 & 0\\ 0 & U_2\end{pmatrix}\,\middle|\, U_1\in U(r),\, U_2\in U(n-r),\, \det U_1\det U_2=1\right\},\end{aligned}$$

so that $G_r(\mathbb{C}^n)$ is identified with a homogeneous space $\tilde{G}/\tilde{K}$.

Set $\tilde{\mathfrak{g}}=\mathfrak{su}(n)$ and

$$\begin{aligned}\tilde{\mathfrak{k}}&=\mathbb{R}\oplus\mathfrak{su}(r)\oplus\mathfrak{su}(n-r)\\ &=\left\{\begin{pmatrix} u_1 & 0\\ 0 & u_2\end{pmatrix}+a\begin{pmatrix} -\frac{1}{r}\sqrt{-1}I_r & 0\\ 0 & \frac{1}{n-r}\sqrt{-1}I_{n-r}\end{pmatrix}\,\middle|\, a\in\mathbb{R},\ \begin{matrix} u_1\in\mathfrak{su}(r)\\ u_2\in\mathfrak{su}(n-r)\end{matrix}\right\},\end{aligned}$$

then $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ are the Lie algebras of $\tilde{G}$ and $\tilde{K}$, respectively. Define a linear subspace $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \;\middle|\; \xi \in M_{n-r,r}(\mathbb{C}) \right\}.$$

Then $\tilde{\mathfrak{m}}$ is identified with the tangent space $T_o(G_r(\mathbb{C}^n))$. The $\tilde{G}$-invariant complex structure J of $G_r(\mathbb{C}^n)$ and the $\tilde{G}$-invariant Kähler metric $\tilde{g}_c$ of $G_r(\mathbb{C}^n)$ of the maximal holomorphic sectional curvature c are given by

$$J\begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix},$$

(1.1) $$\tilde{g}_{c_o}(X, Y) = -\frac{2}{c} trXY, \quad X, Y \in \tilde{\mathfrak{m}}.$$

Notice that $\tilde{g}_c$ satisfies

(1.2) $$\tilde{g}_{c_o} = -\frac{2}{c}\frac{1}{2n} B_{\tilde{\mathfrak{g}}} = -\frac{2}{c}\frac{L(\tilde{\mathfrak{g}})}{2} B_{\tilde{\mathfrak{g}}}$$

on $\tilde{\mathfrak{m}}$, where $B_{\tilde{\mathfrak{g}}}$ is the Killing form of $\tilde{\mathfrak{g}}$, and $L(\tilde{\mathfrak{g}})$ is the squared length of the longest root of $\tilde{\mathfrak{g}}$ relative to the Killing form.

We denote by X^* an vector field on $\tilde{M}$ generated by $X \in \tilde{\mathfrak{g}}$, i.e.,

$$(X^*)_p = \left[\frac{d}{dt} \exp tX \cdot p\right]_{t=0}, \quad p = g\tilde{o} \in \tilde{M}, \quad g \in \tilde{G}.$$

The Riemannian connection $\tilde{\nabla}$ is described in terms of the Lie derivative as follows:

(1.3) $$(L_{X^*} - \tilde{\nabla}_{X^*})_{\tilde{o}}\tilde{Y} = \begin{cases} -ad(X)\tilde{Y}_{\tilde{o}}, & \text{if } X \in \tilde{\mathfrak{k}}, \\ 0, & \text{if } X \in \tilde{\mathfrak{m}}, \end{cases}$$

where $\tilde{Y}$ is a vector field on $\tilde{M}$.

In the case of $r = 2$, the complex 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits another geometric structure named the quaternionic Kähler structure $\mathfrak{J}$. $\mathfrak{J}$ is a $\tilde{G}$-invariant subbundle of $End(T(G_2(\mathbb{C}^n)))$ of rank 3, where $End(T(G_2(\mathbb{C}^n)))$ is the $\tilde{G}$-invariant vector bundle of all linear endmorphisms of the tangent bundle $T(G_2(\mathbb{C}^n))$. Under the identification of $T_o(G_2(\mathbb{C}^n))$ with $\tilde{\mathfrak{m}}$, the fiber $\mathfrak{J}_o$ at the origin o is given by

$$\mathfrak{J}_o = \left\{ J_{\tilde{\varepsilon}} = ad(\tilde{\varepsilon}) \mid \tilde{\varepsilon} \in \tilde{\mathfrak{k}}_q \right\},$$

where $\tilde{\mathfrak{k}}_q$ is an ideal of $\tilde{\mathfrak{k}}$ defined by

$$\tilde{\mathfrak{k}}_q = \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; u_1 \in \mathfrak{su}(2) \right\} \cong \mathfrak{su}(2).$$

Define a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Then ε_1, ε_2 and ε_3 satisfy

$$[\varepsilon_1, \varepsilon_2] = 2\varepsilon_3, \quad [\varepsilon_2, \varepsilon_3] = 2\varepsilon_1, \quad [\varepsilon_3, \varepsilon_1] = 2\varepsilon_2.$$

Set $\tilde{\varepsilon}_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & 0 \end{pmatrix}$ and $J_i = J_{\tilde{\varepsilon}_i}$ for $i = 1, 2, 3$. Then the basis $\{J_1,\ J_2,\ J_3\}$ is a canonical basis of $\mathfrak{J}_o$, satisfying

$$J_i^2 = -id_{\tilde{\mathfrak{m}}} \quad \text{for } i = 1, 2, 3,$$
$$J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,$$
$$\tilde{g}_{c_o}(J_iX,\ J_iY) = \tilde{g}_{c_o}(X,\ Y), \quad \text{for } X, Y \in \tilde{\mathfrak{m}} \text{ and } i = 1, 2, 3.$$

Since J is given by

$$J = ad(\tilde{\varepsilon}_{\mathbb{C}}), \quad \tilde{\varepsilon}_{\mathbb{C}} = \frac{r(n-r)}{n} \begin{pmatrix} -\frac{1}{r}\sqrt{-1}I_r & 0 \\ 0 & \frac{1}{n-r}\sqrt{-1}I_{n-r} \end{pmatrix}$$

on $\mathfrak{m}$, and since $\tilde{\varepsilon}_{\mathbb{C}}$ is an element of the center of $\tilde{\mathfrak{k}}$, J is commutable with $\mathfrak{J}$.

Moreover, the property

$$trJJ' = 0 \tag{1.4}$$

holds for any $J' \in \mathfrak{J}$.

In [2], J. Berndt showed that the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by

$$\begin{aligned} \tilde{R}(X,\ Y)Z = \frac{c}{8}\Big[& \tilde{g}_c(Y, Z)X - \tilde{g}_c(X, Z)Y \\ & + \tilde{g}_c(JY, Z)JX - \tilde{g}_c(JX, Z)JY + 2\tilde{g}_c(X, JY)JZ \\ & + \sum_{k=1}^{3}\Big\{\tilde{g}_c(J_kY, Z)J_kX - \tilde{g}_c(J_kX, Z)J_kY + 2\tilde{g}_c(X, J_kY)J_kZ\Big\} \\ & + \sum_{k=1}^{3}\Big\{\tilde{g}_c(JJ_kY, Z)JJ_kX - \tilde{g}_c(JJ_kX, Z)JJ_kY\Big\}\Big] \end{aligned} \tag{1.5}$$

for any vector fields X, Y and Z of $\tilde{M}$.

Let $HM(n, \mathbb{C})$ be the set of all Hermitian (n, n)-matrices over $\mathbb{C}$, which can be identified with $\mathbb{R}^{n^2}$. For $X, Y \in HM(n, \mathbb{C})$, the natural inner product is given by

$$(X, Y) = \frac{2}{c}\, trXY. \tag{1.6}$$

$GL(n, \mathbb{C})$ acts on $HM(n, \mathbb{C})$ by $X \longmapsto BXB^*$, $B \in GL(n, \mathbb{C})$, $X \in HM(n, \mathbb{C})$. Then the action of $SU(n)$ leaves the inner product (1.6) invariant. Define two linear subspaces of $HM(n, \mathbb{C})$ as follows:

$$HM_0 = \{X \in HM(n, \mathbb{C}) \,|\, trX = 0\},$$
$$HM_{\mathbb{R}} = \{aI \,|\, a \in \mathbb{R}\},$$

where I is the n-identity matrix. Both of them are invariant under the action of $SU(n)$, and irreducible. We get the orthogonal decomposition of $HM(n, \mathbb{C})$ as follows:

$$HM(n, \mathbb{C}) = HM_0 \oplus HM_{\mathbb{R}}.$$

It is well-known that HM_0 (resp. $HM_{\mathbb{R}}$) is identified with the first eigenspace $V_1(G_r(\mathbb{C}^n))$ (resp. the set of all constant functions, i.e. $V_0(G_r(\mathbb{C}^n))$).

The first standard imbedding Ψ of $G_r(\mathbb{C}^n)$ is defined by

$$\Psi(\pi(Z)) = ZZ^* \in HM(n, \mathbb{C}), \quad Z \in M_r(\mathbb{C}^n).$$

Ψ is $SU(n)$-equivariant and the image N of $G_r(\mathbb{C}^n)$ under Ψ is given by

$$N = \Psi(G_r(\mathbb{C}^n)) = \{ A \in HM(n,\mathbb{C}) \mid A^2 = A,\ trA = r \}, \tag{1.7}$$

so that it is contained fully in a hyperplane

$$HM_r = \{A \in HM(n,\mathbb{C}) \,|\, trA = r\} = \left\{A + \frac{r}{n}I \,\middle|\, A \in HM_0\right\}$$

of $HM(n,\mathbb{C})$. The tangent bundle TN and the normal bundle $T^{\perp}N$ are given by

$$\begin{aligned} T_A N =& \{X \in HM(n,\mathbb{C}) \,|\, XA + AX = X\} \subset HM_0, \\ T_A^{\perp} N =& \{Z \in HM(n,\mathbb{C}) \,|\, ZA = AZ\}. \end{aligned} \tag{1.8}$$

In particular, at the origin $A_o = \Psi(o) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, we can obtain

$$\begin{aligned} T_{A_o}N &= \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \,\middle|\, \xi \in M_{n-r,r}(\mathbb{C}) \right\}, \\ T_{A_o}^{\perp}N &= \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \,\middle|\, Z_1 \in HM(r,\mathbb{C}),\ Z_2 \in HM(n-r,\mathbb{C}) \right\}. \end{aligned} \tag{1.9}$$

The complex structure J acts on $T_{A_o}N$ as

$$J\begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0 \end{pmatrix}. \tag{1.10}$$

If $r = 2$, then the quaternionic Kähler structure $\mathfrak{J}$ acts on $T_{A_o}N$ as

$$J_{\bar{\varepsilon}}\begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon\xi^* \\ -\xi\varepsilon & 0 \end{pmatrix}, \qquad \varepsilon \in \mathfrak{su}(2). \tag{1.11}$$

Let $\tilde{\sigma}$ and $\tilde{H}$ denote the second fundamental form and the mean curvature vector of Ψ, respectively. Then, for $A \in N$ and $X, Y \in T_A N$, we can see

$$\tilde{\sigma}_A(X,Y) = (XY + YX)(I - 2A), \tag{1.12}$$

$$\tilde{H}_A = \frac{c}{2r(n-r)}(rI - nA) \tag{1.13}$$

and $\tilde{\sigma}$ satisfies the following:

$$\tilde{\sigma}_A(JX, JY) = \tilde{\sigma}_A(X,Y), \tag{1.14}$$

$$(\tilde{\sigma}_A(X,Y),\ A) = -(X,\ Y). \tag{1.15}$$

Denote by $S^{n^2-2}(\frac{c}{2}\frac{n}{r(n-r)})$ the hypersphere in HM_r centered at $\frac{r}{n}I$ with radius $\sqrt{\frac{2}{c}\frac{r(n-r)}{n}}$. Then we see that Ψ is a minimal immersion of $G_r(\mathbb{C}^n)$ into $S^{n^2-2}(\frac{c}{2}\frac{n}{r(n-r)})$, and that the center of mass of $\Psi(G_r(\mathbb{C}^n))$ is $\frac{r}{n}I$. In fact, Ψ satisfies the equation $\Delta\Psi = cn(\Psi - \frac{r}{n}I)$. Moreover, all coefficients of $\Psi - \frac{r}{n}I$ span the first eigenspace $V_1(G_r(\mathbb{C}^n))$.

