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Global existence for the Klein-Gordon-Zakharov equations

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1 Introduction and Results.

In this note we present results on the Klein-Gordon-Zakharov equations, which are based on [15, 16, 20]. We consider the Cauchy problem of the Klein-Gordon-Zakharov equations in three space dimensions:

$$\partial_t^2 u - \Delta u + u = -nu, \quad t>0, \quad x\in \mathbf{R}^3, \tag{1}$$

$$\partial_t^2 n - c^2\Delta n = \Delta|u|^2, \quad t>0, \quad x\in \mathbf{R}^3, \tag{2}$$

$$\begin{aligned} &u(0,x)=u_0(x), \quad \partial_t u(0,x)=u_1(x), \\ &n(0,x)=n_0(x), \quad \partial_t n(0,x)=n_1(x), \end{aligned} \tag{3}$$

The propagation speed in equation (1) is normalized as unit, while that in equation (2) is denoted by c. Equations (1) and (2) describe the interaction of the Langmuir wave and the ion acoustic wave in a plasma (see Dendy [4] and Zakharov [21]). The function u denotes the fast time scale component of electric field raised by electrons and the function n denotes the deviation of ion density from its equilibrium. The functions u and n are real vector valued and real scalar valued, respectively. We introduce known results on (1)-(3) dividing into two cases (i) $c=1$ and (ii) $c\neq 1$, that is, whether or not the propagation speeds are the same.

Before proceeding, we give notation. For $1\leq p\leq\infty$ and a nonnegative integer m, let L^p and $W^{m,p}$ denote the standard L^p and Sobolev spaces on $\mathbf{R}^3$, respectively. We put $H^m=W^{m,2}$. For $m\in\mathbf{R}$, we let $\dot{H}^m=(-\Delta)^{-m/2}L^2$. We put $\omega=(1-\Delta)^{1/2}$ and $\omega_0=(-\Delta)^{1/2}$. We put $\partial_j=\partial/\partial x_j$ for $j=1,2,3$. Let $\Gamma=(\Gamma_j; j=1,\cdots,10)$ denote the generators of the Poincaré group $(\partial_t,\partial_1,\partial_2,\partial_3,L_1,L_2,L_3,\Omega_{12},\Omega_{23},\Omega_{13})$, where

$$\begin{aligned} L_j &= x_j\partial_t + t\partial_j, \quad j=1,2,3, \\ \Omega_{ij} &= x_i\partial_j - x_j\partial_i, 1\leq i<j\leq 3, \end{aligned}$$

and we put

$$\partial = (\partial_t, \partial_1, \partial_2, \partial_3).$$

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_{10})$, we put

$$\Gamma^\alpha = \Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}.$$

For $m \geq 0$ and $s \geq 0$, we define the weighted Sobolev space $H^{m,s}$ on $\mathbf{R}^3$ as follows:

$$H^{m,s} = \{v \in L^2; (1+|x|^2)^{s/2}(1-\Delta)^{m/2} v \in L^2\}$$

We put $H^m \equiv H^{m,0}$ for $m \geq 0$.

For a function $u(t,x)$, we denote by $\tilde{u}(\tau,\xi)$ the Fourier transform in both t and x of u. For $s, b \in \mathbf{R}$, we define the spaces $X^\pm_{b,s}$ and $Y^\pm_{b,s}$ as follows:

$$\begin{aligned} X^\pm_{b,s} &= \{u \in \mathcal{S}'(\mathbf{R}^4);\ ||u||_{X^\pm_{b,s}} < +\infty\}, \\ ||u||_{X^\pm_{b,s}} &= \left(\int_R \int_{R^3} (1+|\xi|)^{2s}(1+|\tau \pm |\xi|\ |)^{2b} |\tilde{u}(\tau,\xi)|^2\ d\xi d\tau\right)^{1/2}, \end{aligned}$$

$$\begin{aligned} Y^\pm_{b,s} &= \{v \in \mathcal{S}'(\mathbf{R}^4);\ ||v||_{Y^\pm_{b,s}} < +\infty\}, \\ ||v||_{Y^\pm_{b,s}} &= \left(\int_R \int_{R^3} (1+|\xi|)^{2s}(1+|\tau \pm c|\xi|\ |)^{2b} |\tilde{v}(\tau,\xi)|^2 d\xi d\tau\right)^{1/2}. \end{aligned}$$

We note that $X^\pm_{b,s}$ and $Y^\pm_{b,s}$ are Banach spaces for $s, b \in \mathbf{R}$.

