数理解析研究所講究録 1405 巻 2004 年 187-196

Multiple solutions for some singular perturbation problem

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0. Introduction

In this paper we consider the existence and multiplicity of solutions of the following nonlinear Schrödinger equations:

$$\begin{aligned} &-\Delta u+(\lambda^2 a(x)+1)u=|u|^{p-1}u \quad \text{in } \mathbf{R}^N, \\ &u(x)\in H^1(\mathbf{R}^N). \end{aligned} \tag{P_λ}$$

Here $p\in(1,\frac{N+2}{N-2})$ if $N\geq 3$, $p\in(1,\infty)$ if $N=1$, 2 and $a(x)\in C(\mathbf{R}^N,\mathbf{R})$ is non-negative on $\mathbf{R}^N$. We consider multiplicity of solutions (including positive and sign-changing solutions) when the parameter λ is very large.

For $a(x)$, we assume

(a1) $a(x)\in C(\mathbf{R}^N,\mathbf{R})$, $a(x)\geq 0$ for all $x\in\mathbf{R}^N$ and the potential well $\Omega=int\, a^{-1}(0)$ is a non-empty bounded open set with smooth boundary $\partial\Omega$ and $a^{-1}(0)=\overline{\Omega}$.

(a2) $0<\liminf_{|x|\to\infty} a(x)\leq \sup_{x\in\mathbf{R}^N} a(x)<\infty$.

When λ is large, the potential well Ω plays important roles and the following Dirichlet problem appears as a limit of (P_λ):

$$\begin{aligned} &-\Delta u+u=|u|^{p-1}u \quad \text{in } \Omega, \\ &u=0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{0.1}$$

We remark that solutions of (P_λ) and (0.1) can be characterized as critical points of

$$\Psi_\lambda(u)=\int_{\mathbf{R}^N}\frac{1}{2}(|\nabla u|^2+(\lambda^2 a(x)+1)u^2)-\frac{1}{p+1}|u|^{p+1}\,dx:\ H^1(\mathbf{R}^N)\to\mathbf{R}, \tag{0.2}$$

$$\Psi_\Omega(u)=\int_\Omega\frac{1}{2}(|\nabla u|^2+u^2)-\frac{1}{p+1}|u|^{p+1}\,dx:\ H_0^1(\Omega)\to\mathbf{R} \tag{0.3}$$

and it is known that (0.3) has an unbounded sequence of critical values (cf. ...)

Bartsch and Wang [**BW2**] and Bartsch, Pankov and Wang [**BPW**] studied such a situation firstly. Their assumptions on $a(x)$ and nonlinearity are more general and as a special case of their results we have

(i) There exists a least energy solution $u_\lambda(x)$ of (P_λ). Moreover $u_{\lambda_n}(x)$ converges strongly to a least energy solution of (0.3) after extracting a subsequence $\lambda_n \to \infty$ ([**BW2**]).

(ii) When $N \geq 3$ and $p \in (1, \frac{N+2}{N-2})$ is close to $\frac{N+2}{N-2}$, there exists at least $\mathrm{cat}(\Omega)$ positive solutions of (P_λ) for large λ ([**BW2**]). Here $\mathrm{cat}(\Omega)$ denotes Lusternik-Schnirelman category of Ω.

(iii) For any $n \in \mathbf{N}$, there exist n pairs of (possibly sign-changing) solutions $\pm u_{1,\lambda}(x), \cdots, \pm u_{n,\lambda}(x)$ of (P_λ) for large $\lambda \geq \lambda(n)$. Moreover they converge to distinct solutions $\pm u_1(x), \cdots, \pm u_n(x)$ of (0.1) after extracting a subsequence $\lambda_n \to \infty$ ([**BPW**]).

Here we remark that in [**BW2**], [**BPW**] they consider mainly the case where Ω is connected.

In this paper we consider the case where Ω consists of 2 connected components:

$$\Omega = \Omega_1 \cup \Omega_2 \tag{0.4}$$

and we consider the multiplicity of positive and sign-changing solutions for large λ.

