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Interior gradient estimate for curvature flow

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Abstract

Our purpose is to understand the anisotropic curvature flow. Especially we like to prove the interior gradient estimate. We establish the interior gradient estimate for general 1-D anisotropic curvature flow. The estimate depends only on the height of the graph and not on the gradient at initial time.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$. A surface given as a graph $u:\Omega\to\mathbf{R}$ is a minimal surface when u satisfies

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0. \tag{1.1}$$

For this equation, the following interior gradient estimates are well-known ([5, 6, 7]): Given a constant M and $\tilde{\Omega}\subset\subset\Omega$, there exists a constant C depending only on M and $\tilde{\Omega}$ such that if $\sup_{\Omega}|u|\leq M$, then $\sup_{\tilde{\Omega}}|\nabla u|\leq C$. The similar estimates are also known for the mean curvature flow equation ([3]). That is, if $u:\Omega\times(0,T)\to\mathbf{R}$ satisfies

$$\frac{u_t}{\sqrt{1+|\nabla u|^2}}=\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{1.2}$$

and $\sup_{\Omega\times[0,T]}|u| \le M$, $\tilde{\Omega} \subset\subset \Omega$, $0 < T_0 < T$, then there exists C such that $\sup_{\tilde{\Omega}\times[T_0,T]}|\nabla u| \le C$. Again, C is a constant depending only on M, $\tilde{\Omega}$ and T_0. Note that C is independent of the gradient at $t = 0$.

One direction to extend those results are to consider general anisotropic curvature problem, namely, to consider the variational problem corresponding to the energy functional

$$F(u) = \int_\Omega a(\nu)\sqrt{1+|\nabla u|^2},$$

where $\nu = (\nabla u, -1)/\sqrt{1+|\nabla u|^2}$ is the unit normal vector to the graph of u and the function $a : \mathbf{R}^{n+1} \to \mathbf{R}^+$ is the surface energy density and should satisfy certain convexity property. The Euler-Lagrange equation is

$$\operatorname{div}_x a_p(\nu) = 0, \tag{1.3}$$

and the curvature flow equation is

$$\frac{u_t}{\sqrt{1+|\nabla u|^2}} = \operatorname{div}_x a_p(\nu). \tag{1.4}$$

The left-hand side of the equation (1.4) corresponds to the normal velocity of the curve $(x, u(x,\cdot))$ while the right-hand side is the weighted anisotropic curvature. This is a gradient flow of the anisotropic surface energy functional

$$\int_\Omega a(\nu)\, ds,$$

where $ds = \sqrt{1+|\nabla u|^2}dx$ and $\nu = (-\nabla u, 1)/\sqrt{1+|\nabla u|^2}$ with homogeneous Dirichlet ($u = 0$) or Neumann ($a_p(-\nabla u, 1) = 0$) boundary conditions, since

$$\frac{d}{dt}\int_\Omega a(\nu)\, ds = \int_\Omega a_p(-\nabla u, 1)\cdot \nabla u_t\, dx = -\int_\Omega |\operatorname{div}_x a_p(-\nabla u, 1)|^2\, ds.$$

We show the interior gradient estimates for general anisotropic curvature flow for one-dimensional case which is independent of the initial time gradient.

2 Main Theorem

Let $r > 0$ be given. The graph $u : [-r, r] \times [0, T] \to \mathbb{R}$ is said to be an anisotropic curvature flow if smooth function u satisfies

$$\frac{u_t}{\sqrt{1+u_x^2}} = (a_p(u_x, -1))_x. \tag{2.1}$$

where $a : \mathbf{R}^2 \to [0, \infty)$ is an anisotropic surface energy density function satisfying the following assumptions:

(a) $a(tp, tq) = t\,a(p, q)$ for all $t > 0$,

(b) a is a convex function,

(c) there exists $\delta_0 > 0$ such that $a(p, q) - \delta_0|(p, q)|$ is a convex function,

(d) a is smooth except at $(0, 0)$.

Under these assumptions, we show

Theorem 1
Suppose u is a smooth solution of (2.1) *on* $[-r, r] \times [0, T]$ *satisfying*

$$\sup_{[-r,r]\times[0,T]} |u| \leq M.$$

Given $0 < s < r$ *and* $0 < t_0 < T$, *there exists a constant* $C > 0$ *depending only on* δ_0, M, t_0, s, r *such that*

$$\sup_{[-(r-s),r-s]\times[t_0,T]} |u_x| \leq C.$$

Note that the estimate is independent the gradient of the initial data. Also we point out that the dependence of C on a is only through the lower bound of the uniform convexity δ_0, but not on the upper bound (such as C^1 bound). Thus, the result in this paper can be extended equally to the non-smooth anisotropic curvature flow problem [4] by approximations.

