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On the homomorphisms between scalar generalized Verma modules

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§ 0. Introduction

In this article, we consider the existence problem of homomorphisms between generalized Verma modules, which are induced from one dimensional representations (such generalized Verma modules are called scalar, cf. [Boe 1985]). In [Matumoto 2003], we classified the homomorphisms between scalar generalized Verma modules with respect to the maximal parabolic subalgebras.

Here, we announce a classification of homomorphisms between scaler generalized Verma modules for cccertain non-maximal paraboplic subalgebras. The proof will appear elsewhere.

§ 1. Notations and Preliminaries

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. We denote by Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Let W be the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and let $\langle\ ,\ \rangle$ be a non-degenerate invariant bilinear form on $\mathfrak{g}$. For $w \in W$, we denote by $\ell(w)$ the length of w as usuall. We also denote the inner product on $\mathfrak{h}^*$ which is induced from the above form by the same symbols $\langle\ ,\ \rangle$. For $\alpha \in \Delta$, we denote by s_α the reflection in W with respect to α. We denote by w_0 the longest element of W. For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$, as usual. We call $\lambda \in \mathfrak{h}^*$ is dominant (resp. anti-dominant), if $\langle\lambda, \check{\alpha}\rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in \mathfrak{h}^*$ regular, if $\langle\lambda, \alpha\rangle \neq 0$, for each $\alpha \in \Delta$. We denote by P the integral weight lattice, namely $\mathsf{P} = \{\lambda \in \mathfrak{h}^* \mid \langle\lambda, \check{\alpha}\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$. If $\lambda \in \mathfrak{h}^*$ is contained in P, we call λ an integral weight. We define $\rho \in \mathsf{P}$ by $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$. Put $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}\ \ [H, X] = \alpha(H)X\}$, $\mathfrak{u} = \sum_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{u}$. Then $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. We denote by Q the root lattice, namely $\mathbb{Z}$-linear span of Δ. We also denote by Q^+ the linear combination of Π with non-negative integral coefficients. For $\lambda \in \mathfrak{h}^*$, we denote by W_λ the integral Weyl group. Namely,

$$W_\lambda = \{w \in W \mid w\lambda - \lambda \in \mathsf{Q}\}.$$

We denote by Δ_λ the set of integral roots.

$$\Delta_\lambda = \{\alpha \in \Delta \mid \langle \lambda, \check{\alpha}\rangle \in \mathbb{Z}\}.$$

It is well-known that W_λ is the Weyl group for Δ_λ. We put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. This is a positive system of Δ_λ. We denote by Π_λ the set of simple roots for Δ_λ^+ and denote by Φ_λ the set of reflection corresponding to the elements in Π_λ. So, $(W_\lambda, \Phi_\lambda)$ is a Coxeter system. We denote by Q_λ the integral root lattice, namely $\mathsf{Q}_\lambda = \mathbb{Z}\Delta_\lambda^+$ and put $\mathsf{Q}_\lambda^+ = \mathbb{N}\Pi_\lambda$.

Next, we fix notations for a parabolic subalgebra (which contains $\mathfrak{b}$). Hereafter, through this article we fix an arbitrary subset Θ of Π. Let $\bar{\Theta}$ be the set of the elements of Δ which are written by linear combinations of elements of Θ over $\mathbb{Z}$. Put $\mathfrak{a}_\Theta = \{H \in \mathfrak{h} \mid \forall\alpha \in \Theta \;\; \alpha(H) = 0\}$, $\mathfrak{l}_\Theta = \mathfrak{h} + \sum_{\alpha\in\bar{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{n}_\Theta = \sum_{\alpha\in\Delta^+\setminus\bar{\Theta}} \mathfrak{g}_\alpha$, $\mathfrak{p}_\Theta = \mathfrak{l}_\Theta + \mathfrak{n}_\Theta$. Then $\mathfrak{p}_\Theta$ is a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{b}$. Conversely, for an arbitrary parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$, there exists some $\Theta \subseteq \Pi$ such that $\mathfrak{p} = \mathfrak{p}_\Theta$. We denote by W_Θ the Weyl group for $(\mathfrak{l}_\Theta, \mathfrak{h})$. W_Θ is identified with a subgroup of W generated by $\{s_\alpha \mid \alpha \in \Theta\}$. We denote by w_Θ the longest element of W_Θ. Using the invariant non-degenerate bilinear form $\langle\ ,\ \rangle$, we regard $\mathfrak{a}_\Theta{}^*$ as a subspace of $\mathfrak{h}^*$. It is known that there is a unique nilpotent (adjoint) orbit (say $\mathcal{O}_{\mathfrak{p}_\Theta}$) whose intersection with $\mathfrak{n}_\Theta$ is Zariski dense in $\mathfrak{n}_\Theta$. $\mathcal{O}_{\mathfrak{p}_\Theta}$ is called the Richardson orbit with respect to $\mathfrak{p}_\Theta$. We denote by $\bar{\mathcal{O}}_{\mathfrak{p}_\Theta}$ the closure of $\mathcal{O}_{\mathfrak{p}_\Theta}$ in $\mathfrak{g}$. Put $\rho_\Theta = \frac{1}{2}(\rho - w_\Theta\rho)$ and $\rho^\Theta = \frac{1}{2}(\rho + w_\Theta\rho)$. Then, $\rho^\Theta \in \mathfrak{a}_\Theta{}^*$.