Let's assume that M is a submanifold of $G_r(\mathbb{C}^n)$ with an immersion φ. Then $F = \Psi \circ \varphi$ is an immersion of M into $HM(n,\mathbb{C})$, and the set of all coefficients of $F - \frac{r}{n}I$ spans the pull-back $\varphi^* V_1(G_r(\mathbb{C}^n))$.

Let $(M,\ g)$ be a Riemannian submanifold of $\tilde{M}$. Denote by ∇ the Riemannian connection of M, and by σ, A and $\nabla^{\perp}$ the second fundamental form, the Weingarten map and th normal connection of M in $G_2(\mathbb{C}^{2l})$ respectively. We have the Gauss' formula and the Weingarten's formula are:

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \quad \tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{1.16}$$

where X, Y and Z are tangent vector fields and ξ is a normal vector field. Moreover, we see

$$g(A_\xi X,\ Y) = \tilde{g}_c(\sigma(X,Y),\ \xi).$$

If M is an Kähler submanifold of $\tilde{M}$, then the following hold.

$$\sigma(X,\ JY) = \sigma(JX,\ Y) = J\sigma(X,\ Y), \tag{1.17}$$

$$A_\xi J = -JA_\xi = -A_{J\xi}. \tag{1.18}$$

M is called a *quaternionic submanifold*, if the tangent space T_pM is invariant under the action of $\mathfrak{J}$ for each p in M. M is called a *totally real submanifold*, if JT_pM is a subspace of the normal space $T^\perp_p M$ for each p in M.

2. Main results and examples

One of the simplest typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [4, 5] determined maximal totally geodesic submanifolds of $G_2(\mathbb{C}^n)$. I. Satake and S. Ihara in [17, 8] determined all (equivariant) holomorphic, totally geodesic imbeddings of a symmetric domain into another symmetric domain. When an ambient symmetric domain is of type $(\mathrm{I})_{p,q}$, taking a compact dual symmetric space, we obtain the complete list of maximal totally geodesic Kähler submanifolds of $G_r(\mathbb{C}^n)$.

Let M be a maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$ given by a Kähler immersion $\varphi : M \longrightarrow G_r(\mathbb{C}^n)$. Since M is a symmetric space, denote by (G, K) the compact symmetric pair of M, and denote by $(\mathfrak{g}, \mathfrak{k})$ its Lie algebra. Then there exists a certain unitary representation $\rho : G \longrightarrow \tilde{G} = SU(n)$, such that $\varphi(M)$ is given by the orbit of $\rho(G)$ through the origin $o = \{\tilde{K}\}$ in $G_r(\mathbb{C}^n)$.

Let $L(\mathfrak{g})$ be the squared length of the longest root of $\mathfrak{g}$ relative to the Killing form $B_\mathfrak{g}$. Tables of the $L(\mathfrak{g})$ constants appear in [7]. The Kähler metric induced by φ is a G-invariant metric corresponding to an $Ad(G)$-invariant inner product

$$\rho^*\left(-\frac{2}{c}\frac{L(\tilde{\mathfrak{g}})}{2}B_{\tilde{\mathfrak{g}}}\right) = -\frac{2}{c}\frac{L(\mathfrak{g})}{2}\, l_\rho\, B_\mathfrak{g} \tag{2.1}$$

on $\mathfrak{g}$, where l_ρ is the index of a linear representation ρ defined by Dynkin. Tables of indices of basic representations of simple Lie algebras appear in [6].

Using Freudenthal's formula with respect to the inner product (2.1), we can calculate the first eigenvalue of the Laplacian of M. (cf. [20])

Summing up these results, we obtain the following.

Theorem 2.1. *Let $M = G/K$ be a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$, ρ a corresponding unitary representation of G to $SU(n)$, and λ_1 the first eigenvalue of the Laplacian with respect to the induced Kähler metric. Then, M, ρ and λ_1 are one of the following (up to isomorphism).*

(1) $M_1 = G_r(\mathbb{C}^{n-1}) \hookrightarrow G_r(\mathbb{C}^n)$, $\quad 1 \leqq r \leqq n-2$,
$\rho_1 =$ natural inclusion *and* $\lambda_1 = c(n-1)$

(2) $M_2 = G_{r-1}(\mathbb{C}^{n-1}) \hookrightarrow G_r(\mathbb{C}^n)$, $2 \leqq r \leqq n-1$,
$\rho_2 =$ natural inclusion *and* $\lambda_1 = c(n-1)$

(3) $M_3 = G_{r_1}(\mathbb{C}^{n_1}) \times G_{r_2}(\mathbb{C}^{n_2}) \hookrightarrow G_{r_1+r_2}(\mathbb{C}^{n_1+n_2})$, $1 \leqq r_i \leqq n_i - 1$, $i = 1, 2$,
$\rho_3 =$ natural inclusion *and* $\lambda_1 = c \min\{n_1, n_2\}$

(4) $M_4 = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_p(\mathbb{C}^{2p})$, $p \geqq 2$,
$\rho_4 =$ natural inclusion *and* $\lambda_1 = c(p+1)$

(5) $M_5 = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_p(\mathbb{C}^{2p})$, $p \geqq 4$,
$\rho_5 =$ natural inclusion *and* $\lambda_1 = c(p-1)$

(6) $M_{6,m} = \mathbb{C}P^p \hookrightarrow G_r(\mathbb{C}^n)$: *the complex projective space,*
$r = \binom{p}{m-1}$, $n = \binom{p+1}{m}$, $2 \leqq m \leqq p-1$,
$\rho_{6,m} =$ the exterior representation of degree m,
and $\lambda_1 = c(p+1)\binom{p-1}{m-1}^{-1}$

(7) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbb{C}^4)$: *the complex quadric,*
$\rho_7 =$ spin representation *and* $\lambda_1 = 3c$

(8) $M_8 = M_{8,2l} = Q^{2l} \hookrightarrow G_r(\mathbb{C}^{2r})$: *the complex quadric,* $r = 2^{l-1}$, $l \geqq 3$,
$\rho_8^{\pm} =$ (two) spin representations *and* $\lambda_1 = c\frac{2l}{2^{l-2}}$

In the above list, notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

A submanifold M of $G_r(\mathbb{C}^n)$ is parallel if the second fundamental form of M is parallel. H. Nakagawa and R. Takagi in [14] classified parallel Kähler submanifolds of a complex projective space $\mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$. K. Tsukada in [22] showed that, in parallel Kähler submanifolds of $G_r(\mathbb{C}^n)$, the above classification is essential. Moreover, if $r \neq 1, n-1$, then a parallel Kähler submanifold M of $G_r(\mathbb{C}^n)$ is a parallel Kähler submanifolds of some totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$, i.e, M is a parallel Kähler submanifold of one of $\{M_i, i = 1, \ldots, 8\}$. Notice that a Hermitian symmetric submanifolds of $G_r(\mathbb{C}^n)$ is parallel.

Another one of the simplest typical examples of submanifolds of $G_r(\mathbb{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [10] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

Theorem 2.2. *Let M be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$, and λ_1 the first eigenvalue of the Laplacian with respect to the induced Kähler metric. Then, M and λ_1 are one of the following (up to isomorphism).*

(1) $M_9 = \mathbb{C}P^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ *and* $\lambda_1 = c(n-1)$

(2) $M_{10} = Q^{n-2} \hookrightarrow \mathbb{C}P^{n-1} = G_1(\mathbb{C}^n)$ *and* $\lambda_1 = c(n-2)$

(3) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbb{C}^4)$ *and* $\lambda_1 = 3c$

(4) $M_{11} = M_{11,l} = Sp(l)/U(2) \cdot Sp(l-2) \hookrightarrow G_2(\mathbb{C}^{2l})$: *Kähler C-space of type* (C_l, α_2), $l \geqq 2$ *and* $\lambda_1 = c(2l-1)$

M_9 and M_7 are totally geodesic. M_9, M_{10} and M_7 are symmetric spaces. If $l = 2$, then M_{11} is congruent to M_7. If $l > 2$, M_{11} is neither symmetric nor parallel.

For each l with $l > 2$, M_{11} is not a symmetric space. Then, it is not easy to calculate the first eigenvalue λ_1 of M_{11}. We will calculate λ_1 of M_{11} in §4.

From these two theorems, we obtain the following proposition:

Proposition 2.3. *Let M be either a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbb{C}^n)$ or a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then, the first eigenvalue λ_1 of M with respect to the induced Kähler metric satisfies*

$$\lambda_1 \leqq c(n-1).$$

Moreover, the equality holds if and only if M is congruent to one of the following:

$$M_1, \quad M_2, \quad M_{4,2} = M_7, \quad M_9, \quad M_{11}.$$

Notice that all manifolds in Theorems 2.1 and 2.2 are Einstein manifolds.

One of the purposes of this paper is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.

In the case that M is a complex hypersurface of $G_r(\mathbb{C}^n)$, we obtain the following result, which is a generalization of A. Ros [16]'s results.

Theorem A ([12]). *Suppose that M is a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$. Then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c\left(n - \frac{n-2}{r(n-r)-1}\right).$$

The equality holds if and only if $r = 1$ or $n-1$, and M is congruent to the totally geodesic complex hypersurface $\mathbb{C}P^{n-2}$ of the complex projective space $\mathbb{C}P^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbb{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^{\perp}M$ of a Kähler hypersurface M of $G_2(\mathbb{C}^n)$, $\mathfrak{J}T^{\perp}M$ is a vector bundle of real rank 6 over M which is a subbundle of the tangent bundle of $G_2(\mathbb{C}^n)$. We consider a Kähler hypersurface M of $G_2(\mathbb{C}^n)$ satisfying the property that $\mathfrak{J}T^{\perp}M$ is a subbundle of the tangent bundle TM of M, i.e, $\mathfrak{J}T^{\perp}M \subset TM$. In §4, we will see that the Kähler hypersurface $M_{11,l}$ satisfies this condition.

For a Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

Theorem B ([12]). *Suppose that M is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$, $n \geqq 4$. If M satisfies the condition $\mathfrak{J}\, T^{\perp}M \subset TM$, then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right).$$

The equality holds if and only if $n = 4$ and M is congruent to the totally geodesic complex hypersurface Q^3 of the complex quadric $Q^4 = G_2(\mathbb{C}^4)$.

Also see [13] about Theorems A and B.

One of the simplest questions is as follows: *What is M satisfying $\mathfrak{J}T^{\perp}M \subset TM$?* Without the assumption of homogeneity, we shall show the following result.

Theorem C ([11]). *If an Einstein Kähler hypersurface M of $G_2(\mathbb{C}^n)$ satisfies the condition $\mathfrak{J}T^{\perp}M \subset TM$, then n is even and M is locally congruent to $M_{11,n/2}$.*

If $n = 4$, then the statement holds without the assumption that M is Einstein. See Proposition 4.3 in §4.

3. The Kähler C-spaces with $b_2 = 1$

In this section, we will consider the first eigenvalue of the Kähler C-space whose second Betti number is equal to 1. First, we review the general theory of Kähler C-spaces. For details, see [3] and [19].