Remark 1 *For example,* $a(p, q) = (p^2 + q^2)^{\frac{1}{2}}$ *is isotropic curvature flow (mean curvature flow) and satisfies above assumptions.* $a(p, q) = (|p|^r + |q|^r)^{\frac{1}{r}}$ $(1 < r < \infty)$ *is anisotropic curvature flow and also satisfies assumptions.*

Remark 2 *In general dimension, if we assume the axis symmetry of the graph of u, we expect to prove the same interior gradient estimate.*

3 Proof

We cite the following theorem due to Angenent [2] which says that the number of zeros of the solution of parabolic equations is nonincresing as time increases.

Lemma 1 *(Angenent [2])*
Suppose $u \in C^\infty([x_1, x_2] \times [0,T])$ satisfies the equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u \tag{3.1}$$

on $[x_1, x_2] \times [0,T]$ and

$$u(x_j, t) = 0 \text{ for } t \in [0,T] \quad j = 1, 2.$$

Here, a, b, c are smooth functions of (x,t) and $a > 0$. Then for all $t \in (0,T]$, the zero set of $x \to u(x,t)$ will be finite, even when counted with multiplicity. The number of zeros of $x \to u(x,t)$ counted with multiplicity is nonincreasing function of t.

Proof of Theorem. Given $0 < s < r$ and $0 < t_0 < T$, we construct a solution v for (2.1) on $[-s, s] \times (0,T]$ with the following properties:

(a) $v(-s,t) = -M - 1$ and $v(s,t) = M + 1$ for $0 < t \le T$,

(b) $v_x > 0$ on $[-s, s] \times (0, T]$,

(c) for any $-s < x \le s$, $\lim_{t\to 0} v(x,t) > M$.

The property (c) means that v has an initial data which is vertical at $x = -s$. We show that the function v has a gradient bound $0 < v_x \le C$ on $[-s, s] \times [t_0, T]$, where C depends only on M, δ_0, s, t_0. We show the existence of such v later in the proof. Assuming such v exists for now, we then prove that any solution with $\sup_{[-r,r]\times[0,T]} |u| \le M$ satisfies $\sup_{[-(r-s),r-s]\times[t_0,T]} u_x \le C$. The same argument using $-u$ will show $\sup_{[-(r-s),r-s]\times[t_0,T]} |u_x| \le C$. For a contradiction, assume that there exists a point $(\bar{x}, \bar{t}) \in [-(r-s), r-s] \times [t_0, T]$ with $u_x(\bar{x}, \bar{t}) > C$. Since $\sup|u| \le M$ and by (a), we may choose λ so that $|\bar{x} - \lambda| < s$ and $v(\bar{x} - \lambda, \bar{t}) = u(\bar{x}, \bar{t})$. With this λ, define $v_\lambda(x,t) = v(x - \lambda, t)$. Since $u_x(\bar{x}, \bar{t}) > C \ge (v_\lambda)_x(\bar{x}, \bar{t})$ and $v_\lambda(\lambda + s, \bar{t}) = v(s, \bar{t}) = M + 1 > u(\lambda + s, \bar{t})$, there has to be at least another point $\bar{x} < \tilde{x} < \lambda + s$ such that $u(\tilde{x}, \bar{t}) = v_\lambda(\tilde{x}, \bar{t})$. Thus $u - v_\lambda$ has at least two zeros at $t = \bar{t}$ on

$\lambda - s < x < \lambda + s$. Function $u - v_\lambda$ satisfies the equation of the type (3.1) on $[\lambda - s, \lambda + s] \times (0, T]$, with non-zero boundary values for all $t > 0$ due to $\sup |u| \le M$ and (a). Thus we may use Lemma 1 and conclude that $u - v_\lambda$ has at least two zeros in x variable for all $\bar{t} > t > 0$. Since $v_\lambda > M$ for x away from $\lambda - s$ and all small t, and since we assume that u is a smooth function up to $t = 0$, this is impossible to satisfy for all small enough t. (See fig. 3 and 4.)

Thus it remains to prove the existence of such v. To do this, we invert the role of independent variable x and dependent variable $y = v(x, t)$. Let $y = w(x, t)$ be the inverse function of v with respect to the space variables, i.e., w satisfies $y = v(w(y, t), t)$ identically. Since the equation is geometric, w should satisfy the similar equation to (2.1) on $[-M - 1, M + 1] \times (0, T]$ with the role of y and x exchanged. Now, the conditions on v in terms of w are

(a') $w(-M - 1, t) = -s$ and $w(M + 1, t) = s$ for $0 < t \le T$,

(b') $w_x > 0$ on $[-M - 1, M + 1] \times (0, T]$,

(c') for any $-M - 1 \le x \le M$, $\lim_{t \to 0} w(x, t) = -s$.