Define

$$\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle\lambda, \check{\alpha}\rangle \in \{1, 2, \ldots\}\}$$
$${}^\circ\mathsf{P}_\Theta^{++} = \{\lambda \in \mathfrak{h}^* \mid \forall\alpha \in \Theta \quad \langle\lambda, \check{\alpha}\rangle = 1\}$$

We easily have

$${}^\circ\mathsf{P}_\Theta^{++} = \{\rho_\Theta + \mu \mid \mu \in \mathfrak{a}_\Theta^*\}.$$

For $\mu \in \mathfrak{h}^*$ such that $\mu + \rho \in \mathsf{P}_\Theta^{++}$, we denote by $\sigma_\Theta(\mu)$ the irreducible finite-dimensional $\mathfrak{l}_\Theta$-representation whose highest weight is μ. Let $E_\Theta(\mu)$ be the representation space of $\sigma_\Theta(\mu)$. We define a left action of $\mathfrak{n}_\Theta$ on $E_\Theta(\mu)$ by $X \cdot v = 0$ for all $X \in \mathfrak{n}_\Theta$ and $v \in E_\Theta(\mu)$. So, we regard $E_\Theta(\mu)$ as a $U(\mathfrak{p}_\Theta)$-module.

For $\mu \in \mathsf{P}_\Theta^{++}$, we define a generalized Verma module ([Lepowsky 1977]) as follows.

$$M_\Theta(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\Theta)} E_\Theta(\mu - \rho).$$

For all $\lambda \in \mathfrak{h}^*$, we write $M(\lambda) = M_\emptyset(\lambda)$. $M(\lambda)$ is called a Verma module. For $\mu \in \mathsf{P}_\Theta^{++}$, $M_\Theta(\mu)$ is a quotient module of $M(\mu)$. Let $L(\mu)$ be the unique highest weight $U(\mathfrak{g})$-module with the highest weight $\mu - \rho$. Namely, $L(\mu)$ is a unique irreducible quotient of $M(\mu)$. For $\mu \in \mathsf{P}_\Theta^{++}$, the canonical projection of $M(\mu)$ to $L(\mu)$ is factored by $M_\Theta(\mu)$.

$\dim E_\Theta(\mu - \rho) = 1$ if and only if $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$. If $\mu \in {}^\circ\mathsf{P}_\Theta^{++}$, we call $M_\Theta(\mu)$ a scalar generalized Verma module.

§ 2. Reductions of the problem

We retain the notation of §1. In particular, Θ is a subset of Π.

2.1 Basic results of Lepowsky

The following result is one of the fundamental results on the existence problem of homomorphisms between scalar generalized Verma modules.

Theorem 2.1.1. *([Lepowsky 1976])*
Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$.
(1) $\dim Hom_{U(\mathfrak{g})}(M_\Theta(\mu), M_\Theta(\nu)) \leqslant 1$.
(2) Any non-zero homomorphism of $M_\Theta(\mu)$ *to* $M_\Theta(\nu)$ *is injective.*

Hence, the existence problem of homomorphisms between scalar generalized Verma modules is reduce to the following problem.
Problem 1 Let $\mu, \nu \in {}^\circ\mathsf{P}_\Theta^{++}$. When is $M_\Theta(\mu) \subseteq M_\Theta(\nu)$?