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{t}^{\mathbb{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $(\ ,\ )$ be an $Ad(G)$-invariant inner product on $\mathfrak{g}$ defined by $-B_{\mathfrak{g}}$, where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. Let $\Sigma \subset (\mathfrak{t}^{\mathbb{C}})^*$ denote the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$. We have a root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha\in\Sigma} \mathfrak{g}^{\mathbb{C}}_{\alpha}, \tag{3.1}$$

where $\mathfrak{g}^{\mathbb{C}}_{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid (adH)X = \alpha(H)X \text{ for any } H \in \mathfrak{t}\}$. Since $\mathfrak{g}$ is compact type, for any $\alpha \in \Sigma$ and $H \in \mathfrak{t}$, $\alpha(H)$ is pure imaginary, so that there exists a unique element $\check{\alpha} \in \mathfrak{t}$ such that, for any $H \in \mathfrak{t}$, the equality $\alpha(H) = \sqrt{-1}(\check{\alpha}, H)$ holds. We identify α with $\check{\alpha}$, so that the root system Σ is identified with a subset $\{\check{\alpha} \mid \alpha \in \Sigma\}$ of $\mathfrak{t}$. Choose a lexicographic order $>$ on Σ and put $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha > 0\}$. Let Π be the fundamental root system of Σ consisting of simple roots with respect to the linear order $>$. Π is identified with its Dynkin diagram. Let $\{\Lambda_\alpha\}_{\alpha\in\Pi} \subset \mathfrak{t}$ be the fundamental weight system of $\mathfrak{g}^{\mathbb{C}}$ corresponding to Π :

$$\frac{2(\Lambda_\alpha, \beta)}{(\beta,\beta)} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Let Π_0 be a subdiagram of Π. We may suppose that the pair (Π, Π_0) is effective, that is, Π_0 contains no irreducible component of Π. Put $\Sigma_0 = \Sigma \cap \{\Pi_0\}_{\mathbb{Z}}$, where $\{\Pi_0\}_{\mathbb{Z}}$ denote the subgroup of $\mathfrak{t}$ generated by Π_0 over $\mathbb{Z}$. Define a subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{u} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha\in\Sigma_0\cup\Sigma^+} \mathfrak{g}^{\mathbb{C}}_{\alpha}. \tag{3.2}$$

Let $G^{\mathbb{C}}$ be the connected complex semisimple Lie group without center, whose Lie algebra is $\mathfrak{g}^{\mathbb{C}}$, and U the connected closed complex subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{u}$. Let G be a compact connected semisimple subgroup of $G^{\mathbb{C}}$ generated by $\mathfrak{g}$ and put $K = G \cap U$. The canonical imbedding $G \longrightarrow G^{\mathbb{C}}$ gives the diffeomorphism of a compact coset space $M = G/K$ to a simply connected complex coset space $G^{\mathbb{C}}/U$. Therefore, the homogeneous space $M = G/K$ is a complex, compact, simply connected manifold called a *generalized flag manifold* or a *Kähler C-space*. Lie algebra $\mathfrak{k}$ of K is given by

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha\in\Sigma_0} \mathfrak{g}^{\mathbb{C}}_{\alpha}. \tag{3.3}$$

Define a subspace $\mathfrak{c}$ of $\mathfrak{t}$ and a cone $\mathfrak{c}^+$ in $\mathfrak{c}$ by

$$\mathfrak{c} = \sum_{\alpha\in\Pi-\Pi_0} \mathbb{R}\Lambda_\alpha,$$

$$\mathfrak{c}^+ = \left\{\theta \in \mathfrak{c} - \{0\} \;\middle|\; (\theta, \alpha) > 0 \text{ for each } \alpha \in \Pi - \Pi_0\right\}, \tag{3.4}$$

respectively. Then we have $\mathfrak{c}^+ = \sum_{\alpha\in\Pi-\Pi_0} \mathbb{R}^+\Lambda_\alpha$, where $\mathbb{R}^+$ denotes the set of positive real numbers.

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to $(\,,\,)$, so that we have a direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ as vector space. The subspace $\mathfrak{m}$ is K-invariant under the adjoint action and identified with the tangent space T_oM of M at the origin $o = \{K\}$. Put $\Sigma_{\mathfrak{m}}^{+} = \Sigma^{+} - \Sigma_0$, $\Sigma_{\mathfrak{m}}^{-} = -\Sigma_{\mathfrak{m}}^{+}$ and define K-invariant subspaces $\mathfrak{m}^{\pm}$ of $\mathfrak{g}^{\mathbf{C}}$ by

$$\mathfrak{m}^{\pm} = \sum_{\alpha \in \Sigma_{\mathfrak{m}}^{\pm}} \mathfrak{g}_{-\alpha}^{\mathbf{C}}. \tag{3.5}$$

Then the complexification $\mathfrak{m}^{\mathbf{C}}$ of $\mathfrak{m}$ is the direct sum $\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^{+} + \mathfrak{m}^{-}$, and $\mathfrak{m}^{\pm}$ is the $\pm\sqrt{-1}$-eigenspace of the complex structure J of M at the origin o.

Denote by $X \longrightarrow \overline{X}$ the complex conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to the real form $\mathfrak{g}$. We can choose root vectors $E_\alpha \in \mathfrak{g}_\alpha^{\mathbf{C}}$ for $\alpha \in \Sigma$ with the following properties and fix them once for all:

$$[E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha, \quad (E_\alpha, E_{-\alpha}) = 1, \quad \overline{E}_\alpha = E_{-\alpha} \quad \text{for } \alpha \in \Sigma. \tag{3.6}$$

Let $\{\omega^\alpha\}_{\alpha\in\Sigma}$ be the linear forms of $\mathfrak{g}^{\mathbf{C}}$ dual to $\{E_\alpha\}_{\alpha\in\Sigma}$, more precisely, the linear forms defined by

$$\omega^\alpha(\mathfrak{t}^{\mathbf{C}}) = \{0\}, \qquad \omega^\alpha(E_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Every G-invariant Kähler metric on M is given by

$$g(\theta) = 2 \sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}} (\theta, \alpha)\, \omega^{-\alpha} \cdot \overline{\omega}^{-\alpha}, \quad \omega^{-\alpha} \cdot \overline{\omega}^{-\alpha} = \frac{1}{2}\left(\omega^{-\alpha} \otimes \overline{\omega}^{-\alpha} + \overline{\omega}^{-\alpha} \otimes \omega^{-\alpha}\right) \tag{3.7}$$

for $\theta \in \mathfrak{c}^{+}$. Note that the inner product $(\,,\,)$ satisfies

$$(\,,\,)_{\mathfrak{m}^{+}\times\overline{\mathfrak{m}^{+}}} = 2 \textstyle\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}} \omega^{-\alpha} \cdot \overline{\omega}^{-\alpha}.$$

We define an element $\delta_{\mathfrak{m}} \in \mathfrak{t}$ by

$$\delta_{\mathfrak{m}} = \frac{1}{2} \textstyle\sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}} \alpha \quad \in \mathfrak{c}^{+}.$$

Then, for the Kähler metric $g(\theta)$, the Ricci tensor Ric and the scalar curvature τ are given respectively by

$$Ric = 4 \sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}} (\delta_{\mathfrak{m}}, \alpha)\, \omega^{-\alpha} \cdot \overline{\omega}^{-\alpha}, \qquad \tau = 4 \sum_{\alpha\in\Sigma_{\mathfrak{m}}^{+}} \frac{(\delta_{\mathfrak{m}}, \alpha)}{(\theta, \alpha)}. \tag{3.8}$$

If $\Pi - \Pi_0$ consists of only one root, say α_r, then the Kähler C-space M is said to be of type $(\mathfrak{g}, \alpha_r)$. The second Betti number b_2 of M is equal to 1. In this case, we obtain

$$\mathfrak{c}^{+} = \mathbb{R}^{+}\Lambda_{\alpha_r},$$

so that there exists a positive real number b with $2\delta_{\mathfrak{m}} = b\Lambda_{\alpha_r}$. Therefore, $(\mathfrak{g}, \alpha_r)$ is a Kähler-Einstein manifold, and the Ricci tensor and the scalar curvature with respect to a Kähler metric $g(a\Lambda_{\alpha_r})$ are given by

$$Ric = \frac{b}{a} g(a\Lambda_{\alpha_r}), \qquad \tau = 2\frac{b}{a} \dim_{\mathbf{C}} M,$$

respectively.

Y. Matsushima and M. Obata showed the following:

Theorem 3.1 ([15]). *Let M be an n-dimensional compact Einstein Kähler manifold of positive scalar curvature τ. Then the first eigenvalue $\lambda_1(M)$ of the Laplacian satisfies that*

$$\lambda_1(M) \geqq \frac{\tau}{n}.$$

The equality holds if and only if M admits a one-parameter group of isometries (i.e., a non-trivial Killing vector field).

This theorem implies the following proposition immediately.

Proposition 3.2. *For the Kähler C-space $M = (\mathfrak{g}, \alpha_r)$ equipped with the Kähler metric $g(a\Lambda_{\alpha_r})$, the first eigenvalue $\lambda_1(M)$ of the Laplacian is given by $\lambda_1(M) = \dfrac{2b}{a}$.*

From now on, we assume that $\mathfrak{g}$ is a compact semisimple simple Lie algebra of type $C_l, l \geqq 2$, and we consider a Kähler C-space of type $(\mathfrak{g}, \alpha_r)$. Then, Π is identified with the Dynkin diagram of type C_l

and Σ^+ is given by

$$\Sigma^+ = \left\{ \begin{array}{l} \alpha_i + \cdots + \alpha_{j-1} \quad (1 \leqq i < j \leqq l+1), \\ (\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \leqq i \leqq j \leqq l-1) \end{array} \right\}.$$

Therefore, we have

$$\begin{aligned} \Sigma_{\mathfrak{m}}^+ = &\{\alpha_i + \cdots + \alpha_r + \cdots + \alpha_j \quad (1 \leqq i \leqq r \leqq j \leqq l)\}, \\ &\cup \{(\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \leqq i \leqq r, \, i \leqq j \leqq l-1)\}. \end{aligned}$$

Immediately, we get

$$\dim_{\mathbb{C}} M = \#\Sigma_{\mathfrak{m}}^+ = \frac{r}{2}(4l - 3r + 1).$$

Then a direct computation gives

$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Sigma_{\mathfrak{m}}^+} \alpha = (2l - r + 1)\left(\sum_{m=1}^{r-1} m\alpha_m + r\sum_{m=r}^{l-1} \alpha_m + \frac{1}{2} r\alpha_l\right).$$

For details, see [12].

The Cartan matrix C of $\mathfrak{g} = C_l$ and its inverse matrix are given by $C = (c_{ij})_{1 \leqq i,j \leqq l}$, $c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, $C^{-1} = (d_{ij})_{1 \leqq i,j \leqq l}$,

$$d_{ij} = \begin{cases} j & \text{if } 1 \leqq j \leqq l-1 \text{ and } j \leqq i \leqq l, \\ i & \text{if } 1 \leqq j \leqq l-1 \text{ and } 1 \leqq i \leqq j, \\ \frac{i}{2} & \text{if } j = l, \end{cases}$$

so that the following holds

$$\Lambda_{\alpha_r} = \sum_{m=1}^{l} d_{rm}\alpha_m = \sum_{m=1}^{r-1} m\alpha_m + r\sum_{m=r}^{l-1} \alpha_m + \frac{1}{2} r\alpha_l.$$

Therefore, we obtain

$$2\delta_{\mathfrak{m}} = (2l - r + 1)\Lambda_{\alpha_r}.$$

Summing up the above consideration, we obtain following.

Theorem 3.3. *For the Kähler C-space M of type (C_l, α_r) equipped with the Kähler metric $g(a\Lambda_{\alpha_r})$, the complex dimension, the scalar curvature τ and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given respectively by*

$$\dim_{\mathbb{C}} M = \frac{r(4l-3r+1)}{2}, \quad \tau = \frac{2(2l-r+1)}{a}\dim_{\mathbb{C}} M, \quad \lambda_1(M) = \frac{2(2l-r+1)}{a}.$$

4. The homogeneous Kähler hypersurface (C_l, α_2)

In this section, we will consider a Kähler C-space of type (C_l, α_r) as a Kähler submanifold of $G_r(\mathbb{C}^{2l})$.

Let's set

$$\mathfrak{g} = \mathfrak{sp}(l) = \left\{ \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix} \;\middle|\; \begin{matrix} A, C \in M_l(\mathbb{C}), \\ A^* = -A,\ {}^tC = C \end{matrix} \right\},$$

then $\mathfrak{g}$ is a compact semisimple Lie algebra of type C_l whose complexification is given by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(l, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;\middle|\; \begin{matrix} A, B, C \in M_l(\mathbb{C}), \\ {}^tB = B,\ {}^tC = C \end{matrix} \right\}.$$

Note that the Killing form $B_{\mathfrak{g}}$ is given by

$$B_{\mathfrak{g}}(X, Y) = 2(l+1)trXY, \quad X, Y \in \mathfrak{g}.$$

For integers i and j with $1 \le i, j \le l$, let E_{ij} be the matrix in $M_l(\mathbb{C})$ whose (i,j)-coefficient is 1 and others are zero. and let's set

$$\theta_i = \frac{\sqrt{-1}}{4(l+1)} \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$$

for $1 \le i, j \le l$. Relative to an abelian subalgebra $\mathfrak{t} = \mathbb{R}\{\theta_i, 1 \leqq i \leqq l\}$, the set Σ^+ of all positive roots is given as $\Sigma^+ = \{\theta_i - \theta_j\ (i < j),\quad \theta_i + \theta_j\ (i \leqq j)\}$. The simple roots α_i numbered as the last section is given by $\alpha_i = \theta_i - \theta_{i+1}\ (1 \leqq i \leqq l-1),\ \alpha_l = 2\theta_l$.