1.1 Case $c = 1$.

When the propagation speeds are the same, many results have been obtained concerning the global existence of small amplitude solutions for the coupled systems of the Klein-Gordon and wave equations with quadratic nonlinearity. Two methods are known to be applicable to solve those systems.

(i) First one is based on the theory of normal forms introduced by Shatah [18], which is an extension of Poincaré's theory. The idea of this method is to transform the original system with quadratic nonlinearity into a new system with cubic nonlinearity. We state our results obtained by applying the argument of normal forms in Ozawa, Tsutaya and Tsutsumi [15]. We have the following theorem :

Theorem 1 *Let $0 < \varepsilon \leq 10^{-2}$. Assume that $u_0 \in H^{52} \cap W^{29,6/(5+2\varepsilon)}$, $u_1 \in H^{51} \cap W^{28,6/(5+2\varepsilon)}$, $n_0 \in H^{51} \cap W^{28,220/217} \cap \dot{H}^{-1}$ and $n_1 \in H^{50} \cap W^{27,220/217} \cap \dot{H}^{-2}$. Then there exists a $\delta > 0$ such that if*

$$\begin{aligned} &\|u_0\|_{H^{52}\cap W^{29,6/(5+2\varepsilon)}} + \|u_1\|_{H^{51}\cap W^{28,6/(5+2\varepsilon)}} \\ &+\|n_0\|_{H^{51}\cap W^{28,220/217}\cap \dot{H}^{-1}} + \|n_1\|_{H^{50}\cap W^{27,220/217}\cap \dot{H}^{-2}} \leq \delta, \end{aligned} \tag{4}$$

(1)-(3) has the unique global solutions (u,n) satisfying

$$u \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{52-j}),$$

$$n \in \left[\bigcap_{j=0}^{2} C^j([0,\infty); H^{51-j})\right] \cap \left[\bigcap_{j=0}^{1} C^j([0,\infty); \dot{H}^{-1-j})\right],$$

$$\sum_{j=0}^{1} \|\partial_t^j u(t)\|_{W^{26-j,6/(1-2\varepsilon)}} = O(t^{-(1+\varepsilon)}) \quad (t \to \infty),$$
$$\sum_{j=0}^{1} \|\partial_t^j n(t)\|_{W^{25-j,220/3}} = O(t^{-107/110}) \quad (t \to \infty),$$

where δ depends only on ε. Furthermore, the above solutions (u,n) of (1)-(3) have the free profiles $u_{+0} \in H^{52}$, $u_{+1} \in H^{51}$, $n_{+0} \in H^{51}$ and $n_{+1} \in H^{50}$ such that

$$\begin{aligned} &\sum_{j=0}^{1} \|\partial_t^j (u(t) - u_+(t))\|_{H^{52-j}} \\ &\qquad + \sum_{j=0}^{1} \|\partial_t^j (n(t) - n_+(t))\|_{H^{51-j}} \longrightarrow 0 \quad (t \to \infty), \end{aligned}$$

where

$$\begin{aligned} u_+(t) &= (\cos \omega t) u_{+0} + (\omega^{-1} \sin \omega t) u_{+1}, \\ n_+(t) &= (\cos \omega_0 t) n_{+0} + (\omega_0^{-1} \sin \omega_0 t) n_{+1}. \end{aligned}$$

Remark 1.

(1) $u_+(t)$ and $n_+(t)$ are the solutions of the free Klein-Gordon equation and the free wave equation with the initial conditions $(u_+(0), \partial_t u_+(0)) = (u_{+0}, u_{+1})$ and $(n_+(0), \partial_t n_+(0)) = (n_{+0}, n_{+1})$, respectively.

(2) In the case of one or two space dimensions, the global existence result for small initial data can be proved more easily than the case of three space dimensions. We do

not need the time decay estimates to show the global existence of solutions in the one and two dimensional cases.

(3) Recently, M. Ohta has proved the blow-up in a finite time for (1)-(3) with large initial data.

The following corollary is an immediate consequence of Theorem 1.