Furthermore, on $[-M - 1, M + 1] \times (0, T]$, w should satisfy

$$\frac{w_t}{\sqrt{1 + w_x^2}} = (a_q(1, w_x))_x. \tag{3.2}$$

Since $\frac{\partial y}{\partial x} = 1/\frac{\partial x}{\partial y}$, we need to show that there exists a constant $C > 0$ such that $w_x > C$ on $[-M, M] \times [t_0, T]$. We solve (3.2) with the following convex initial data. Let $\Gamma \in C^\infty([-M - 1, M + 1])$ (See fig.2 and 4.) be

- $\Gamma(x) = -s$ for $x \in [-M - 1, M]$,
- $\Gamma(M + 1) = s$, $\Gamma''(M + 1) = 0$,
- $\Gamma(x) \ge -s$, $\Gamma'(x) \le 3s$, $\Gamma''(x) \ge 0$ for $x \in [M, M + 1]$.

Let w be the unique smooth solution of (3.2) with the initial data Γ and the boundary data (a'). Since any functions $c_1 + c_2 x$ are solutions of (3.2), one obtains the gradient estimate

$$0 \le w_x \le 3s \tag{3.3}$$

on $[-M-1, M+1] \times [0,T]$, by using these functions as barriers and the standard maximum principle applied to w_x. Also, note that the convexity of w is preserved, i.e., $w_{xx} \geq 0$. This is seen by differentiating the equation with respect to t and then applying the maximum principle to w_t. $w_t = 0$ on the boundary and $w_t = a_{qq} w_{xx} \geq 0$ for $t = 0$ imply $w_t \geq 0$. The equation then yields $w_{xx} \geq 0$ on $[-M-1, M+1] \times [0,T]$.

Now, (3.3) implies that $a_{qq}(-1, w_x) \geq c(s, \delta_0)$(call this δ)> 0 by assumption (c). We claim that the solution of

$$\begin{cases} z_t = \delta z_{xx} & [-M-1, M+1] \times [0,T], \\ z(\pm(M+1), t) = \pm s & t \in [0,T], \\ z(x,0) = \Gamma(x) & x \in [-M-1, M+1] \end{cases}$$

satisfies $w \geq z$ on $[-M-1, M+1] \times [0,T]$.(See fig.2) This is because of the following combined with the standard maximum principle:

$$(w-z)_t = a_{qq}(-1, w_x) w_{xx} - \delta z_{xx} = a_{qq}(-1, w_x)(w-z)_{xx} + (a_{qq}(-1, w_x) - \delta) z_{xx}$$

$$\geq a_{qq}(-1, w_x)(w-z)_{xx}.$$

In the last line, we used $z_{xx} \geq 0$, which follows by the same reason for $w_{xx} \geq 0$ before, and $a_{qq}(-1, w_x) \geq \delta$. We next claim that for $t_0 \leq t$, there exists $c = c(t_0, s, \delta) > 0$ such that $z_x \geq c$ on $[-M-1, M+1] \times [t_0, T]$. z_x satisfies again the heat equation with non-negative initial data and the homogeneous Neumann data, and thus by the strong maximum principle (or extending the solution to $\mathbb{R}$ by a suitable reflection argument and then using the representation formula with the heat kernel) we have such c. Since $w_{xx} \geq 0$, for (x,t) with $t \geq t_0$, we have

$$w_x(x,t) \geq w_x(-M-1, t) \geq z_x(-M-1, t) \geq c$$

as the result. Note that we are using $w \geq z$ and $w = z$ on the boundary $x = -M-1$. This completes the proof.