We have studied the multiplicity of positive solutions in our previous paper [**DT**], it is shown that there exist positive solutions $u_{1,\lambda}(x), u_{2,\lambda}(x), u_{3,\lambda}(x)$ of (P_λ) for large λ such that after extracting a subsequence $\lambda_n \to \infty$,

$$u_{1,\lambda_n}(x) \to \begin{cases} u_1(x) & \text{in } \Omega_1\,, \\ 0 & \text{in } \mathbf{R}^N \setminus \Omega_1, \end{cases} \qquad u_{2,\lambda_n}(x) \to \begin{cases} u_2(x) & \text{in } \Omega_2\,, \\ 0 & \text{in } \mathbf{R}^N \setminus \Omega_2, \end{cases}$$

$$u_{3,\lambda_n}(x) \to \begin{cases} u_1(x) & \text{in } \Omega_1\,, \\ u_2(x) & \text{in } \Omega_2, \\ 0 & \text{in } \mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2), \end{cases}$$

strongly in $H^1(\mathbf{R}^N)$. Here $u_i(x)$ is a least energy solution of

$$\begin{aligned} -\Delta u + u = u^p \quad & \text{in } \Omega_i, \\ u = 0 \qquad & \text{in } \partial\Omega_i. \end{aligned} \tag{0.5}$$

In particular, (P_λ) has at least 3 positive solutions for large λ. See [**DT**] for the case Ω consists of multiple connected components: $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$.

We remark that a solution $u_i(x)$ of (0.5) is said to be a least energy solution if and only if

$$\Psi_{i,D}(u_i) = \inf\{\Psi_{i,D}(u);\ u(x) \in H_0^1(\Omega_i) \text{ is a non-trivial solution of (0.5)}\},$$

holds. Here $\Psi_{i,D}(u)$ is defined by

$$\Psi_{i,D}(u) = \int_{\Omega_i} \frac{1}{2}(|\nabla u|^2 + u^2) - \frac{1}{p+1}|u|^{p+1}\, dx : H_0^1(\Omega_i) \to \mathbf{R}. \tag{0.6}$$

("D" stands for Dirichlet boundary conditions.) It is natural to ask the existence of a sequence of solutions of (P_λ) converging to solutions of (0.5) in each Ω_i, which may not be least energy solutions.

1. Results

First we deal with positive solutions. Our first theorem is the following

Theorem 1.1. *Assume (a1)–(a2),* (0.4) *and* $N \geq 3$. *Then there exists a* $p_1 \in (1, \frac{N+2}{N-2})$ *and* $\lambda_1 \geq 1$ *such that for* $p \in (p_1, \frac{N+2}{N-2})$ *and* $\lambda \geq \lambda_1$, (P_λ) *possesses at least* $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2) + \mathrm{cat}(\Omega_1 \times \Omega_2)$ *positive solutions.*

Remark 1.2. Since $\mathrm{cat}(\Omega_1 \cup \Omega_2) = \mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$, the argument of Bartsch-Wang [**BW2**] ensures $\mathrm{cat}(\Omega_1) + \mathrm{cat}(\Omega_2)$ positive solutions, which converges to a positive solution of (0.3) in one of components and to 0 elsewhere after extracting a subsequence $\lambda_n \to \infty$. We remark that our Theorem 1.1 ensures additional $\mathrm{cat}(\Omega_1 \times \Omega_2)$ positive solutions. We can also observe that these solutions converge to positive solutions in both components Ω_1, Ω_2.

Next we study the multiplicity of sign-changing solutions. When Ω consists of 2 components, we have two limit problems (0.5), which are corresponding to $\Psi_{i,D} : H_0^1(\Omega_i) \to \mathbf{R}$ $(i = 1, 2)$. It is well-known that each functional has an unbounded sequences of critical points $(u_j^{(i)}(x))_{j=1}^\infty \subset H_0^1(\Omega_i)$ $(i = 1, 2)$. A natural question is to ask for a given pair $(u_{j_1}^{(1)}(x), u_{j_2}^{(2)}(x))$ whether (P_λ) has a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ converging to $u_{j_i}^{(i)}(x)$ in Ω_i and to 0 elsewhere. Here we try to give a partial answer to this problem. More precisely, we try to find a solution $u_\lambda(x) \in H^1(\mathbf{R}^N)$ which converges to $(u_1^{(1)}(x), u_j^{(2)}(x))$ after extracting a subsequence $\lambda_n \to \infty$. Here $u_1^{(1)}(x)$ is a mountain pass solution of (0.5) in Ω_1 and $u_j^{(2)}(x)$ is a minimax solution of (0.5) in Ω_2.