Σ_0 and $\Sigma_{\mathfrak{m}}^+$ are given by

$$\Sigma_0 = \left\{ \begin{matrix} \pm(\theta_i - \theta_j) & (1 \leqq i < j \leqq r \text{ or } r+1 \leqq i < j \leqq l), \\ \pm(\theta_i + \theta_j) & (r+1 \leqq i \leqq j \leqq l) \end{matrix} \right\},$$

$$\Sigma_{\mathfrak{m}}^+ = \left\{ \begin{matrix} \theta_i - \theta_j & (1 \leqq i \leqq r \text{ and } r+1 \leqq j \leqq l), \\ \theta_i + \theta_j & (1 \leqq i \leqq r \text{ and } i \leqq j \leqq l) \end{matrix} \right\}.$$

Choosing suitable root vectors, from (3.2) and (3.3), we can get

$$\mathfrak{u} = \left\{ \begin{pmatrix} A & A'' & B & B'' \\ 0 & A' & {}^tB'' & B' \\ 0 & 0 & -{}^tA & 0 \\ 0 & C' & -{}^tA'' & -{}^tA' \end{pmatrix} \;\middle|\; \begin{matrix} A, B \in M_r(\mathbb{C}), \\ A', B', C' \in M_{l-r}(\mathbb{C}), \\ A'', B'' \in M_{r,l-r}(\mathbb{C}), \\ {}^tB = B,\ {}^tB' = B',\ {}^tC' = C' \end{matrix} \right\},$$

$$\begin{aligned} \mathfrak{k} &= \mathfrak{g} \cap \mathfrak{u} \\ &= \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -\overline{C'} \\ 0 & 0 & \overline{A} & 0 \\ 0 & C' & 0 & \overline{A'} \end{pmatrix} \;\middle|\; \begin{matrix} A \in M_r(\mathbb{C}), \\ A', C' \in M_{l-r}(\mathbb{C}), \\ A^* = -A,\ A'^* = -A',\ {}^tC' = C' \end{matrix} \right\} \\ &= \mathfrak{u}(r) + \mathfrak{sp}(l-r). \end{aligned}$$

Therefore, the Kähler C-space M of type (C_l, α_r) is identified with the homogeneous space $G/K = Sp(l)/U(r)\cdot Sp(l-r)$.

For $x, y \in M_{l-r,r}(\mathbb{C})$ and $z \in M_r(\mathbb{C})$ with ${}^tz = z$, define

$$\eta(x,y,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & {}^ty & 0 & -{}^tx \\ y & 0 & 0 & 0 \end{pmatrix}.$$

Note that, if $r = l$, then we ignore x and y, and $\eta(x,y,z)$ and $\eta(0,0,z)$ denote a matrix $\begin{pmatrix} 0_l & 0_l \\ z & 0_l \end{pmatrix}$, $z \in M_l(\mathbb{C})$, ${}^tz = z$. (3.5) implies

$$\begin{aligned} \mathfrak{m} &= \{\eta(x,y,z) - \eta(x,y,z)^*\}, \\ \mathfrak{m}^+ &= \{\eta(x,y,z)\}. \end{aligned}$$

From (3.7), the G-invariant Kähler metric corresponding to $a\Lambda_{\alpha_r}$ is given by

(4.1) $$g(a\Lambda_{\alpha_r})(X,X) = 2a\, tr(x^*x + y^*y + \bar{z}z), \quad X = \eta(x,y,z) - \eta(x,y,z)^* \in \mathfrak{m}.$$

The natural inclusion $Sp(l) \to SU(2l)$ defines an immersion φ of M into $\tilde{M} = G_r(\mathbb{C}^{2l}) = \tilde{G}/\tilde{K} = SU(2l)/S(U(r)\cdot U(2l-r))$ by

$$\varphi(g\cdot K) = g\cdot \tilde{K}, \quad g \in G.$$

Under identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, the image of $X = \eta(x,y,z) - \eta(x,y,z)^* \in \mathfrak{m}$ is

$$\varphi_*(X) = \begin{pmatrix} 0 & -x^* & -\bar{z} & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix},$$

so that we have

(4.2) $$\tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c}\, tr(x^*x + y^*y + \bar{z}z),$$

where c is the maximal holomorphic sectional curvature of $G_r(\mathbb{C}^{2l})$. Therefore, Theorem 3.3, (4.1) and (4.2) imply the following.

Theorem 4.1. *For the Kähler C-space $M = Sp(l)/U(r)\cdot Sp(l-r)$ of type (C_l, α_r) equipped with the Kähler metric $g(\frac{2}{c}\Lambda_{\alpha_r})$, M is immersed in $G_r(\mathbb{C}^{2l})$ by the Kähler immersion φ. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by*

$$\dim_{\mathbb{C}} M = \frac{r(4l-3r+1)}{2}, \quad \lambda_1(M) = c\,(2l-r+1).$$

In particular, if $r = 2$, then $M = Sp(l)/U(2)\cdot Sp(l-2)$ is a Kähler hypersurface of $G_2(\mathbb{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by

$$\lambda_1(M) = c\,(2l-1).$$

For $z \in M_r(\mathbb{C})$, define an unit vector ν at the origin o of $G_2(\mathbb{C}^{2l})$ by

$$\nu(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{m}}, \quad \frac{4}{c}\, tr\, z^*z = 1.$$

Then $\nu(z)$ is tangent to M if and only if z is symmetric.

The Kähler hypersurface $M = (C_l, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{2l})$.

Proposition 4.2. *The Kähler hypersurface $M = Sp(l)/U(2)\cdot Sp(l-2)$ of $G_2(\mathbb{C}^{2l})$ satisfies*

$$\mathfrak{J}\, T^{\perp}M \subset TM \quad (\Longleftrightarrow J\xi \perp \mathfrak{J}\xi \text{ for any } \xi \in T^{\perp}M), \tag{4.3}$$

where TM and $T^{\perp}M$ are the tangent bundle and the normal bundle of M, respectively.

Proof. Let ν_o be an unit normal vector of M at o defined by

$$\nu_o = \nu(z_o), \quad z_o = \frac{1}{2}\sqrt{\frac{c}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that the normal space $T_o^{\perp}M$ is given by

$$T_o^{\perp}M = \mathbb{R}\left\{\nu_o,\, J\nu_o = \nu(\sqrt{-1}z_o)\right\}.$$

Then we see

$$\begin{aligned} \mathfrak{J}_o\, T_o^{\perp}M =& \mathbb{R}\{J_i\nu_o,\, J_iJ\nu_o, \quad i = 1,2,3\} \\ =& \mathbb{R}\left\{\nu(z_o\varepsilon_i),\, \nu(\sqrt{-1}z_o\varepsilon_i), \quad i = 1,2,3\right\}, \end{aligned}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 1. It is easy to check that $z_o\varepsilon_i$ and $\sqrt{-1}z_o\varepsilon_i$ are symmetric, so that we obtain

$$\mathfrak{J}_o\, T_o^{\perp}M \subset T_oM.$$

Since the quaternionic Kähler structure $\mathfrak{J}$ is $\tilde{G}$-invariant, and since the immersion φ is G-equivariant, (4.3) holds at any point of M. □

If the ambient space is $G_2(\mathbb{C}^4)$, then the condition (4.3) determines a Kähler hypersurface as follows:

Proposition 4.3. *Suppose that a Kähler hypersurface M of $Q^4 = G_2(\mathbb{C}^4)$ satisfies the condition*

$$\mathfrak{J}\, T^{\perp}M \subset TM.$$

Then M is totally geodesic. Moreover, if M is compact, then M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

Proof. Denote by $\tilde{\nabla}$ the Riemannian connection of Q^4, and denote by ∇, σ, A and $\nabla^{\perp}$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of M, respectively. It is well-known that Gauss' formula and Weingarten's formula hold:

$$\begin{aligned} \tilde{\nabla}_XY &= \nabla_XY + \sigma(X,Y), \\ \tilde{\nabla}_X\xi &= -A_\xi X + \nabla^{\perp}_X\xi, \end{aligned} \tag{4.4}$$

for $X, Y \in TM$ and $\xi \in T^{\perp}M$. The metric condition implies

$$\tilde{g}_c(\sigma(X,Y),\xi) = \tilde{g}_c(A_\xi X, Y). \tag{4.5}$$

Relative to the complex structure J, σ and A satisfy

$$\sigma(X, JY) = J\sigma(X,Y), \quad A_\xi \circ J = -J \circ A_\xi = -A_{J\xi}. \tag{4.6}$$

For a local unit normal vector field ξ, we define local vector fields as follow: $e_i = J_i\xi$, $i = 1,2,3$, where J_1, J_2 and J_3 are a local canonical basis of $\mathfrak{J}$. Then, under the assumption of this proposition, $\{e_1, e_2, e_3,\ Je_1, Je_2, Je_3,\ \xi,\ J\xi\}$ is a local orthonormal frame field of Q^4 such that $\{e_1, e_2, e_3,\ Je_1, Je_2, Je_3\}$ is a tangent frame of M. For $X \in TM$, (4.4) implies

$$\begin{aligned}\nabla_X e_i + \sigma(X, e_i) = \tilde{\nabla}_X e_i &= (\tilde{\nabla}_X J_i)\xi + J_i(\tilde{\nabla}_X \xi)\\ &= (\tilde{\nabla}_X J_i)\xi - J_i A_\xi X + J_i(\nabla^\perp_X \xi).\end{aligned} \tag{4.7}$$

Since $\mathfrak{J}$ is parallel with respect to the connection $\tilde{\nabla}$, we have $\tilde{\nabla}_X J_i \in \mathfrak{J}$, so that the normal component of (4.7) is

$$\begin{aligned}\sigma(X, e_i) &= -\,\tilde{g}_c(J_i A_\xi X,\ \xi)\xi - \tilde{g}_c(J_i A_\xi X,\ J\xi)J\xi\\ &= g_c(A_\xi X,\ e_i)\xi + g_c(A_\xi X,\ Je_i)J\xi,\end{aligned}$$

where g_c is the induced Kähler metric of M. On the other hand, (4.5) and (4.6) imply

$$\begin{aligned}\sigma(X, e_i) &= \tilde{g}_c(\sigma(X, e_i),\ \xi)\xi + \tilde{g}_c(\sigma(X, e_i),\ J\xi)J\xi\\ &= g_c(A_\xi X,\ e_i)\xi - g_c(A_\xi X,\ Je_i)J\xi.\end{aligned}$$

From these two equations, we get

$$g_c(A_\xi X,\ Je_i) = 0. \tag{4.8}$$

Instead of X, applying to JX, we have

$$g_c(A_\xi X,\ e_i) = g_c(-A_\xi JX,\ Je_i) = 0.$$

Therefore, we have $A_\xi = 0$, or $\sigma = 0$, so that M is totally geodesic. By B. Y. Chen and T. Nagano [4]'s results, if M is compact, M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$. □

5. Upper bounds for λ_1 of submanifolds in $G_r(\mathbb{C}^n)$

In this section, we prove Theorem A and Theorem B.

Let M be a compact connected Kähler hypersurface of $G_r(\mathbb{C}^n)$ immersed by a immersion φ. Denote by Δ, the Laplacian on M. It is well-known that every $HM(n,\mathbb{C})$-valued function F satisfies

$$(\Delta F,\ \Delta F)_{L^2} - \lambda_1(\Delta F,\ F)_{L^2} \geqq 0. \tag{5.1}$$

The equality holds if and only if F is a sum of eigenfunctions with respect to eigenvalues 0 and λ_1. It is equivalent to that there exists a constant vector $C \in HM(n,\mathbb{C})$ such that $\Delta(F - C) = \lambda_1(F - C)$.