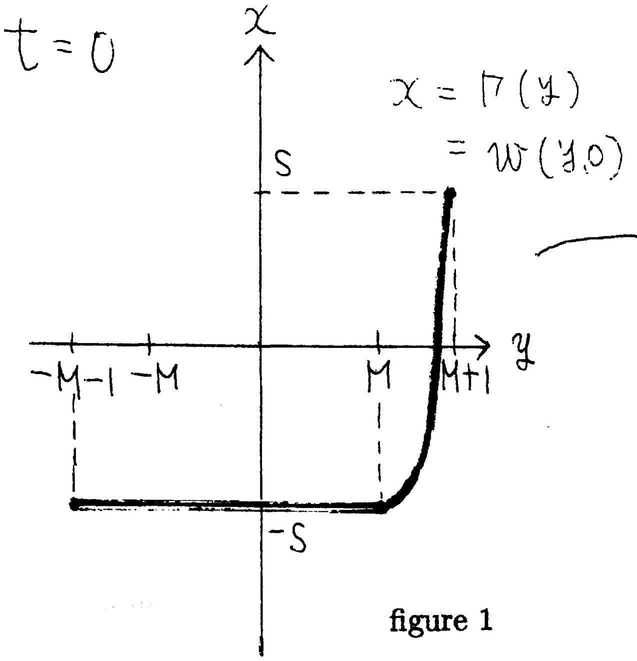
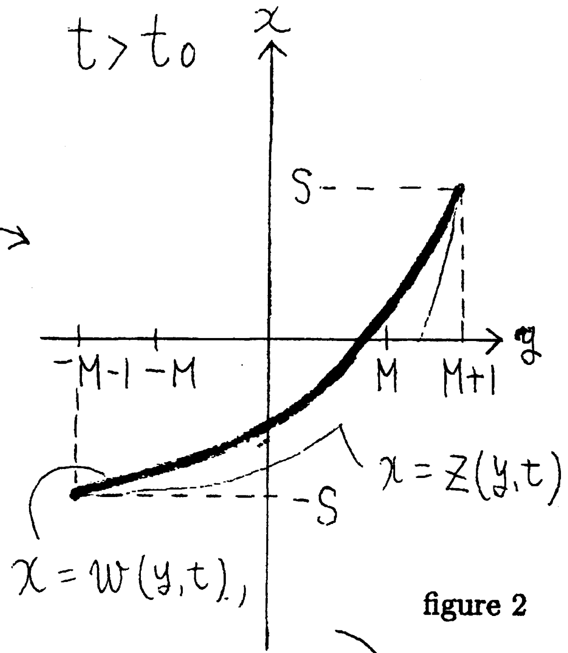


figure 1



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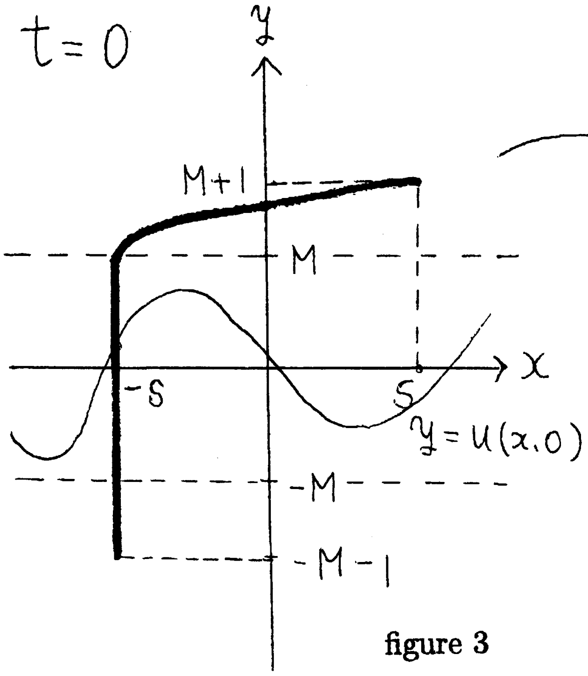

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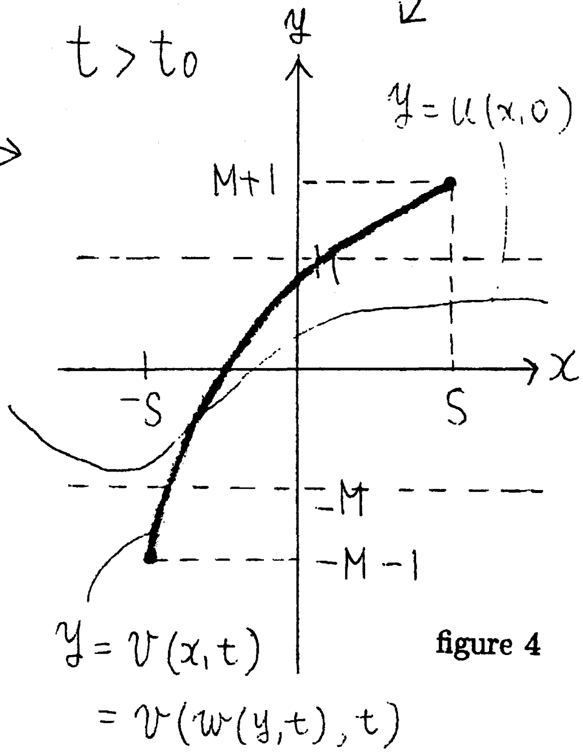

figure 4

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