To find an unbounded sequence of critical values of a functional $I(u) \in C^1(E, \mathbf{R})$ defined on an infinite dimensional Hilbert space E, $\mathbf{Z}_2$-symmetry of $I(u)$ — $I(\pm u) = I(u)$ for all $u \in E$ — plays an important role. We remark that $\Psi_\lambda(u) \in C^1(H^1(\mathbf{R}^N), \mathbf{R})$ and a functional $\tilde{\Psi}(u_1, u_2) = \Psi_{1,D}(u_1) + \Psi_{2,D}(u_2) \in C^1(H_0^1(\Omega_1) \times H_0^1(\Omega_2), \mathbf{R})$, which is corresponding to (0.5) in Ω_1 and Ω_2, have different symmetries; $\Psi_\lambda(u)$ is $\mathbf{Z}_2$-symmetric and $\tilde{\Psi}(u_1, u_2)$ is $(\mathbf{Z}_2)^2$-symmetric, that is,

$$\Psi_\lambda(su) = \Psi_\lambda(u) \qquad \text{for all } s \in \mathbf{Z}_2 = \{-1, 1\},\ u \in H^1(\mathbf{R}^N),$$
$$\tilde{\Psi}(s_1 u_1, s_2 u_2) = \tilde{\Psi}(u_1, u_2) \text{ for all } s_1, s_2 \in \{-1, 1\},\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2).$$

Note that $\mathbf{Z}_2$-action on $\Psi_\lambda(u)$ is corresponding to the following $\mathbf{Z}_2$-action on $\tilde{\Psi}(u_1, u_2)$

$$\tilde{\Psi}(su_1, su_2) = \tilde{\Psi}(u_1, u_2) \quad \text{for all } s \in \{-1, 1\},\ (u_1, u_2) \in H_0^1(\Omega_1) \times H_0^1(\Omega_2)$$

and there are no symmetries of $\Psi_\lambda(u)$ corresponding to the $\mathbf{Z}_2$-symmetry of $\tilde{\Psi}(u_1, u_2)$:

$$\tilde{\Psi}(u_1, \pm u_2) = \tilde{\Psi}(u_1, u_2). \tag{1.1}$$

We also remark that solutions $(u_1^{(1)}(x), u_j^{(2)}(x))$ are obtained using group action (1.1). Thus to construct solutions $u_\lambda(x)$ converging to $(u_1^{(1)}(x), u_j^{(2)}(x))$, we need to develop a kind of perturbation theory from symmetries and in this paper we use ideas from Ambrosetti [**A**], Bahri-Berestycki [**BB**], Struwe [**St**] and Rabinowitz [**R**] (See also Bahri-Lions [**BL**], Tanaka [**T**] and Bolle [**B**]). In [**A, BB, St, R, BL, T**], perturbation theories are developed for

$$-\Delta u = |u|^{p-1}u + f(x) \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain. They successfully showed the existence of unbounded sequence of solutions for all $f(x) \in L^2(\Omega)$ for a certain range of p.

Now we can give our second result.

Theorem 1.3. *Assume (a1)–(a2) and* (0.4). *Then $\Psi_{1,D}(u)$ and $\Psi_{2,D}(u)$ have critical values $c_{min}^{1,D}$ and $\{c_k^{2,D}\}_{k=1}^\infty$ with the following property: For any $k \in \mathbf{N}$ there exists $\lambda_2(k) \geq 1$ such that for any $\lambda \geq \lambda_2(k)$, (P_λ) has a solution $u_\lambda(x)$ such that*

(i) $\Psi_\lambda(u_\lambda) \to c_{min}^{1,D} + c_k^{2,D}$ as $\lambda \to \infty$.