Denote by H the mean curvature vector of the isometric immersion $\Phi = \Psi \circ \varphi$. Then, since M is minimal in $G_r(\mathbb{C}^n)$, (1.13) implies

$$\begin{aligned}2(r(n-r)-1)H_A &= 2r(n-r)\tilde{H}_A - \tilde{\sigma}_A(\xi,\xi) - \tilde{\sigma}_A(J\xi, J\xi)\\ &= c(rI - nA) - \tilde{\sigma}_A(\xi,\xi) - \tilde{\sigma}_A(J\xi, J\xi),\end{aligned} \tag{5.2}$$

where A is a position vector of $\Phi(M)$ in $HM(n,\mathbb{C})$, and ξ is a local unit normal vector field of φ. Using (1.15) and (5.2), we get

$$(H_A,\ A) = -1. \tag{5.3}$$

$HM(n,\mathbb{C})$-valued function Φ satisfies $\Delta\Phi = -2(r(n-r)-1)H$, so that (5.1) and (5.3) imply the following. The equality condition dues to T. Takahashi's theorem in [18].

Lemma 5.1.

$$2(r(n-r)-1)\int_M (H_A, H_A)\,dv_M - \lambda_1 vol(M) \geqq 0. \tag{5.4}$$

The equality holds if and only if Φ is a minimal immersion of M into some round sphere in $HM(n,\mathbb{C})$, more precisely, there exists some positive constant R and some constant vector $C \in HM(n,\mathbb{C})$ such that H_A satisfies

$$H_A = \frac{1}{R^2}(C - A). \tag{5.5}$$

Lemma 5.2. *If the equality holds in (5.4), then M is contained in a totally geodesic submanifold of $G_r(\mathbb{C}^n)$ which is product of Grassmann manifolds, more precisely, there exist integers k_i, r_i, $i = 1, \cdots, m$ such that*

$$0 \leqq r_i \leqq k_i, \qquad r_1 \geqq r_2 \geqq \cdots \geqq r_m,$$

$$\sum_{i=1}^m r_i = r, \qquad \sum_{i=1}^m k_i = n,$$

$$M \subset G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}) \subset G_r(\mathbb{C}^n). \tag{5.6}$$

Notice that $G_0(\mathbb{C}^{k_i}) = G_{k_i}(\mathbb{C}^{k_i}) = \{one\ point\}$.

Proof. Assume that the equality holds in (5.4).

Since M is minimal in $G_r(\mathbb{C}^n)$, H is normal to $G_r(\mathbb{C}^n)$. Then, from (1.8) and (5.5), we get

$$CA = AC, \tag{5.7}$$

where C is a constant vector in Lemma 5.1. Since $SU(n)$ acts on $G_r(\mathbb{C}^n)$ transitively, without loss of generality, we can assume that C is a diagonal matrix as follows:

$$C = \begin{pmatrix} c_1 I_{k_1} & & & 0 \\ & c_2 I_{k_2} & & \\ & & \ddots & \\ 0 & & & c_m I_{k_m} \end{pmatrix}, \quad k_i > 0, \quad c_i \neq c_j \ (i \neq j). \tag{5.8}$$

Notice that

$$n = k_1 + k_2 + \cdots + k_m.$$

Define a linear subspace L of $HM(n,\mathbb{C})$ by $L = \{Z \in HM(n,\mathbb{C}) \mid ZC = CZ\}$, so that

$$L = \left\{ \begin{pmatrix} Z_1 & & & 0 \\ & Z_2 & & \\ & & \ddots & \\ 0 & & & Z_m \end{pmatrix} \middle| \ Z_i \in M_{k_i}(\mathbb{C}) \right\}.$$

From (5.7), M is contained in $G_r(\mathbb{C}^n) \cap L$.

For each integer r_i with $0 \leqq r_i \leqq k_i$, $\sum_{i=1}^m r_i = r$, let's define connected subsets of $G_r(\mathbb{C}^n)$ by

$$W_{r_1,\cdots,r_m} = \left\{ \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix} \middle| \ \begin{matrix} A_i \in M_{k_i}(\mathbb{C}), \\ A_i^2 = A_i, \quad tr\, A_i = r_i \end{matrix} \right\}.$$

So, $G_r(\mathbb{C}^n) \cap L$ is a disjoint union of all $W_{r_1,\cdots,r_m}$'s. Since M is connected, M is contained in suitable one of $W_{r_1,\cdots,r_m}$'s, saying $W_{r_1,\cdots,r_m}$. By the definition, we see

$$W_{r_1,\cdots,r_m} = G_{r_1}(\mathbb{C}^{k_1}) \times G_{r_2}(\mathbb{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbb{C}^{k_m}).$$

Without loss of generality, we can choose a diagonal matrix C with respect to which the inequalities $r_1 \geqq r_2 \geqq \cdots \geqq r_m$ hold. □

From (1.12), (1.14) and (5.2), we get

$$H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c} (\Psi_* \xi)^2 (I - 2A) \right\}. \tag{5.9}$$

Using (1.6) and (1.7), we see

$$(H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ nr(n-r) - 2\, tr\, \frac{4}{c} r\, (\Psi_*\xi)^2 \left(I + \frac{n-2r}{r} A \right) + tr\, \frac{16}{c^2} (\Psi_*\xi)^2 (I-2A)(\Psi_*\xi)^2 (I-2A) \right\}. \tag{5.10}$$

Since the immersion Ψ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r,r}(\mathbb{C})$ satisfying $A_o = g_A A g_A^*$ and

$$\sqrt{\frac{c}{4}} \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} = g_A (\Psi_* \xi)\, g_A^* \,. \tag{5.11}$$

Since the inner product (,) is $\tilde{G}$-equivariant and ξ is unit, we have $tr\, v_A^* v_A = tr\, v_A v_A^* = 1$. After translating by g_A, together with (5.11), (5.10) implies

$$(H_A, H_A) = \frac{c}{2(r(n-r)-1)^2} \{ n(r(n-r) - 2) + 2\, tr\ (v_A^* v_A v_A^* v_A) \}. \tag{5.12}$$

Lemma 5.3. *For $v \in M_{n-r,r}(\mathbb{C})$ with $tr\, v^* v = 1$, the following inequality holds*

$$tr\, v^* v v^* v \leqq 1. \tag{5.13}$$

Moreover, the following three conditions are equivalent to each other.

(1) *The equality holds in* (5.13).

(2) *The hermitian r-matrix v^*v is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.

(3) *The hermitian $(n-r)$-matrix vv^* is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

If the equality holds in (5.13), *then there exists* $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ *such that $v' = QvP^*$ satisfies*

$$v'^* v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \textit{and} \quad v' v'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}.$$

Proof. Lemma 5.3 follows from that both of hermitian matrices v^*v and vv^* are similar to diagonal matrices with non-negative eigenvalues. □

Form (5.12) and Lemma 5.3, the following lemma is immediately obtained, which is used to prove Theorem A.

Lemma 5.4.

$$(H_A, H_A) \leqq \frac{c}{2(r(n-r)-1)}\left\{n - \frac{n-2}{r(n-r)-1}\right\}. \tag{5.14}$$

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose v_A satisfying

$$v_A^* v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \textit{and} \quad v_A v_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}. \tag{5.15}$$

proof of Theorem A. (5.4) and (5.14) imply

$$\lambda_1 \leqq c\left(n - \frac{n-2}{r(n-r)-1}\right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemmas 5.1 and 5.4 hold.

Assume $m = 1$. Then, (5.5) and (5.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(r(n-r)-1)}\left\{(rI - nA) - \frac{4}{c}(\Psi_* \xi)^2 (I - 2A)\right\}.$$

After translating by g_A, together with (5.11) and (5.15), we obtain

$$\begin{aligned} \frac{1}{R^2}(c_1 - 1)I_r &= \frac{c}{2(r(n-r)-1)}\left\{(r-n)I_r + \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}\right\}, \\ \frac{1}{R^2} c_1 I_{n-r} &= \frac{c}{2(r(n-r)-1)}\left\{rI_{n-r} - \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}\right\}. \end{aligned}$$

The first equation implies $r = 1$, and the second one implies $n - r = 1$. So, we have $n = 2$ and $r = 1$. This contradicts that M is a complex hypersurface.

Since $m \geqq 2$, from Lemma 5.2, M is contained in a proper totally geodesic submanifold of $G_r(\mathbb{C}^n)$. On the other hand, M is of complex codimension 1 in $G_r(\mathbb{C}^n)$. Consequently, either $r = 1$ or $r = n - 1$ occurs, and M is a totally geodesic complex hypersurface of a complex projective space $\mathbb{C}P^{n-1} \cong G_1(\mathbb{C}^n) \cong G_{n-1}(\mathbb{C}^n)$. □

Proof of Theorem B. Let's assume that M is a compact connected Kähler hypersurface of $G_2(\mathbb{C}^n)$ satisfying the condition $J\xi \perp \mathfrak{J}\xi$. Since both of the complex structure and the quaternionic Kähler structure are $\tilde{G}$-invariant, we obtain, at the origin A_o,

$$J\begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} \perp J_i\begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{5.16}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 1. Set

$$v_A = \begin{pmatrix} v_A' & v_A'' \end{pmatrix}, \quad v_A', v_A'' \in M_{n-2,1}(\mathbb{C}) \cong \mathbb{C}^{n-2}.$$

Using (1.10) and (1.11), (5.16) implies that $|v_A'| = |v_A''|$ and $v_A' \perp v_A''$. Combing these with $tr\, v_A^* v_A = 1$, we obtain $|v_A'| = |v_A''| = \frac{1}{\sqrt{2}}$, so that

$$v_A^* v_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.17}$$

Together with (5.17), (5.12) implies

$$(H_A, H_A) = \frac{c}{2(2n-5)}\left\{n - \frac{n-1}{2n-5}\right\}.$$

Therefore, form Lemma 5.1, we obtain

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemma 5.1 holds.

Computing dimensions of manifolds in (5.6), we have

$$2n - 5 \leqq \sum_{i=1}^{m} r_i(k_i - r_i). \tag{5.18}$$

From $\sum_{i=1}^{m} r_i = 2$ and $r_1 \geqq r_2 \geqq \cdots \geqq r_m$, the following two cases occur:

$$\text{Case I} \;:\; r_1 = r_2 = 1, \quad r_3 = \cdots = r_m = 0,$$
$$\text{Case II} \;:\; r_1 = 2, \quad r_2 = \cdots = r_m = 0.$$

In Case I, (5.18) implies $2n-5 \leqq k_1 + k_2 - 2 \leqq n-2$, so $n \leqq 3$. This is contradiction.

Therefore, Case II occurs. Then, (5.18) implies $2n - 5 \leqq 2(k_1 - 2)$, so that we have $n = k_1, \quad m = 1, \quad k_2 = \cdots = k_m = 0$. (5.5) and (5.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(2n-5)}\left\{(2I - nA) - \frac{4}{c}(\Psi_*\xi)^2(I - 2A)\right\}.$$

After translating by g_A, together with (5.11) and (5.17), we obtain

$$\frac{1}{R^2}(c_1 - 1) = \frac{c}{2(2n-5)}\left\{2 - n + \frac{1}{2}\right\},$$
$$\frac{1}{R^2}c_1 I_{n-2} = \frac{c}{2(2n-5)}\{2I_{n-2} - v_A v_A^*\}.$$

The second equation implies

$$v_A v_A^* = dI_{n-2}, \quad d = 2 - \frac{2(2n-5)}{c}\frac{c_1}{R^2}. \tag{5.19}$$

From (5.17), we have

$$d v_A = dI_{n-2} v_A = (v_A v_A^*) v_A = v_A (v_A^* v_A) = \frac{1}{2} v_A,$$

so that $d = \frac{1}{2}$. Consequently, taking traces of both sides of (5.19), we obtain $n = 4$.

Therefore, from Proposition 4.3, M is congruent to Q^3. □

6. The second fundamental form of $Sp(l)/U(2) \cdot Sp(l-2)$

In this section, we will consider a Kähler C-space $M_{11,l} = Sp(l)/U(2) \cdot Sp(l-2)$ as a Kähler submanifold of $G_2(\mathbb{C}^{2l})$, and determine its second fundamental form (cf. [3], [19]). We will use the same notations as those in the section 4.

Let us set $G = Sp(l)$ and $K = U(2) \cdot Sp(l-2)$. Then K is a closed subgroup of G. The Lie algebra $\mathfrak{g}$ of G and the Lie algebra $\mathfrak{k}$ of K are given by $\mathfrak{g} = \mathfrak{sp}(l)$ and $\mathfrak{k} = \mathfrak{u}(2) + \mathfrak{sp}(l-2)$.