2.2 Reduction to the integral infinitesimal character setting

Since the both $\nu \in W\mu$ and $\nu - \mu \in Q^+$ are necessary condition for the above problem, we can reformulate our problem as follows.
Problem 2 Let $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$ be dominant. Let $x, y \in W_\lambda$ be such that $x\lambda, y\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. When is $M_\Theta(x\lambda) \subseteq M_\Theta(y\lambda)$?
We fix $\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. Then, we can construct a suralgebra $\mathfrak{g}'$ of $\mathfrak{h}$ such that the corresponding Coxter system is $(W_\lambda, \Phi_\lambda)$. Since $\Theta \subseteq \Pi_\lambda$ holds, we can construct corresponding parabolic subalgebra $\mathfrak{p}'_\Theta$ of $\mathfrak{g}'$. For $\mu \in \mathsf{P}_\Theta^{++}$, we denote by $M'_\Theta(\mu)$ the corresponding generalized Verma module of $\mathfrak{g}'$. We consider the category $\mathcal{O}$ in the sense of [Bernstein-Gelfand-Gelfand 1976] corresponding to our particular choice of positive root system. More precisely, we denote by $\mathcal{O}$ (respectively $\mathcal{O}'$) "the category $\mathcal{O}$" for $\mathfrak{g}$ (respectively $\mathfrak{g}'$). We denote by $\mathcal{O}_\lambda$ (respectively, $\mathcal{O}'_\lambda$) the full subcategory of $\mathcal{O}$ (respectively $\mathcal{O}'$) consisting of the objects with a generalized infinitesimal character λ. Soegel's celebrated theorem ([Soegel 1990] Theorem 11) says that there is a Category equivalence between $\mathcal{O}_\lambda$ and $\mathcal{O}'_\lambda$. Under the equivalence a Verma module $M(x\lambda) \quad (x \in W_\lambda)$ corresponds to $M'(x\lambda)$. From Lepowsky's generalized BGG resolutions of the generalized Verma modules and their rigidity, we easily see $M_\Theta(x\lambda)$ corresponds to $M'_\Theta(x\lambda)$ under Soegel's category equivalence. So, we have the following lemma as a corollary of Soergel's theorem.

Lemma 2.2.1. *Let* $\lambda \in \mathfrak{h}^*$ *be dominant. Let* $x, y \in W_\lambda$ *be such that* $x\lambda, y\lambda \in {}^\circ\mathsf{P}_\Theta^{++}$. *Then, the following two conditions are equivalent.*
(1) $M_\Theta(x\lambda) \subseteq M_\Theta(y\lambda)$.
(2) $M'_\Theta(x\lambda) \subseteq M'_\Theta(y\lambda)$.

This lemma tells us that we may reduce Problem 2 to the case that λ is integral.
We put

$$W(\Theta) = \{w \in W \mid w\Theta = \Theta\}.$$

§ 3. Excellent parabolic subalgebras

3.1 θ-accepable positive roots

Hereafter, we fix a subset Θ of Π. For $\alpha \in \Delta$, we put

$$\Delta(\alpha) = \{\beta \in \Delta \mid \exists c \in \mathbb{R} \ \ \beta|_{\mathfrak{a}_\Theta} = c\alpha|_{\mathfrak{a}_\Theta}\},$$

$$\Delta^+(\alpha) = \Delta(\alpha) \cap \Delta^+,$$

$$U_\alpha = \mathbb{C}S + \mathbb{C}\alpha \subseteq \mathfrak{h}^*.$$

Then $(U_\alpha, \Delta(\alpha), \langle\ ,\ \rangle)$ is a subroot system of $(\mathfrak{h}^*, \Delta, \langle\ ,\ \rangle)$. The set of simple roots for $\Delta^+(\alpha)$ is denoted by $\Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} = 0$, then $S = \Pi(\alpha)$. If $\alpha|_{\mathfrak{a}_\Theta} \neq 0$, then $\Pi(\alpha)$ is written as $S \cup \{\tilde{\alpha}\}$. If $\alpha \in \Delta$ satisfies $\alpha|_{\mathfrak{a}_\Theta} \neq 0$ and $\alpha = \tilde{\alpha}$, then we call α Θ-reduced. For $\alpha \in \Delta^+$, we denote by $W_\Theta(\alpha)$ the Weyl group of $(\mathfrak{h}^*, \Delta(\alpha))$. Clearly, $W_\Theta \subseteq W_\Theta(\alpha) \subseteq W$. We denote by w^α the longest element of $W_\Theta(\alpha)$. We call $\alpha \in \Delta$ Θ-acceptable iff $w^\alpha w_\Theta = w_\Theta w^\alpha$. We denote by Δ^Θ (resp. Δ_r^Θ) the set of Θ-acceptable roots (resp. Θ-reduced Θ-acceptable roots). Put $(\Delta^\Theta)^+ = \Delta^+ \cap \Delta^\Theta$ and $(\Delta_r^\Theta)^+ = \Delta^+ \cap \Delta_r^\Theta$. For $\alpha \in \Delta^\Theta$, we define