(ii) For any given sequence $\lambda_\ell \to \infty$, we can extract a subsequence $\lambda_{n_\ell} \to \infty$ such that $u_{\lambda_{n_\ell}}$ converges to a function $u(x)$ strongly in $H^1(\mathbf{R}^N)$. Moreover $u(x)$ satisfies (0.5) *in $\Omega_1 \cup \Omega_2$, $u|_{\mathbf{R}^N \setminus (\Omega_1 \cup \Omega_2)} \equiv 0$ and $u(x) > 0$ in Ω_1.*

(iii) Moreover if the set of critical values of either $\Psi_{1,D}(u)$ or $\Psi_{2,D}(u)$ are discrete in a neighborhood of $c_{min}^{1,D}$ or $c_k^{2,D}$, then we have

$$\Psi_{1,D}(u|_{\Omega_1}) = c_{min}^{1,D}, \quad \Psi_{2,D}(u|_{\Omega_2}) = c_k^{2,D}.$$

Remark 1.4. It seems that discreteness of critical values of $\Psi_{i,D}(u)$ is not known; However we don't know any example that the set of critical values has interior points. We also remark that if the least energy solution of $\Psi_{1,D}(u)$ is non-degenerate — for example it holds for $\Omega = \{x \in \mathbf{R}^n;\ |x| < R\}$ $(R > 0)$ —, then critical values of $\Psi_{1,D}(u)$ are isolated in a neighborhood of $c^{1,D}_{min}$ and the assumption of (iii) holds.

When $N = 1$, we have a stronger result. We write $\Omega_1 = (a_1, b_1)$, $\Omega_2 = (a_2, b_2)$. For any j_1, $j_2 \in \mathbf{N}$ and $s_i \in \{-1, +1\}$ there exist unique solutions $u_i(x) = u_i(j_i, s_i; x)$ of (0.1) in Ω_i which possesses exactly j_i zeros in $\Omega_i = (a_i, b_i)$ and $s_i u_i'(a_i) > 0$. We have the following

Theorem 1.5. *Assume $N = 1$ and $\Omega_i = (a_i, b_i)$ $(i = 1, 2)$. Then for any j_1, $j_2 \in \mathbf{N}$ and $s_i \in \{-1, +1\}$ there exists a solution $u_\lambda(x)$ for large λ such that*

$$u_\lambda(x) \to u(x) \quad \textit{strongly in } H^1(\mathbf{R})$$

as $\lambda \to \infty$, where $u|_{\Omega_i}(x) = u_i(j_i, s_i; x)$ and $u|_{\mathbf{R}\setminus(\Omega_1 \cup \Omega_2)}(x) = 0$.

In the following section, we give a variational formulation and give an idea of the proofs of Theorem 1.3. We refer [**ST**] for details of proofs of Theorems 1.1, 1.3 and 1.5.

2. An idea of the proof

(a) Reduction to a problem on an infinite dimensional torus

To find critical points of $\Psi_\lambda(u)$, we reduce our problem to a variational problem on an infinite dimensional torus. For $i = 1, 2$, we choose bounded open subset Ω_i' with smooth boundary such that

$$\Omega_i \subset\subset \Omega_i', \quad (i = 1, 2), \qquad \overline{\Omega_1'} \cap \overline{\Omega_2'} = \emptyset.$$

First we take local mountain pass approach due to del Pino and Felmer [**DF**] to find solutions concentrating only on $\Omega_1 \cup \Omega_2$. We choose a function $f(\xi) \in C^1(\mathbf{R}, \mathbf{R})$ such that for some $0 < \ell_1 < \ell_2$

$$f(\xi) = |\xi|^{p-1}\xi \quad \text{for } |\xi| \le \ell_1,$$
$$0 \le f'(\xi) \le \frac{2}{3} \quad \text{for all } \xi \in \mathbf{R},$$
$$f(\xi) = \frac{1}{2}\xi \quad \text{for } |\xi| \ge \ell_2.$$

We set

$$g(x, \xi) = \begin{cases} |\xi|^{p-1}\xi & \text{if } x \in \Omega_1' \cup \Omega_2', \\ f(\xi) & \text{if } x \in \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2'), \end{cases}$$
$$G(x, \xi) = \int_0^\xi g(x, s)\, ds.$$

In what follows we will try to find critical points of

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + (\lambda^2 a(x)+1)u^2\,dx - \int_{\mathbf{R}^N} G(x,u)\,dx$$
$$= \frac{1}{2}\|u\|^2_{\lambda,\mathbf{R}^N} - \int_{\mathbf{R}^N} G(x,u)\,dx.$$