For $x,y \in M_{l-2,2}(\mathbb{C})$ and $z \in M_2(\mathbb{C})$ with ${}^t z = z$, define $X(x,y,z) = \eta(x,y,z) - \eta(x,y,z)^*$. Then we have $\mathfrak{m} = \{X(x,y,z)\}$, $\mathfrak{m}^+ = \{\eta(x,y,z)\}$ and $\mathfrak{m}^- = \{{}^t\eta(x,y,z)\}$.

For $X = X(x,y,z)$, $X' = X(x',y',z') \in \mathfrak{m}$, define a Hermitian inner product g_o on $\mathfrak{m}$ by

$$g_o(X,X') = \frac{4}{c}\mathrm{Re}\, tr(x'^* x + y'^* y + \overline{z'} z),$$

then g_o is $ad(\mathfrak{k})$-invariant, so that g_o induces a G-invariant Kähler metric g on $M_{11,l}$. $(M_{11,l},\ J,\ g)$ is an Einstein Kähler manifold.

The natural inclusion $G \to \tilde{G}$ defines a G-equivariant Kähler immersion φ of $M_{11,l}$ into $\tilde{M} = G_2(\mathbb{C}^{2l})$, by $\varphi(g\cdot K) = g \cdot \tilde{K},\ g \in G$. The complex codimension of φ is 1, so that $M_{11,l}$ is a complex hypersurface of $G_2(\mathbb{C}^{2l})$.

For $X = X(x,y,z) \in \mathfrak{m}$, let's set

$$X_{\tilde{\mathfrak{k}}}(x,y,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\overline{y} & 0 \\ 0 & {}^t y & 0 & -{}^t x \\ 0 & 0 & \overline{x} & 0 \end{pmatrix}, \quad X_{\tilde{\mathfrak{m}}}(x,y,z) = \begin{pmatrix} 0 & -x^* & -z^* & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix}.$$

Denote by φ_*, the differential of φ. Then, the image of the tangent space $T_o(M_{11,l})$ is given by

$$\varphi_{*o}T_o(M_{11,l}) = \varphi_{*o}\mathfrak{m} = \{X_{\tilde{\mathfrak{m}}}(x,y,z)\} \subset \tilde{\mathfrak{m}} = T_{\tilde{o}}(G_2(\mathbb{C}^n)). \tag{6.1}$$

For $z \in M_2(\mathbb{C})$ with ${}^t z = -z$, set

$$\xi(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we can identify the normal space $T_o^{\perp}(M_{11,l})$ with the subspace

$$\mathfrak{m}^{\perp} = \{\xi(z)\} \tag{6.2}$$

of $\tilde{\mathfrak{m}}$. Since φ is G-equivariant, the normal space at $g\cdot o$ is given by

$$T_{g\cdot o}^{\perp}(M_{11,l}) = \left\{ \left[\frac{d}{dt} g\exp(t\xi)\cdot\tilde{o}\right]_{t=0} \,\middle|\, \xi \in \mathfrak{m}^{\perp} \right\}.$$

For $X = X(x,y,z) \in T_o(M_{11,l})$, the curve $c(t) = \exp(tX)\cdot\tilde{o}$ is a curve in $M_{11,l}$, so that the vector field X^* generated by X is tangent to $M_{11,l}$. Define a unit normal vector field along $c(t)$ by

$$\xi(t) = (\exp tX)_{*\tilde{o}}\xi_0, \quad \xi_0 = \xi(z_0), \quad z_0 = \sqrt{\frac{c}{8}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(1.3) implies

$$\left(L_{X^*}\xi(t) - \tilde{\nabla}_{X^*}\xi(t)\right)_{\tilde{o}} = -\left[X_{\tilde{\mathfrak{k}}}(x,y,z),\ \xi_0\right].$$

By the definition of the Lie derivative,

$$(L_{X^*}\xi(t))_{\tilde{o}} = \left[X^*,\ \xi(t)\right]_{\tilde{o}} = \left[\frac{d}{dt}\exp(-tX)_{*c(t)}\xi(t)\right]_{t=0} = \left[\frac{d}{dt}\xi_0\right]_{t=0} = 0,$$

so that we obtain

$$\tilde{\nabla}_{\varphi_{*o}X}\xi(t) = \left[X_{\tilde{\mathfrak{k}}}(x,y,z),\, \xi_0\right] = \begin{pmatrix} 0 & -z_0{}^t y & 0 & z_0{}^t x \\ -\overline{y}z_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \overline{x}z_0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{m}}.$$

From (6.1) and (6.2), we obtain the following.

Proposition 6.1. *$\tilde{\nabla}_{\varphi_{*o}X}\xi(t)$ is tangent to $M_{11,l}$. Moreover, the unit normal vector field $\xi(t)$ is parallel at o, and the Weingarten map satisfies*

$$A_{\xi_0}X(x,y,z) = X(\overline{y}z_0, -\overline{x}z_0, 0) \tag{6.3}$$

for any $X(x,y,z) \in \mathfrak{m}$.

Define three subspaces of $T_o(M_{11,l})$ by

$$V_0(o,\,\xi_0) = \left\{X(0,0,z) \,\middle|\, {}^t z = z,\ z \in M_2(\mathbb{C})\right\},$$
$$V_+(o,\,\xi_0) = \left\{X(x,y,0) \,\middle|\, x = (x_1,x_2),\ y = (-\overline{x_2}, \overline{x_1}),\ x_i \in M_{l-2,1}(\mathbb{C})\right\}$$

and

$$V_-(o,\,\xi_0) = \left\{X(x,y,0) \,\middle|\, x = (x_1,x_2),\ y = (\overline{x_2}, -\overline{x_1}),\ x_i \in M_{l-2,1}(\mathbb{C})\right\}.$$

We have the eigenspace decomposition of the tangent space $T_p(M_{11,l})$ as follows.

Proposition 6.2. *For any point $p \in M_{11,l}$ and any unit normal vector $\xi \in T_p^{\perp}(M_{11,l})$, there exist three subspaces V_0, V_+ and V_- of $T_p(M_{11,l})$, such that the following properties hold.*

(1) *V_0 is a J-invariant 0-eigenspace of A_ξ satisfying*

$$V_0 = \mathfrak{J}_p T_p^{\perp}(M_{11,l}).$$

(2) *$V_\pm$ are $\mathfrak{J}$-invariant $\pm\sqrt{\frac{c}{8}}$-eigenspaces of A_ξ satisfying*

$$JV_+ = V_-.$$

(3) *The eigenspace decomposition*

$$T_p(M_{11,l}) = V_0 \oplus V_+ \oplus V_-$$

holds.

Proof. In the case that $p = o$ and $\xi = \xi_0$, put $V_0 = V_0(o,\,\xi_0)$ and $V_\pm = V_\pm(o,\,\xi_0)$. By simple calculation of matrices, we can easily see that V_0, V_+ and V_- satisfy the properties of this proposition.

In the case that $p = o$ and ξ is arbitrary, (1.18) implies this proposition.

Since the structures J and $\mathfrak{J}$ are $\tilde{G}$-invariant, and since the immersion φ is G-equivariant, this proposition holds for arbitrary p and ξ. □

7. A SECOND FUNDAMENTAL FORM OF AN EINSTEIN KÄHLER HYPERSURFACE

In this section, we study an Einstein Kähler hypersurface of $G_2(\mathbb{C}^n)$, and under some assumption, determine its second fundamental form.

Let M be a Kähler hypersurface of $\tilde{M} = G_2(\mathbb{C}^n)$. The complex dimension m of M is equal to $2n-5$. Let p be any fixed point of M, and ξ be a local unit normal vector field around p, and set $\xi_1 = \xi$, $\xi_2 = J\xi$, so that $\{\xi_1, \xi_2\}$ is a local orthonormal frame field of the normal bundle $T^\perp M$.

Denote by R the curvature tensor field of M. Then we have the Gauss equation

$$(7.1) \qquad g(R(X,Y)Z,W) = \sum_{\alpha=1}^{2}\Big\{g(A_{\xi_\alpha}X,W)g(A_{\xi_\alpha}Y,Z) - g(A_{\xi_\alpha}X,Z)g(A_{\xi_\alpha}Y,W)\Big\} + \tilde{g}_c(\tilde{R}(X,Y)Z,W)$$

for any tangent vector fields X, Y, Z and W of M.

For any vector field X along M, denote by X^T and $X^\perp$, the tangential part of X and the normal part of X, respectively. Then, we obtain the following.

Lemma 7.1. *The Ricci curvature tensor Ric satisfies*

$$(7.2) \qquad Ric(Y,Z) = -2g(A_\xi^2Y,Z) + \frac{c}{8}\Big\{(2m+2)g(Y,Z) + 3\sum_{k=1}^{3} g\big((J_kY)^T,(J_kZ)^T\big) - \sum_{k=1}^{3} g\big((JJ_kY)^T,(JJ_kZ)^T\big) + 2\sum_{k=1}^{3}\tilde{g}_c(J\xi,J_k\xi)\,\tilde{g}_c(JJ_kY,Z)\Big\}$$

for any tangent vector fields Y and Z.

Proof. Let $\{e_1,\cdots,e_{2m}\}$ be a local orthonormal basis of TM. Note that A_{ξ_α} is symmetric. Moreover, from (1.18), $trA_{\xi_\alpha} = 0$ and $A_{\xi_1}^2 = A_{\xi_2}^2 = A_\xi^2$. So we get, from (7.1),

$$(7.3) \qquad Ric(Y,Z) = \sum_{i=1}^{2m} g(R(e_i,Y)Z,e_i) = -2g(A_\xi^2Y,Z) + \sum_{i=1}^{2m}\tilde{g}_c(\tilde{R}(e_i,Y)Z,e_i).$$

Since $\{e_1,\cdots,e_{2m}\ \xi,\ J\xi\}$ is a local orthonormal frame of $T\tilde{M}$, (1.4) implies

$$(7.4) \qquad \sum_{i=1}^{2m}\tilde{g}_c(JJ_ke_i,e_i) = -\tilde{g}_c(JJ_k\xi,\xi) - \tilde{g}_c(JJ_k(J\xi),J\xi) = 2\tilde{g}_c(J\xi,J_k\xi).$$

Combining (7.3), (1.5) and (7.4), we see that (7.2) holds. □

From now on, we assume that $\mathfrak{J}T^\perp M$ is a vector subbundle of the tangent bundle TM, i.e,

$$(7.5) \qquad \mathfrak{J}T^\perp M \subset TM.$$

This condition is equivalent to the condition that $J_p\nu \perp \mathfrak{J}_p\nu$, where p is any point of M and ν is any normal vector at p.

Set $V_0 = \mathfrak{J}T^\perp M$. For any unit normal vector ξ, $\{J_1\xi, J_2\xi, J_3\xi, JJ_1\xi, JJ_2\xi, JJ_3\xi\}$ is an orthonormal basis of V_0, i.e.,

$$(7.6) \qquad V_0 = Span_{\mathbb{R}}\{J_1\xi, J_2\xi, J_3\xi, JJ_1\xi, JJ_2\xi, JJ_3\xi\},$$

so that V_0 is J-invariant. Let's define V be the orthogonal complement of V_0 in TM. Then we have an orthogonal decomposition

$$TM = V_0 \oplus V.$$

It is easy to see that V is J-invariant and $\mathfrak{J}$-invariant.

For a fiber bundle $\mathfrak{F}$, denote by $\Gamma(\mathfrak{F})$ the linear space of all smooth sections of $\mathfrak{F}$.

Lemma 7.2.