$$\sigma_\alpha = w^\alpha w_\Theta = w_\Theta w^\alpha.$$

Clearly, ${\sigma_\alpha}^2 = 1$, $\sigma_\alpha = \sigma_{\tilde{\alpha}}$. If $\alpha|_{\mathfrak{a}_\Theta} = 0$, then $\sigma_\alpha = 1$. If $\alpha \in \Delta$ is orthogonal to all the elements in Θ, then we can easily see α is Θ-reduced and $s_\alpha = \sigma_\alpha$. For $\alpha \in \Delta$, we put

$$V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \alpha\rangle = 0\}.$$

For $\alpha \in \Delta^\Theta$, we put $\hat{\alpha} = \tilde{\alpha}|_{\mathfrak{a}_\Theta} \in \mathfrak{a}_\Theta^*$. We can easily see:

Lemma 3.1.1. *Let $\alpha \in \Delta_r^\Theta$. Then, we have*

(1) σ_α *preserves* $\mathfrak{a}_\Theta^*$.
(2) $\sigma_\alpha \in W(\Theta)$. *In particular,* $\sigma_\alpha \rho_\Theta = \rho_\Theta$.
(3) $\sigma_\alpha \hat{\alpha} = -\hat{\alpha}$.
(4) $\sigma_\alpha|_{\mathfrak{a}_\Theta^*}$ *is the reflection with respect to* V_α.

3.2 Excellent parabolic subalgebras

We retain the notations in the previous section.

Let $\Theta \subseteq \Pi$.

A parabolic subalgebra $\mathfrak{p}_\Theta$ is called excellent, if all the roots are Θ-acceptable.

Remark If $\mathfrak{p}_\Theta$ is a complexified minimal parabolic subalgebra of a real form of $\mathfrak{g}$ such that the m-part of the Langlands decomposition of $\mathfrak{p}_\Theta$ is semisimple, then all the roots are Θ-acceptable and σ_α is a reflection with respect to a restricted root $\hat{\alpha}$ for each $\alpha \in \Delta_r^\Theta$.

If $\mathfrak{g}$ is a classical algebra, we can classify excellent parabolic subalgebras of $\mathfrak{g}$ as follows.

(1) Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (the case of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is similar) and let k be a positive integer dividing n. We consider the following parabolic subalgebras.

$\mathfrak{p}(A_{n-1,k})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{n/k}.$$

Then, $\mathfrak{p}(A_{n-1,k})$ is excellent. Conversely any excellent parabolic subalgebra is conjugate to $A_{n,k}$ for some k.

(2) Let $\mathfrak{g}$ be a complex simple Lie algebra of the type X_n. Here, X means one of B, C, and D. Let k and ℓ be positive integers such that k divides $n - \ell$.

We consider the following parabolic subalgebras.

$\mathfrak{p}(X_{n,k,\ell})$: a parabolic subalgebra of $\mathfrak{g}$ whose Levi part is isomorphic to

$$\overbrace{\mathfrak{gl}(k, \mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}(k, \mathbb{C})}^{(n-\ell)/k} \oplus X_\ell.$$

Here, X_ℓ means that the complex simple Lie algebra of the type X_ℓ. X_0 means the zero Lie algebra.

$\mathfrak{p}(X_{n,k,\ell})$ is excellent unless $X = D$, $\ell = 0$, and k is an odd number greater than 1. Any excellent parabolic subalgebra is conjugate to one of such $\mathfrak{p}(X_{n,k,\ell})$s.

§ 4. Main result

4.1 Elementary homomorphisms

Here, we review some notion in [Matumoto 1993] §3. Hereafter, $\mathfrak{g}$ means a reductive Lie algebra over $\mathbb{C}$ and retain the notations in §1-3. We fix a subset Θ of Π and $\alpha \in \Delta$.