We can observe that $\Phi_\lambda(u) \in C^2(H^1(\mathbf{R}^N),\mathbf{R})$ satisfies $(PS)_c$ condition for all $c \in \mathbf{R}$. Moreover we have

Lemma 2.1. *Suppose that $(u_\lambda(x))_{\lambda\geq\lambda_0}$ is a family of critical points of $\Phi_\lambda(u)$ and assume that there exists constants m, $M > 0$ independent of λ such that*

$$m \leq \Phi_\lambda(u_\lambda) \leq M \quad \text{for all } \lambda \geq 1.$$

Then we have

(i) $\left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} m \leq \|u_\lambda\|^2_{\lambda,\mathbf{R}^N} \leq \left(\frac{1}{2}-\frac{1}{p+1}\right)^{-1} M$ *for all* $\lambda \geq 1$.

(ii) There exists $\lambda(M) \geq 1$ such that for $\lambda \geq \lambda(M)$, $u_\lambda(x)$ satisfies $|u_\lambda(x)| \leq \ell_1$ for $x \in \mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')$. In particular, $g(x,u_\lambda(x)) = |u_\lambda(x)|^{p-1}u_\lambda(x)$ holds in $\mathbf{R}^N$ and $u_\lambda(x)$ is a solution of the original problem (P_λ).

(iii) After extracting a subsequence $\lambda_n \to \infty$, there exists $u \in H^1(\mathbf{R}^N)$ such that

$$\|u_{\lambda_n} - u\|_{\lambda_n,\mathbf{R}^N} \to 0 \quad \text{as } n \to \infty.$$

Moreover $u(x)$ satisfies $u(x) \equiv 0$ in $\mathbf{R}^N \setminus (\Omega_1' \cup \Omega_2')$ and

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \Omega_i, \tag{2.1}$$
$$u = 0 \qquad \text{on } \partial\Omega_i \tag{2.2}$$

for $i = 1, 2$. It also holds $\Phi_{\lambda_n}(u_{\lambda_n}) \to \Psi_{1,D}(u|_{\Omega_1}) + \Psi_{2,D}(u|_{\Omega_2})$ as $n \to \infty$.

Here and after we use notation

$$\|u_\lambda\|^2_{\lambda,O} = \int_O |\nabla u|^2 + (\lambda^2 a(x)+1)u^2\,dx$$

for an open set $O \subset \mathbf{R}^N$ and $\lambda > 0$.

Identifying $H^1(\Omega_1' \cup \Omega_2')$ and $H^1(\Omega_1') \oplus H^1(\Omega_2')$, we write $u = (u_1, u_2) \in H^1(\Omega_1' \cup \Omega_2')$ if $u_1 = u|_{\Omega_1'}$, $u_2 = u|_{\Omega_2'}$ holds. We define for $u = (u_1,u_2) \in H^1(\Omega_1' \cup \Omega_2')$

$$I_\lambda(u_1,u_2) = \inf_{w\in H^1(\mathbf{R}^N),\, w=(u_1,u_2) \text{ on } \Omega_1'\cup\Omega_2'} \Phi_\lambda(w),$$

Now we set

$$\Sigma_{i,\lambda} = \{v \in H^1(\Omega_i'); \ \|v\|_{\lambda,\Omega_i'} = 1\} \quad \text{for } i = 1,2$$

and define

$$J_\lambda(v_1, v_2) = \sup_{s,t>0} I_\lambda(sv_1, tv_2) : \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda} \to \mathbf{R}.$$

We can observe that for any $M > 0$ there exists $\lambda(M) \geq 1$ such that for any $\lambda \geq \lambda(M)$

- For any $(v_1, v_2) \in [J_\lambda \leq M]_{\Sigma_{1,\lambda}\oplus\Sigma_{2,\lambda}}$, $(s,t) \mapsto I_\lambda(sv_1, tv_2)$ has a unique maximizer.
- $[J < M]_{\Sigma_{1,\lambda}\oplus\Sigma_{2,\lambda}} \to \mathbf{R} :\ (v_1, v_2) \mapsto J_\lambda(v_1, v_2)$ is of class C^1 and its critical points are corresponding to critical points of $I_\lambda(u)$.