Corollary 2 *Let $0 < \varepsilon \leq 10^{-2}$ and let m be a positive integer with $m \geq 52$. Assume that $u_0 \in H^m \cap W^{29,6/(5+2\varepsilon)}$, $u_1 \in H^{m-1} \cap W^{28,6/(5+2\varepsilon)}$, $n_0 \in H^{m-1} \cap W^{28,220/217} \cap \dot{H}^{-1}$, $n_1 \in H^{m-2} \cap W^{27,220/217} \cap \dot{H}^{-2}$ and (u_0, u_1, n_0, n_1) satisfy (4). Then the solutions (u, n) given by Theorem 1 satisfy*

$$u \in \bigcap_{j=0}^{m} C^j([0,\infty); H^{m-j}), \qquad n \in \bigcap_{j=0}^{m-1} C^j([0,\infty); H^{m-1-j}).$$

In addition, if $u_0, u_1, n_0, n_1 \in \cap_{j=1}^{\infty} H^j$, then we have

$$u(t,x), \quad n(t,x) \in C^{\infty}([0,\infty) \times \mathbf{R}^3).$$

The existence and uniqueness of local solutions for (1)-(3) follows from the standard iteration argument. The crucial part of the proofs of Theorem 1 and Corollary 2 is to establish the a priori estimates of the solutions for (1)-(3). We use the argument of normal forms of Shatah [18] to transform the quadratic nonlinearity into the cubic one. However, in our problem the transformed cubic nonlinearity is represented in terms of the integral operator with singular kernels. The singularity of the integral kernels makes it difficult to solve (1)-(3). This is different from the case of the system containing only the Klein-Gordon equations, where the integral kernels of the resulting integral operators are regular (see [18]). Therefore, our main task in the proof of Theorem 1 is to evaluate the singularity of the integral kernels of the transformed cubic nonlinearity. Then we can apply the usual $L^p - L^q$ estimate to the transformed system.

(ii) Another method to solve (1)-(3) is to use the invariant Sobolev space with respect to the generators of the Lorentz group. This was developed by Klainerman [11]. He also introduced the notion of the null condition to prove the existence of global solutions for the wave equations with quadratic nonlinearity. We note that the null condition technique is based on the Lorentz invariance of the equations.

In Theorem 1, one needs the high regularity assumptions on the data to ensure the global existence. Moreover, the global solution n of (1)-(3) given by Theorem 1 must belong to the homogeneous Sobolev space $\dot{H}^{-1}$ of negative index. In this part we show that there exist the global solutions of (1)-(3) for small initial data using the invariant Sobolev space but without applying the null condition technique and improve the regularity requirements on the initial data. We do not need the null condition technique due to the nonlinearity of (1)-(2). The nonlinear terms in (1)-(2) do not seem to satisfy the null condition as in [1] or [5].

We have the following theorem concerning the global existence of solutions to (1)-(3).

Theorem 3 *Let $0 < \varepsilon < 1/6$ and $k \geq 4$. Assume that $u_0 \in H^{k+5,k+4}$, $u_1 \in H^{k+4,k+4}$, $n_0 \in H^{k+4,k+4}$ and $n_1 \in H^{k+3,k+4}$. Then there exists a $\delta > 0$ such that if*

$$\|u_0\|_{H^{k+5,k+4}} + \|u_1\|_{H^{k+4,k+4}} + \|n_0\|_{H^{k+4,k+4}} + \|n_1\|_{H^{k+3,k+4}} \leq \delta,$$

then (1)-(3) has the unique global solutions (u, n) satisfying

$$\begin{aligned} &u \in \bigcap_{j=0}^{k+5} C^j([0,\infty); H^{k+4-j}), \\ &n \in \bigcap_{j=0}^{k+4} C^j([0,\infty); H^{k+4-j}), \\ &\sum_{|\alpha|=k+4} \sup_{t\geq 0} (1+t)^{-\varepsilon} \{\|\partial_t \Gamma^\alpha u(t)\|_{L^2} + \|\omega \Gamma^\alpha u(t)\|_{L^2}\} \\ &+ \sum_{|\alpha|\leq k+4} \sup_{t\geq 0} (1+t)^{-\varepsilon} \|\Gamma^\alpha u(t)\|_{L^2} + \sum_{|\alpha|\leq k+4} \sup_{t\geq 0} \|\Gamma^\alpha n(t)\|_{L^2} \\ &+ \sum_{|\alpha|\leq k} \sup_{\substack{t\geq 0 \\ x\in R^3}} \{|(1+t+|x|)^{3/2-2\varepsilon} \Gamma^\alpha u(t,x)| \\ &\qquad\qquad + |(1+t+|x|)\Gamma^\alpha n(t,x)|\} < \infty. \end{aligned} \tag{5}$$

Remark 2.

(1) We see that the solutions (u, n) of (1)–(3) given by Theorem 3 asymptotically approach the free solutions as $t \to \infty$ since the right hand sides of (1)–(2) are integrable in time by (5).