We denote by $\omega_\alpha \in \mathfrak{a}_\Theta^* \subseteq \mathfrak{h}^*$ the fundamental weight for α with respect to the basis $\Pi(\alpha) = \Theta \cup \{\alpha\}$. Namely ω_α satisfies that $\langle \omega_\alpha, \beta \rangle = 0$ for $\beta \in \Theta$, $\langle \beta, \check{\alpha} \rangle = 1$, and $\omega_\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Here, $\mathfrak{c}(\mathfrak{g}(\alpha))$ is the center of $\mathfrak{g}(\alpha)$. We see that there is some positive real number a such that $\omega_\alpha = a\alpha|_{\mathfrak{a}_\Theta}$, since $\alpha|_{\mathfrak{h}\cap\mathfrak{c}(\mathfrak{g}(\alpha))} = 0$. Hence, we have $V_\alpha = \{\lambda \in \mathfrak{a}_\Theta^* \mid \langle \lambda, \omega_\alpha \rangle = 0\}$.

For $\alpha \in (\Delta_r^\Theta)^+$, we define

$$\mathfrak{g}(\alpha) = \mathfrak{h} + \sum_{\beta \in \Delta(\alpha)} \mathfrak{g}_\beta, \qquad \mathfrak{p}_\Theta(\alpha) = \mathfrak{g}(\alpha) \cap \mathfrak{p}_\Theta.$$

Then, $\mathfrak{g}(\alpha)$ is a reductive Lie subalgebra of $\mathfrak{g}$ whose root system is $\Delta(\alpha)$ and $\mathfrak{p}_\Theta(\alpha)$ is a maximal parabolic subalgebra of $\mathfrak{g}(\alpha)$.

Put $\rho(\alpha) = \frac{1}{2}\sum_{\beta \in \Delta^+(\alpha)} \beta$, For $\nu \in \mathfrak{a}_\Theta^*$, we denote by $\mathbb{C}_\nu$ the one-dimensional $U(\mathfrak{p}_\Theta(\alpha))$-module corresponding to ν. For $\nu \in \mathfrak{a}_\Theta^*$ we define a generalized Verma module for $\mathfrak{g}(\alpha)$ as follows.

$$M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + \nu) = U(\mathfrak{g}(\alpha)) \otimes_{U(\mathfrak{p}_\Theta(\alpha))} \mathbb{C}_{\nu - \rho(\alpha)}.$$

Then, we have:

Theorem 4.1.1. *([Matumoto 2003]) Let ν be an arbitrary element in V_α and let c be either 1 or $\frac{1}{2}$. Assume that $M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta - nc\omega_\alpha) \subseteq M_\Theta^{\mathfrak{g}(\alpha)}(\rho_\Theta + nc\omega_\alpha)$ for all $n \in \mathbb{N}$. Then, we have $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha) \subseteq M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ for all $n \in \mathbb{N}$.*

We call the above homomorphism of $M_\Theta(\rho_\Theta + \nu - nc\omega_\alpha)$ into $M_\Theta(\rho_\Theta + \nu + nc\omega_\alpha)$ an elementary homomorphism.

The following working hypothesis is proposed in [Matumoto 2003].

Working Hypothesis *An arbitrary nontrivial homomorphism between scalar generalized Verma modules is a composition of elementary homomorphisms.*

The working hypothesis in the case of the Verma modules is nothing but the result of Bernstein-Gelfand-Gelfand.

I would like to propose :

Conjecture *For an excellent parabolic subalgebra,the above working hypothesis is affirmative.*

4.2 Main result

Now, we state our main result.

Theorem 4.2.1. *Let $\mathfrak{g}$ be a classical Lie algebra and let $\mathfrak{p}_\Theta$ be one of the following cases.*

(a) $\mathfrak{p}(A_{n-1,k})$ $(k|n)$,
(b) $\mathfrak{p}(B_{n,2k,m})$ $(k \leqslant m)$,
(c) $\mathfrak{p}(B_{n,2k+1,m})$ $(k \geqslant m)$,
(d) $\mathfrak{p}(C_{n,2k,m})$ $(k \leqslant m)$,
(e) $\mathfrak{p}(C_{n,2k+1,m})$ $(k \geqslant m)$,
(f) $\mathfrak{p}(D_{n,2k-1,m})$ $(k \leqslant m)$,
(g) $\mathfrak{p}(D_{n,2k,m})$ $(k \geqslant m)$.

Then, any homomorphism between scalar generalized Verma modules with integral infinitesimal characters is a composition of elementary homomorphisms.

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