Here we use notation:

$$[J_\lambda < M]_{\Sigma_{1,\lambda}\oplus\Sigma_{2,\lambda}} = \{(v_1, v_2) \in \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda};\ J_\lambda(v_1, v_2) < M\}.$$

(b) Comparison functionals

To find critical points of $J_\lambda(v_1, v_2) : \Sigma_{1,\lambda} \oplus \Sigma_{2,\lambda} \to \mathbf{R}$ the following observation is useful. We use notation:

$$J_{i,\lambda}(v_i) = \sup_{s>0} I_\lambda(sv_i) : \Sigma_{i,\lambda} \to \mathbf{R}. \tag{2.3}$$

Lemma 2.2. There exists $c_\lambda > 0$ such that

$$c_\lambda \to 0 \qquad \text{as } \lambda \to \infty,$$
$$|J_\lambda(v_1, v_2) - J_{1,\lambda}(v_1) - J_{2,\lambda}(v_2)| < c_\lambda,$$
$$\left|J_\lambda'(v_1, v_2)(h_1, h_2) - J_{1,\lambda}'(v_1)h_1 - J_{2,\lambda}'(v_2)h_2\right| < c_\lambda(\|h_1\|_{\lambda,\Omega_1'} + \|h_2\|_{\lambda,\Omega_2'})$$

for all $(v_1, v_2) \in [J_\lambda < M]_{\Sigma_{1,\lambda}\oplus\Sigma_{2,\lambda}}$ and $(h_1, h_2) \in T_{v_1}\Sigma_{1,\lambda} \oplus T_{v_1}\Sigma_{1,\lambda}$. ∎

We remark that

$$\Sigma_{i,\lambda} \to \mathbf{R} :\ v_i \mapsto J_{i,\lambda}(v_i)$$

are even functionals and the existence of infinite many critical points can be obtained through minimax arguments. By Lemma 2.2, we regards $J_\lambda(v_1, v_2)$ as a perturbation of $J_{1,\lambda}(v_1) + J_{2,\lambda}(v_2)$.

(c) Minimax methods for $J_{i,\lambda}(v_i)$

We define minimax values $c_{min}^{1,\lambda}$, $b_n^{2,\lambda}$ $(n \in \mathbf{N})$ by

$$c_{min}^{1,\lambda} = \inf_{v_1 \in \Sigma_{1,\lambda}} J_{1,\lambda}(v_1), \tag{2.4}$$

$$b_n^{2,\lambda} = \inf_{\gamma \in \Gamma_n^\lambda} \max_{\theta \in S^n} J_{2,\lambda}(\gamma(\theta)), \tag{2.5}$$

where $S^n = \{\theta = (\theta_1, \cdots, \theta_{n+1});\ |\theta| = 1\}$ and

$$\Gamma_n^\lambda = \{\gamma \in C(S^n, \Sigma_{2,\lambda});\ \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^n\}.$$

We have

Lemma 2.3.
(i) $c_{min}^{1,\lambda}$ *is a critical value of* $J_{1,\lambda}(v_1)$.
(ii) $b_n^{2,\lambda}$ *is a critical value of* $J_{2,\lambda}(v_2)$.
(iii) $b_1^{2,\lambda} \leq b_2^{2,\lambda} \leq \cdots \leq b_n^{2,\lambda} \leq b_{n+1}^{2,\lambda} \leq \cdots$,
(iv) $b_n^{2,\lambda} \to \infty$ *as* $n \to \infty$. ∎

We are interested in the limit $\lim_{\lambda\to\infty} c_{min}^{1,\lambda}$ and $\lim_{\lambda\to\infty} b_n^{2,\lambda}$. Here appears the limit problem $\Psi_{i,D}(u_i)$ defined in (0.6).

In an analogous way to (2.3), (2.4)–(2.5), we set

$$\Sigma_{i,D} = \{u \in H_0^1(\Omega_i);\ \|u\|_{H_0^1(\Omega_i)} = 1\}$$

and consider a functional defined by

$$J_{i,D}(v) = \max_{t>0} \Psi_{i,D}(tv) : \Sigma_{i,D} \to \mathbf{R}.$$

We define as in (2.4)–(2.5)

$$c_{min}^{1,D} = \inf_{v\in\Sigma_{1,D}} J_{1,D}(v), \tag{2.6}$$

$$b_n^{2,D} = \inf_{\gamma\in\Gamma_n^D} \max_{\theta\in S^{n-1}} J_{2,D}(\gamma(\theta)), \tag{2.7}$$

where $S^{n-1} = \{\theta \in \mathbf{R}^n;\ |\theta| = 1\}$ and

$$\Gamma_n^D = \{\gamma \in C(S^{n-1}, \Sigma_{2,D});\ \gamma(-\theta) = -\gamma(\theta) \text{ for all } \theta \in S^{n-1}\}.$$