(2) Compared to Theorem 1, the regularity assumptions on the initial data has improved significantly. Instead, we need some spatial decay on the data.

The following corollary follows easily from the proof of Theorem 3.

Corollary 4 *In addition to all the assumptions in Theorem 3, if* u_0, u_1, n_0, $n_1 \in \bigcap_{m\geq 1} H^m$, *then the solutions* (u, n) *given by Theorem 3 satisfy*

$$u(t,x),\ n(t,x) \in C^\infty([0,\infty)\times \mathbf{R}^3).$$

We can prove Theorem 3 by using two methods : the decay estimate of the inhomogeneous linear Klein-Gordon equation by Georgiev [6] and the Sobolev inequality in the Minkowski space by Klainerman [10, 12] and Hörmander [8]. See for details in [20].

1.2 Case $c \neq 1$.

We may assume that $0 < c < 1$ without loss of generality. In fact, this condition is natural from a physical point of view since the propagation speed in (1) is about one thousand times as large as that in (2).

It seems impossible to use the method of normal forms because the integral kernels have stronger singularity that in the case $c = 1$. Those kernels are difficult to handle. The Lorentz invariance method does not seem useful, either since the operator L_j does not commute with the d'Alembertian $\partial_t^2 - c^2\Delta$ with $c \neq 1$. We use another method to show the global existence for the case $c \neq 1$.

We note that the solutions u and n of equations (1) and (2) formally satisfy the following energy identity:

$$E(u(t), \partial_t u(t), n(t), \partial_t n(t)) = E(u_0, u_1, n_0, n_1),$$

where

$$\begin{aligned} & E(u, \partial_t u, n, \partial_t n) \\ = {} & \frac{1}{2}\,||\nabla u||_{L^2}^2 + \frac{1}{2}\,||u||_{L^2}^2 + \frac{1}{2}\,||\partial_t u||_{L^2}^2 \\ & + c^2||n||_{L^2}^2 + ||\partial_t n||_{\dot{H}^{-1}}^2 + \mathrm{Re}\int_{R^3} n(t,x)|u(t,x)|^2 dx. \end{aligned} \tag{6}$$

Here and hereafter, $\hat{v}(\xi)$ denotes the Fourier transform of $v(x)$ in the spatial variables.

The following theorem is about the time local well-posedness in $H^1 \oplus L^2 \oplus L^2 \oplus H^{-1}$ of (1)-(3).

Theorem 5 *Assume that* $0 < c < 1$ *holds. Let* $(u_0, u_1, n_0, n_1) \in H^1 \oplus L^2 \oplus L^2 \oplus H^{-1}$ *[resp.* $H^1 \oplus L^2 \oplus L^2 \oplus \dot{H}^{-1}$*]. Assume that* $1/2 < b < 1$ *and* b *is close enough to* $1/2$*. Then there exists a* $T > 0$ *such that the problem (1)-(3) has unique solutions* (u, n) *on the time interval* $[-T, T]$ *satisfying*

$$u \in C([-T,T]; H^1) \cap C^1([-T,T]; L^2), \tag{7}$$

$$n \in C([-T,T]; L^2) \cap C^1([-T,T]; H^{-1}) \tag{8}$$

$$[\textit{resp.} \quad C([-T,T]; L^2) \cap C^1([-T,T]; \dot{H}^{-1})],$$

$$u \pm i(1-\Delta)^{-1/2}\partial_t u \in X^{\pm}_{b,1},$$

$$n \pm i(1-c^2\Delta)^{-1/2}\partial_t n \in Y^{\pm}_{b,0}$$

$$[\textit{resp.} \quad n \pm i(c\omega_0)^{-1}\partial_t n \in Y^{\pm}_{b,0}],$$

where T *depends only on* $||u_0||_{H^1}, ||u_1||_{L^2}, ||n_0||_{L^2}$ *and* $||n_1||_{H^{-1}}$ *[resp.* $||n_1||_{\dot{H}^{-1}}$*]. In addition, if* $n_1 \in \dot{H}^{-1}$*, then the solutions* (u, n) *satisfy the energy identity:*

$$E(u(t), \partial_t u(t), n(t), \partial_t n(t)) = E(u_0, u_1, n_0, n_1), \qquad t \in [-T, T]. \tag{9}$$

Furthermore, the solutions as above depend continuously on the initial data in the topology of (7) and (8) on the time interval $[-T', T']$ *for* $0 < T' < T$.