(1) *V_0 is a subspace of* 0*-eigenspace of A_ξ, i.e., $A_\xi Y = 0$ for any $Y \in \Gamma(V_0)$.*

(2) *For any $X \in \Gamma(TM)$, $Y \in \Gamma(V)$ and $J' \in \Gamma(\mathfrak{J})$,*

$$g(\nabla_X Y,\ J'\xi) = -g(A_\xi X,\ J'Y). \tag{7.7}$$

Proof. For any $X \in \Gamma(TM)$ and $J' \in \Gamma(\mathfrak{J})$, since $J'\xi$ is a section of V_0, (1.16) implies

$$\begin{aligned}\nabla_X(J'\xi) + \sigma(X,\ J'\xi) &= \tilde{\nabla}_X(J'\xi) = (\tilde{\nabla}_X J')\xi + J'(\tilde{\nabla}_X \xi)\\ &= (\tilde{\nabla}_X J')\xi - J'A_\xi X + J'\nabla^\perp_X \xi.\end{aligned} \tag{7.8}$$

Since $\mathfrak{J}$ is parallel, $\tilde{\nabla}_X J' \in \mathfrak{J}$. Thus, under our assumption (7.5), we see that $(\tilde{\nabla}_X J')\xi$ and $J'\nabla^\perp_X \xi$ are tangent to M. Therefore, the normal component of (7.8) is given by

$$\begin{aligned}\sigma(X,\ J'\xi) &= -\tilde{g}_c\,(J'A_\xi X,\ \xi)\,\xi - \tilde{g}_c\,(J'A_\xi X,\ J\xi)\,J\xi\\ &= g\,(A_\xi X,\ J'\xi)\,\xi + g\,(A_\xi X,\ J'J\xi)\,J\xi\\ &= \tilde{g}_c\,(\sigma(X,\ J'\xi),\ \xi)\,\xi + \tilde{g}_c\,(\sigma(X,\ J'J\xi),\ \xi)\,J\xi,\end{aligned}$$

which, from (1.17), is equivalent to

$$\tilde{g}_c\,(\sigma(X,\ J'\xi),\ \xi)\,\xi - \tilde{g}_c\,(\sigma(X,\ J'\xi),\ J\xi)\,J\xi,$$

so that we have

$$\tilde{g}_c\,(\sigma(X,\ J'\xi),\ J\xi) = 0. \tag{7.9}$$

Exchanging X for $JX \in \Gamma(TM)$, we get $\tilde{g}_c\,(\sigma(JX,\ J'\xi),\ J\xi) = 0$, so that

$$\tilde{g}_c\,(\sigma(X,\ J'\xi),\ \xi) = 0. \tag{7.10}$$

From (7.9) and (7.10), we get $\sigma(X,\ J'\xi) = 0$. Therefore, (1.17) and (7.6) imply $\sigma(X,\ Y) = 0$ for any $Y \in \Gamma(V_0)$, namely, $A_\xi Y = 0$.

Next, we consider the V-component of (7.8). The assumption (7.5) implies that $(\tilde{\nabla}_X J')\xi$ and $J'\nabla^\perp_X \xi$ are sections of V_0, so that, for any $Y \in \Gamma(V)$, we get

$$g\,(\nabla_X(J'\xi),\ Y) = -\tilde{g}_c\,(J'A_\xi X,\ Y)\,.$$

Since $J'\xi \perp Y$, this implies (7.7) immediately. □

For any tangent vector field X of M, denote by X_0 and X_V, the V_0-component of X and V-component of X, respectively. Then, we obtain the following.

Lemma 7.3. *Under the assumption* (7.5)*, the Ricci curvature tensor Ric satisfies*

$$Ric(Y,\ Z) = -2g(A_\xi^2 Y_V,\ Z_V) + \frac{c}{8}\Big\{(4n-4)\,g(Y_0,\ Z_0) + (4n-2)\,g(Y_V,\ Z_V)\Big\} \tag{7.11}$$

for any tangent vector fields Y and Z.

Proof. Lemma 7.2 (1) implies that

$$g(A_\xi^2 Y,\ Z) = g(A_\xi^2 Y_V,\ Z) = g(A_\xi^2 Y_V,\ Z_V). \tag{7.12}$$

Since V is $\mathfrak{J}$-invariant, $J_k Y_V$ is a section of V, so that

$$\begin{aligned}(J_k Y)^\perp = (J_k Y_0)^\perp &= \tilde{g}_c(J_k Y_0,\ \xi)\,\xi + \tilde{g}_c(J_k Y_0,\ J\xi)\,J\xi \\ &= -g(Y_0,\ J_k\xi)\,\xi - g(Y_0,\ J_k J\xi)\,J\xi.\end{aligned}$$

Then, we get

$$\begin{aligned}g\big((J_kY)^T,\ (J_kZ)^T\big) &= \tilde{g}_c\big(J_kY,\ J_kZ\big) - \tilde{g}_c\big((J_kY)^\perp,\ (J_kZ)^\perp\big) \\ &= g\big(Y,\ Z\big) - g(Y_0,\ J_k\xi)\,g(Z_0,\ J_k\xi) - g(Y_0,\ J_kJ\xi)\,g(Z_0,\ J_kJ\xi),\end{aligned}$$

so that, from (7.6), we have

$$\begin{aligned}\sum_{k=1}^{3} g\big((J_kY)^T,\ (J_kZ)^T\big) &= 3g(Y,\ Z) - g(Y_0,\ Z_0) \\ &= 2g(Y_0,\ Z_0) + 3g(Y_V,\ Z_V).\end{aligned} \tag{7.13}$$

Exchanging Y and Z for JY and JZ respectively, we get

$$\sum_{k=1}^{3} g\big((JJ_kY)^T,\ (JJ_kZ)^T\big) = 2g(Y_0,\ Z_0) + 3g(Y_V,\ Z_V). \tag{7.14}$$

Since $J\xi \perp J_k\xi$, combining (7.2), (7.12), (7.13) and (7.14), we see that (7.11) holds. □

In the next stage, we consider the Codazzi's equation

$$g\big((\nabla_X A)_\xi Y - (\nabla_Y A)_\xi X,\ Z\big) = \tilde{g}_c(\tilde{R}(X,\ Y)Z,\ \xi) \tag{7.15}$$

for any tangent vector fields X, Y and Z of M.

Let μ be a non-zero eigenvalue of A_ξ, and Y be an eigenvector corresponding to μ. We can assume that μ is a local smooth function on M, and Y is a local smooth section of TM. Then, for any $X \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_X A)_\xi Y &= \nabla_X(A_\xi Y) - A_{\nabla_X^\perp \xi} Y - A_\xi(\nabla_X Y) \\ &= d\mu(X)Y + \mu\nabla_X Y - A_{\nabla_X^\perp \xi} Y - A_\xi(\nabla_X Y),\end{aligned}$$

so that, from Lemma 7.2 (1), since Y is a local section of V, we see

$$\begin{aligned}g\big((\nabla_X A)_\xi Y,\ J'\xi\big) &= \mu g\big(\nabla_X Y,\ J'\xi\big) - g\big(A_{\nabla_X^\perp \xi} Y,\ J'\xi\big) - g\big(A_\xi(\nabla_X Y),\ J'\xi\big) \\ &= \mu g\big(\nabla_X Y,\ J'\xi\big) - g\big(Y,\ A_{\nabla_X^\perp \xi} J'\xi\big) - g\big(\nabla_X Y,\ A_\xi J'\xi\big) \\ &= \mu g\big(\nabla_X Y,\ J'\xi\big)\end{aligned}$$

for any $J' \in \Gamma(\mathfrak{J})$. By Lemma 7.2 (2), we see

$$g\big((\nabla_X A)_\xi Y,\ J'\xi\big) = -\mu g\big(A_\xi X,\ J'Y\big).$$

If X is also an eigenvector of A_ξ corresponding to a non-zero eigenvalue λ, we get

$$g\big((\nabla_X A)_\xi Y,\ J'\xi\big) = -\lambda\mu g\big(X,\ J'Y\big) = \lambda\mu g\big(J'X,\ Y\big) \tag{7.16}$$

and

$$g\big((\nabla_Y A)_\xi X,\ J'\xi\big) = \lambda\mu g\big(J'Y,\ X\big) = -\lambda\mu g\big(J'X,\ Y\big). \tag{7.17}$$

On the other hand, from (1.5), we can see that, for above X and Y,

$$\tilde{g}_c(\tilde{R}(X, Y)J'\xi, \xi) = \frac{c}{4}\sum_{k=1}^{3}\tilde{g}_c(X, J_kY)\,\tilde{g}_c(J_kJ'\xi, \xi).$$

Since $\{J_1, J_2, J_3\}$ is a basis of $\mathfrak{J}$, there exist real numbers a^l, $l = 1, 2, 3$, such that $J' = \sum_{l=1}^{3} a^l J_l$, so that we see $\tilde{g}_c(J_kJ'\xi, \xi) = \sum_{l=1}^{3} a^l \tilde{g}_c(J_kJ_l\xi, \xi) = -a^k$ and

$$\tilde{g}_c(\tilde{R}(X, Y)J'\xi, \xi) = -\frac{c}{4}\sum_{k=1}^{3}\tilde{g}_c(X, a^kJ_kY) = \frac{c}{4}g(J'X, Y). \tag{7.18}$$

From (7.15), (7.16), (7.17) and (7.18), we obtain the following.

Lemma 7.4. *Under the assumption* (7.5), *the equality*

$$\left(\lambda\mu - \frac{c}{8}\right) g(J'X, Y) = 0 \tag{7.19}$$

holds, where X and Y are eigenvectors of A_ξ corresponding to non-zero eigenvalues λ and μ respectively, and J' is any section of $\mathfrak{J}$.

The following proposition is a goal of this section.

Proposition 7.5. *If an Einstein Kähler hypersurface M of $G_2(\mathbb{C}^n)$ satisfies the condition $\mathfrak{J}T^{\perp}M \subset TM$, then, for any point $p \in M$ and any unit normal vector $\xi \in T_p^{\perp}M$, there exist three subspaces V_0, V_+ and V_- of T_pM such that the following properties hold :*

(1) *V_0 is a J-invariant 0-eigenspace of A_ξ satisfying*

$$V_0 = \mathfrak{J}_pT_p^{\perp}M.$$

(2) *$V_\pm$ are $\mathfrak{J}_p$-invariant $\pm\sqrt{\frac{c}{8}}$-eigenspaces of A_ξ satisfying*

$$JV_+ = V_-.$$

(3) *The eigenspace decomposition*

$$T_pM = V_0 \oplus V_+ \oplus V_-$$

holds.

Moreover, n must be even.

Proof. Let $A_\xi|_V$ be the restriction of A_ξ to V. Denote by ρ, the scalar curvature of M. Since the Ricci curvature Ric satisfies the Einstein condition $Ric = \frac{\rho}{2m}g$, Lemma 7.3 implies

$$g(A_\xi^2Y_V, Z_V) = \frac{c}{16}\left\{(4n - 4 - \frac{4\rho}{cm})\,g(Y_0, Z_0) + (4n - 2 - \frac{4\rho}{cm})\,g(Y_V, Z_V)\right\} \tag{7.20}$$

for any tangent vector fields Y and Z. Choosing Y and Z as $Y = Z \in V_0$, we get $\rho = cm(n-1) = c(n-1)(2n-5)$. Therefore, (7.20) implies

$$g(A_\xi^2Y_V, Z_V) = \frac{c}{8}\,g(Y_V, Z_V),$$

equivalently, all eigenvalues of $A_\xi|_V$ are $\pm\sqrt{\frac{c}{8}}$. In particular, 0 is not an eigenvalue of $A_\xi|_V$, which, together with Lemma 7.2 (1), implies that V_0 is a 0-eigenspace of A_ξ. Denote by $V_\pm$, eigenspaces corresponding to $\pm\sqrt{\frac{c}{8}}$ respectively. Then V is a diagonal sum of subspaces $V_\pm : V = V_+ \oplus V_-$. From (1.18), we easily see $JV_+ = V_-$.

For any $X \in V_+$, $Y \in V_-$ and $J' \in \mathfrak{J}_p$, Lemma 7.4 implies $g(J'X, Y) = 0$. Since $J'X \in V$, we get $J'X \in V_+$, so that V_+ is $\mathfrak{J}_p$-invariant. Similarly, we can see that V_- is also $\mathfrak{J}_p$-invariant.

Since the real dimension of V_0 is 6, we have $dim_{\mathbb{R}} V = 2m - 6 = 4n - 16$ and $dim_{\mathbb{R}} V_\pm = \frac{1}{2} dim_{\mathbb{R}} V = 2n - 8$. Since $V_\pm$ are $\mathfrak{J}_p$-invariant, $2n - 8$ is a multiple of 4, so that n is even. □

8. A Focal Variety

Let M be an Einstein Kähler hypersurface M of $\tilde{M} = G_2(\mathbb{C}^n)$ satisfies the condition $\mathfrak{J}T^\perp M \subset TM$. By Proposition 7.5, n must be even, so that we put $n = 2l$. In this section, we study the first focal set of M, and prove our main theorem.