We can easily observe that $c_{min}^{1,D}$ and $b_n^{2,D}$ are critical values of $\Psi_{1,D}(u)$, $\Psi_{2,D}(u)$ and

$$\begin{aligned} &b_1^{2,D} \leq b_2^{2,D} \leq b_3^{2,D} \leq \cdots \leq b_n^{2,D} \leq b_{n+1}^{2,D} \leq \cdots, \\ &b_n^{2,D} \to \infty \ (n \to \infty). \end{aligned} \tag{2.8}$$

Moreover we have

Proposition 2.4. *Let* $c_{min}^{1,\lambda}$ *(*$b_n^{2,\lambda}$, $c_{min}^{1,D}$, $b_n^{2,D}$ *respectively) be a critical value of* $J_{1,\lambda}(v_1)$ *(*$J_{2,\lambda}(v_2)$, $J_{1,D}(v_1)$, $J_{2,D}(v_2)$ *respectively) defined in* (2.4)–(2.7). *Then we have*
(i) $c_{min}^{1,\lambda} \to c_{min}^{1,D}$ *as* $\lambda \to \infty$.
(ii) $b_n^{2,\lambda} \to b_n^{2,D}$ *as* $\lambda \to \infty$.

By (2.8), there exists a sequence $n(1) < n(2) < \cdots < n(k) < n(k+1) < \cdots$ such that

$$b_{n(k)}^{2,D} < b_{n(k)+1}^{2,D}. \tag{2.9}$$

We also define another set of minimax values by

$$c_k^{2,D} = \inf_{\sigma\in\Lambda_k} \max_{\theta\in S_+^{n(k)}} J_{2,D}(\sigma(\theta)), \tag{2.10}$$

where $S_+^{n(k)} = \{\theta = (\theta_1, \cdots, \theta_{n(k)}, \theta_{n(k)+1});\ \theta \in S^{n(k)+1}, \theta_{n(k)+1} \geq 0\}$ and

$$\Lambda_k = \{\sigma \in C(S_+^{n(k)}, \Sigma_{2,D});\ \sigma|_{S^{n(k)}} \in \Gamma_{n(k)}^{2,D},\ \inf_{\theta\in S^{n(k)}} \Psi_{2,D}(\sigma(\theta)) < b_{n(k)}^{2,D} + \delta_k\}.$$

Here $\delta_k > 0$ is a number satisfying $\delta_k < \frac{1}{2}(b_{n(k)+1}^{2,D} - b_{n(k)}^{2,D})$. We can also see that $c_k^{2,D}$ is a critical value of $\Psi_{2,D}(u)$ and $c_k^{2,D} \to \infty$ as $k \to \infty$. Although the definition of $c_k^{2,D}$ is rather complicated, it has a virtue that $c_k^{2,D}$ can be used to find critical points in presence of *non-odd perturbation.* More precisely, assume (2.9), then there exists $\tilde{\delta}_k > 0$ such that if a perturbed functional $\tilde{J}(v) : \Sigma_{2,D} \to \mathbf{R}$ satisfies

$$|\tilde{J}(v) - J_{2,D}(v)| < \tilde{\delta}_k \qquad \text{for all } v \in \Sigma_{2,D}.$$

Then, setting $\tilde{c}_k = \inf_{\sigma\in\Lambda_k} \max_{\theta\in S_+^{n(k)}} \tilde{J}(\sigma(\theta))$, we can observe that $\tilde{c}_k$ is a critical value of $\tilde{J}(v)$. This virtue also enables us to deal with a perturbation of $J_{1,\lambda}(v_1)d + J_{2,\lambda}(v_2)$ and we can obtain Theorem 1.3. s

Remark 2.5. The numbers $c_{min}^{1,D}$ and $\{c_k^{2,D}\}_{k=1}^{\infty}$ in the statement of Theorem 1.3 are given in (2.6) and (2.10).

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