Corollary 6 *Assume that* $0 < c < 1$ *holds. Let* $(u_0, u_1, n_0, n_1) \in H^1 \oplus L^2 \oplus L^2 \oplus \dot{H}^{-1}$*. Assume that* $1/2 < b < 1$ *and* b *is close enough to* $1/2$*. Then there exists an* $\eta > 0$ *such that if*

$$||u_0||_{H^1} + ||u_1||_{L^2} + ||n_0||_{L^2} + ||n_1||_{\dot{H}^{-1}} \leq \eta,$$

the solutions (u, n) *given by Theorem 5 extends globally in time.*

Remark 3.

(i) We note that

$$\dot{H}^{-1} \subset H^{-1}.$$

Since the energy functional E defined in (6) includes the $\dot{H}^{-1}$ norm of $\partial_t n$, we need $n_1 \in \dot{H}^{-1}$ for the proof of Corollary 6.

(ii) Corollary 6 is an immediate consequence of Theorem 5 and the Sobolev embedding theorem. In fact, we can obtain the *a priori* estimate of $(u, \partial_t u, n, \partial_t n) \in H^1 \oplus L^2 \oplus L^2 \oplus \dot{H}^{-1}$ from the energy identity (9) and the Sobolev embedding theorem for small initial data. This leads to the global existence result. The constant η in Corollary 6 depends only on c and the best constant relevant to the Sobolev embedding $H^1 \hookrightarrow L^4$, but not on b.

Outline of Proof of Theorem 5.

We suppose that

$$(u_0, u_1, n_0, n_1) \in H^1 \oplus L^2 \oplus L^2 \oplus H^{-1}.$$

We first put

$$\begin{aligned} u_\pm &= u \pm i\, \omega^{-1} \partial_t u, \\ n_\pm &= n \pm i(c\omega)^{-1} \partial_t n, \end{aligned}$$

where $\omega = (1 - \Delta)^{1/2}$. Then (1)-(3) are rewritten as follows:

$$(i\partial_t \mp \omega) u_\pm = \pm(4\omega)^{-1}(n_+ + n_-)(u_+ + u_-), \tag{10}$$

$$(i\partial_t \mp c\omega) n_\pm = \pm(4c)^{-1} \omega_0^2 \omega^{-1} |u_+ + u_-|^2 \mp c(2\omega)^{-1}(n_+ + n_-), \tag{11}$$

$$u_\pm(0) = u_{\pm 0}, \quad n_\pm(0) = n_{\pm 0}, \tag{12}$$

where

$$\begin{aligned} u_{\pm 0} &= u_0 \pm i\, \omega^{-1} u_1, \\ n_{\pm 0} &= n_0 \pm i(c\, \omega)^{-1} n_1. \end{aligned}$$

We note that

$$(u_{\pm 0},\ n_{\pm 0}) \in H^1 \oplus L^2.$$

We try to solve (10)-(12) locally in time. For that purpose, we consider the following integral equations associated with (10)-(12):

$$u_\pm(t) \quad = \quad \varphi_T(t)\, W_\pm(t) u_{\pm 0}$$

$$\mp i(4\omega)^{-1}\varphi_T(t)\int_0^t W_\pm(t-s)(n_+(s)+n_-(s))(u_+(s)+u_-(s))ds, \qquad t\in\mathbf{R}, \qquad (13)$$

$$\begin{aligned} n_\pm(t) &= \varphi_T(t)W_{c\pm}(t)n_{\pm 0} \\ &\mp i\,\varphi_T(t)\int_0^t W_{c\pm}(t-s)[(4c)^{-1}\omega_0^2\omega^{-1}|u_+(s)+u_-(s)|^2 \\ &-c(2\omega)^{-1}(n_+(s)+n_-(s))]ds, \qquad t\in\mathbf{R}, \qquad (14)\end{aligned}$$

where $W_\pm(t)=e^{\mp it\omega}$, $W_{c\pm}(t)=e^{\mp ict\omega}$, T is a positive constant to be chosen small in the process of the proof and φ_T is a function in $C^\infty(\mathbf{R})$ such that $\varphi_T(t)=0$ for $|t|\geq 2T$ and $\varphi_T(t)=1$ for $|t|\leq T$. We note that the solutions of (13)-(14) in a suitable class is a solution of (10)-(12) on the time interval $[-T,T]$.

We use the Fourier restriction norm method to show the well-posedness of (13)-(14) for small $T>0$. For the scheme of the Fourier restriction norm method, see Bourgain [2, 3] (see also Kenig, Ponce and Vega [9] and Ginibre, Tsutsumi and Velo [7]).