We will use the same notations as those in the section 7. Moreover, for any point $p \in M$ and any unit normal vector ξ, define subspaces of $T_p\tilde{M}$ by

$$\begin{aligned} V_{0,+} &= \ \mathfrak{J}\xi = Span_{\mathbb{R}} \left\{ J_1\xi, J_2\xi, J_3\xi \right\}, \\ V_{0,-} &= J\mathfrak{J}\xi = Span_{\mathbb{R}} \left\{ JJ_1\xi, JJ_2\xi, JJ_3\xi \right\}, \\ \perp_+ &= Span_{\mathbb{R}} \left\{ \xi \right\}, \\ \perp_- &= Span_{\mathbb{R}} \left\{ J\xi \right\}. \end{aligned}$$

By direct computation, (1.5) implies the following. Also see [2, Theorem 3].

Lemma 8.1. *Let $\tilde{R}_\xi$ be the curvature operator with respect to ξ, i.e, $\tilde{R}_\xi$ is defined by $\tilde{R}_\xi(X) = \tilde{R}(X, \xi)\xi$ for any $X \in T_p\tilde{M}$. Let κ be an eigenvalue of $\tilde{R}_\xi$, and T_κ be an eigenspace corresponding to κ. Then, we have the following complete table.*

κ	T_κ
0	$\perp_+ \oplus V_{0,-}$
$\frac{c}{8}$	$V_+ \oplus V_-$
$\frac{c}{2}$	$\perp_- \oplus V_{0,+}$

Let $U^\perp M$ be the unit normal bundle of M with a natural projection π , i.e., $U^\perp M$ is the subbundle of all unit normal vectors of M. For $\xi \in U^\perp M$, let $\gamma_\xi(t)$ be the geodesic of $G_2(\mathbb{C}^n)$, such that $\gamma_\xi(0) = \pi(\xi)$ and $\gamma'_\xi(0) = \xi$. For $r > 0$, define a smooth map F_r from $U^\perp M$ into $G_2(\mathbb{C}^n)$ by $F_r(\xi) = \gamma_\xi(r)$. If r is sufficiently small, the image $N_r = F_r(U^\perp M)$ is a tube around M with radius r, which is a real hypersurface of $G_2(\mathbb{C}^n)$. If $rank\left(F_{r*}\right)_\xi < dim_{\mathbb{R}}\tilde{M} - 1$ for some r and ξ, a point $F_r(\xi)$ is called a "focal point". $F_r(\xi)$ is called the first focal point if $F_t(\xi)$ is not a focal point for any t with $0 < t < r$.

Let $\xi(s)$ be a curve in $U^\perp M$ with $\xi(0) = \xi$ and $\xi'(0) = \hat{X} \in T_\xi(U^\perp M)$. Define a smooth map ψ by $\psi(t, s) = F_t(\xi(s))$, and define a vector field $Z(t)$ along γ_ξ by

$$Z(t) = \left(F_{t*}\right)_\xi \hat{X} = \left[\frac{d}{ds} F_t(\xi(s))\right]_{s=0} = \left[\frac{\partial}{\partial s}\psi\right]_{s=0}.$$

Since ψ is a variation of a geodesic γ_ξ, $Z(t)$ is a Jacobi field along γ_ξ, i.e, $Z(t)$ satisfies the Jacobi equation

$$\tilde{\nabla}_t^2 Z(t) + \tilde{R}(Z(t), \gamma'_\xi(t))\gamma'_\xi(t) = 0.$$

$Z(t)$ must satisfy the initial condition $Z(0) = \pi_{*\xi}\hat{X}$, $Z'(0) = \left[\tilde{\nabla}_s \xi(s)\right]_{s=0}$. We remark that the image $(F_{t*})_\xi \left(T_\xi(U^\perp M)\right)$ are spanned by above Jacobi fields.

To get a basic Jacobi field, set $Z(t) = f(t)P(t)$, where P is a parallel vector field along γ_ξ, and f is a smooth function. Since $\gamma'_\xi(t)$ and the curvature tensor $\tilde{R}$ are also parallel, the function f satisfies $f''(t)P(t) + f(t)\,\tau_t\big(\tilde{R}(P(0), \xi)\xi\big) = 0$, where τ_t is a parallel displacement along $\gamma_\xi(t)$. In particular, if $P(0) \in T_\kappa$ and $P(0) \neq 0$, then f satisfies $f''(t) + \kappa f(t) = 0$.

Lemma 8.2. *For each of the cases below, there exists a curve $\xi(s)$ in $U^\perp M$, such that f satisfies*

$$f''(t) + \kappa f(t) = 0, \quad f(0)P(0) = \pi_{*\xi}\xi'(0), \quad f'(0)P(0) = \left[\tilde{\nabla}_s \xi(s)\right]_{s=0}. \tag{8.1}$$

(1) *$P(0) \in \perp_-$ and $f(t) = \sqrt{\frac{2}{c}}\, \sin \sqrt{\frac{c}{2}}t$.*
(2) *$P(0) \in V_{0,+}$ and $f(t) = \cos \sqrt{\frac{c}{2}}t$.*
(3) *$P(0) \in V_{0,-}$ and $f(t) \equiv 1$.*
(4) *$P(0) \in V_+$ and $f(t) = \sqrt{2}\cos\left(\sqrt{\frac{c}{8}}t + \frac{\pi}{4}\right)$.*
(5) *$P(0) \in V_-$ and $f(t) = \sqrt{2}\cos\left(\sqrt{\frac{c}{8}}t - \frac{\pi}{4}\right)$.*

Proof. In the case (1), there exists $a \in \mathbb{R}$, such that $P(0) = aJ\xi$. Set $\xi(s) = \cos as \cdot \xi + \sin as \cdot J\xi$. Then, we see $\pi_{*\xi}\xi'(0) = 0$ and $\left[\tilde{\nabla}_s \xi(s)\right]_{s=0} = aJ\xi$. From Lemma 8.1, we have $\kappa = \frac{c}{2}$. Therefore, the equation (8.1) is equivalent to $f'' + \frac{c}{2}f = 0$, $f(0) = 0$, $f'(0) = 1$, which has a unique solution $f(t) = \sqrt{\frac{2}{c}}\, \sin \sqrt{\frac{c}{2}}t$.

In other cases, $X = P(0)$ is tangent to M. Let $c(s)$ be a curve in M with $c'(0) = X$, and $\xi(s)$ be a parallel normal vector field along $c(s)$, satisfying $\xi(0) = \xi$. Then, we see $\pi_{*\xi}\xi'(0) = X$ and $\left[\tilde{\nabla}_s \xi(s)\right]_{s=0} = -A_\xi X$.

Let's assume $X \in V_+$. Lemma 8.1 implies $\kappa = \frac{c}{8}$, and Proposition 7.5 implies $\left[\tilde{\nabla}_s \xi(s)\right]_{s=0} = -\sqrt{\frac{c}{8}}X$. Therefore, the equation (8.1) is equivalent to $f'' + \frac{c}{8}f = 0$, $f(0) = 1$, $f'(0) = -\sqrt{\frac{c}{8}}$, which has a unique solution $f(t) = \sqrt{2}\cos\left(\sqrt{\frac{c}{8}}t + \frac{\pi}{4}\right)$, so that the case (4) is proved.

The remaining cases are similarly proved. □

Let's set $r_1 = \sqrt{\frac{2}{c}}\frac{\pi}{2}$. Then, any point of N_{r_1} is the first focal point, the image of $(F_{r_1 *})_\xi$ is a vector space $\tau_{r_1}(\perp_- \oplus V_{0,-} \oplus V_-)$, and $rank(F_{r_1 *})_\xi = \frac{1}{2}dim_{\mathbb{R}}\tilde{M}$, so that the first focal set N_{r_1} is a submanifold of $\tilde{M}$. The tangent space of N_{r_1} at $q = F_{r_1}(\xi)$ is given by

$$T_q N_{r_1} = \tau_{r_1}(\perp_- \oplus V_{0,-} \oplus V_-),$$

which is $\mathfrak{J}$-invariant. It is easy to see that the real dimension of N_{r_1} is equal to $\frac{1}{2}dim_{\mathbb{R}}\tilde{M}$. Moreover, the normal space of N_{r_1} at q is given by

$$T_q^\perp N_{r_1} = \tau_{r_1}(\perp_+ \oplus V_{0,+} \oplus V_+),$$

so that we see

$$JT_q N_{r_1} = T_q^\perp N_{r_1}.$$

Therefore, we obtain the following.

Proposition 8.3. *The first focal set N_{r_1} of M is a quaternionic Kähler, totally real submanifold of $G_2(\mathbb{C}^{2l})$. The real dimension of N_{r_1} is one half of $dim_{\mathbb{R}}G_2(\mathbb{C}^{2l})$.*

In [21], H. Tasaki showed that any complete, quaternionic Kähler, totally real submanifold of $G_2(\mathbb{C}^{2l})$ is congruent to a quaternionic projective space. Then, for some fixed $q \in N_{r_1}$, there exists a quaternionic projective space $\mathbb{H}P^{l-1}$, such that $q \in \mathbb{H}P^{l-1}$ and $T_q N_{r_1} = T_q \mathbb{H}P^{l-1}$. In [1], Alekseevskii proved that a quaternionic submanifold in a quaternionic Kähler manifold is totally geodesic. Therefore, N_{r_1} is a open portion of $\mathbb{H}P^{l-1}$.

By Proposition 6.2, $M_{11,l}$ satisfies the same assumption as M. Then, the first focal set of $M_{11,l}$ is congruent to $\mathbb{H}P^{l-1}$ up to the automorphism of $G_2(\mathbb{C}^{2l})$, so that M and $M_{11,l}$ are locally congruent. Therefore, we complete the proof of Theorem C.

References

1. D. V. Alekseevskii, *Compact quaternion spaces*, Functional Anal. Appl. **2** (1968), 109–114.
2. J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Sem. Mat. Univ. Politec. Torino **55** (1997), 19–83.
3. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458–538.
4. B. Y. Chen and T. Nagano, *Totally geodesic submanifolds of symmetric spaces, I*, Duke Math. J. **44** (1977), 745–755.
5. ——, *Totally geodesic submanifolds of symmetric spaces, II*, Duke Math. J. **45** (1978), 405–425.
6. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. series 2 **6** (1957), 111–244.
7. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.
8. S. Ihara, *Holomorphic imbeddings of symmetric domains*, J. Math. Soc. Japan **19** (1967), 261–302.
9. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry I, II*, John Wiley and Sons, Interscience, New York, 1969.
10. K. Konno, *Homogeneous hypersurfaces in Kähler C-spaces with $b_2 = 1$*, J. Math. Soc. Japan **40** (1988), 687–703.
11. Y. Miyata, *A characterization of certain Einstein Kähler hypersurfaces in a complex Grassmann manifold of 2-planes*, Tokyo J. Math.(to appear).
12. ——, *Spectral geometry of Kähler hypersurfaces in a complex Grassmann manifold*, Tokyo J. Math.(to appear).
13. ——, *Spectral geometry of Kähler hypersurfaces in the complex Grassmann manifold*, RIMS Kokyuroku **995** (1997), 95–110.
14. H. Nakagawa and R. Takagi, *On locally symmetric Kaehler submanifolds in a complex projective space*, J. Math. Soc. Japan **28** (1976), 638–667.
15. M. Obata, *Riemannian manifolds admitting a solution of a certain system of differential equations*, Proc. U.S.-Japan Sem. in Differential Geom., Kyoto, Japan (1965), 101–114.
16. A. Ros, *On spectral geometry of Kaehler submanifolds*, J. Math. Soc. Japan **36** (1984), 433–448.
17. I. Satake, *Holomorphic imbeddings of symmetric domains into a Siegel space*, Amer. J. Math. **87** (1965), 425–461.
18. T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.
19. M. Takeuchi, *Homogeneous Kähler submanifolds in complex projective spaces*, Japan. J. Math. **4** (1978), 171–219.
20. ——, *Modern Spherical Functions*, Translations of Mathematical Monographs, vol. 135, Amer. Math. Soc., 1994.
21. H. Tasaki, *Quaternionic submanifolds in quaternionic symmetric spaces*, Tohoku Math. Journ. **38** (1986), 513–538.
22. K. Tsukada, *Parallel Kaehler submanifolds of Hermitian symmetric spaces*, Math. Z. **190** (1985), 129–150.