If we try to apply the Fourier restriction norm method to (13)-(14), we have only to prove the following proposition:

Proposition 7 *There exist two positive constants a_0 and b_0 such that for a and b with $a_0\leq a<1/2<b\leq b_0$,*

$$|\langle\, v, wu\,\rangle|\leq\ C||v||_{X_a}||w||_{Y_b}||\omega u||_{X_b}, \qquad (15)$$

$$|\langle\, w,\ uv\,\rangle|\leq\ C||w||_{Y_a}||\omega u||_{X_b}||v||_{X_b}, \qquad (16)$$

where X_b and Y_b denote either of $X^\pm_{b,0}$ and either of $Y^\pm_{b,0}$, respectively, and

$$\langle\, f,g\,\rangle=\int_{R^4} f(t,x)\ \overline{g(t,x)}\ dtdx.$$

Remark 4.

The duality argument with (15)-(16) implies that

$$||nu||_{X_{-a}}\leq C||n||_{Y_b}||\omega u||_{X_b}, \qquad (17)$$

$$||\omega|u|^2||_{Y_{-a}}\leq C||\omega u||_{X_b}||\omega u||_{X'_b} \qquad (18)$$

for a and b with $a_0\leq a<1/2<b\leq b_0$, where X_b and X'_b denote either of $X^\pm_{b,0}$ and Y_b denotes either of $Y^\pm_{b,0}$. Estimates (17) and (18) enable us to apply the Fourier restriction norm method to (13)-(14). See [16] for the proof of Proposition 7.

References

[1] A. Bachelot, *Problème de Cauchy globale pour des systèmes de Dirac-Klein-Gordon*, Ann. Inst. Henri Poincaré, Physique théorique **48**, 387–422 (1988).

[2] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I Schrödinger equations*, Geom. Funct. Anal. **3**, 107–156 (1993).

[3] J. Bourgain, *Fourier restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. II The KdV equation*, Geom. Funct. Anal. **3**, 209–262 (1993).

[4] R.O. Dendy, Plasma Dynamics, Oxford University Press, Oxford. 1990.

[5] V. Georgiev, *Small amplitude solutions of the Maxwell-Dirac equations*, Indiana Univ. Math. J. **40**, 845–883 (1991).

[6] V. Georgiev, *Decay estimates for the Klein-Gordon equations*, Commun. Part. Diff. Eqs. **17**, 1111–1139 (1992).

[7] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal. **151**, 384-436 (1997).

[8] L. Hörmander, *On Sobolev spaces associated with some Lie algebras*, Report 4, 1985, Inst. Mittag-Leffler.

[9] C.E. Kenig, G. Ponce and G. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, Duke Math. J. **71**, 1–21 (1993).

[10] S. Klainerman, *Uniform decay estimates and the Lorentz invariance of the classical wave equations*, Comm. Pure Appl. Math. **38**, 321–332 (1985).

[11] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lect. in Appl. Math. **23**, 293–326 (1986).

[12] S. Klainerman, *Remarks on the global Sobolev inequalities in the Minkowski space* R^{n+1}, Comm. Pure Appl. Math. **40**, 111–117 (1987).

[13] H. Lindblad, *A sharp counterexample to the local existence of low-regularity solutions to nonlinear wave equations*, Duke Math. J. **72**, 503–539 (1993).

[14] H. Lindblad, *Counterexamples to local existence for semi-linear wave equations*, Amer. J. Math. **118**, 1–16 (1996).

[15] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Normal Form and Global Solutions for the Klein-Gordon-Zakharov Equations*, Ann. Inst. Henri Poincaré, Analyse non linéaire **12**, 459-503 (1995).

[16] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Well-posedness in energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations with different propagation speeds in three space dimensions*, Math. Ann. **313**, 127-140 (1999).

[17] G. Ponce and T. Sideris, *Local regularity of nonlinear wave equations in three space dimensions*, Comm. Part. Diff. Eqs. **18**, 169–177 (1993).

[18] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38**, 685–696 (1985).

[19] S.G. Thornhill and D. ter Haar, *Langmuir turbulence and modulational instability*, Phys. Reports (Sect. C of Phys. Lett.) **43**, 43–99 (1978).

[20] K. Tsutaya *Global existence of small amplitude solutions for the Klein-Gordon-Zakharov equations*, Nonlinear Analysis, TMA **27**, 1373–1380 (1996).

[21] V.E. Zakharov, Collapse of Langmuir waves, Sov. Phys. JETP **35**, 908–914 (1972).