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C. H. LEE'S RESULTS ON EXPONENTIAL CALCULUS OF MINIMUM TYPE PSEUDODIFFERENTIAL OPERATORS AND THEIR APPLICATION TO MICROLOCAL ENERGY METHODS

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1. INTRODUCTION

In this article we introduce Dr. Chang Hoon Lee's recent results on exponential calculus of minimum type pseudodifferential operators. Further we show their application to microlocal energy methods for microfunctions developed in [K3].
Indeed, the exponential calculus of hermitian pseudo-differential operators of so-called 'minimum type' plays an essential role in the theory above. However, the calculus used in the paper above is based on the early results by T. Aoki and it was clear that the results should be improved on the basis of Aoki's latest results.

For example, Aoki's theory is concerning the composition calculus of the operators as follows:

$$\exp(P(x, \partial_x)),$$

where $P(x, \partial_x)$ is an operator of order less than 1, for example, $(-\Delta_x)^\theta$ $(0 < \theta < 1/2)$. On the other hand, operators of minimum type are defined on the product space $X \times Y$ of complex manifolds X, Y.

Precisely speaking, let $z = (z_1, ..., z_n), w = (w_1, ..., w_m)$ be the local coordinates of X, Y respectively, and $(z, \xi), (w, \eta)$ be the local coordinates of the cotangent bundles T^*X, T^*Y. Then, the operators on $X \times Y$ are expressed as their symbols; that is, some equivalence class of analytic functions : $P(z, w, \xi, \eta)$: on $T^*(X \times Y) = T^*X \times T^*Y$. $P(z, w, \xi, \eta)$ is a holomorpphic function defined on a conic open set of $T^*X \times T^*Y$, which satisfies the following estimates as $|\xi| + |\eta| \to +\infty$;

$$|P(z, w, \xi, \eta)| \leq C_\varepsilon \exp(\varepsilon(|\xi| + |\eta|)) \quad (\forall \varepsilon > 0).$$

Here, we restrict ourselves to the operators of product type; that is, the definition domain of $P(z, w, \xi, \eta)$ is of the form $V \times W$, where V, W be conic open sets of T^*X, T^*Y respectively. Then we have one difficulty concerning the boundedness of symbols. That is, even if P is of order less than 0 (for example : ξ_1/η_1 :), the symbol is not bounded on $|\xi| > 1, |\eta| > 1$. On the other hand, the operator

$$\xi\eta/(\xi^2 + \eta^2)$$

is bounded on $V\times W$ for suitable V, W. In 1985, K. Kataoka introduced minimum type operators as a generalization of such good operators of product type which satisfiy the following estimates:

$$|P(z,w,\xi,\eta)| \le C\min\{|\xi|^{\kappa}, |\eta|^{\kappa}\} \quad (0<\kappa=2\theta-1<1).$$

For example,

$$(\xi\bar{\eta})^{\theta}/\sqrt{\xi^2+\bar{\eta}^2} \quad (1/2<\theta<1).$$

Indeed, the exponentials of hermitian positive operators of minimum type on $V\times V^*$ played an essential role in the theory of energy method for microfunctions (here, V^* is the complex conjugate of V).

Though Kataoka's paper could not treat the cases of the growth orders $1/2\le\theta<1$ concerning the exponential calculus of minimum type pseudodifferential operators, Dr. Lee succeeded in extending the results to all the cases. Dr. Lee's arguments are deeply based on T. Aoki's proofs. However, his proofs never goes straight way. Indeed, he needed many ideas on modifications of Aoki's construction of many quantities and of inequalities.

2. C. H. Lee's main results

We state the 3 main results by C. H. Lee without proofs. The complete versions will be published soon as

"*Exponential calculus of pseudodifferential operators of minimum type I, II*".

Let X and Y be n- and m- dimensional complex manifolds, respectively. Set

$$S^*X := (T^*X - X)/\mathbb{R}^+,\ S^*Y := (T^*Y - Y)/\mathbb{R}^+.$$

We define the mapping γ as

$$\gamma\colon \overset{\circ\circ}{T^*}(X\times Y) \ni (z,w;\xi,\eta) \longmapsto \left(z;\frac{\xi}{|\xi|}\right)\times\left(w;\frac{\eta}{|\eta|}\right)\in S^*X\times S^*Y,$$

where

$$\overset{\circ\circ}{T^*}(X\times Y) := T^*(X\times Y)\backslash\{(T^*X\times Y)\cup(X\times T^*Y)\}.$$

For $d_1, d_2>0$ and an open subset U of $S^*X\times S^*Y$, we use the notation

$$\gamma^{-1}(U;d_1,d_2) := \gamma^{-1}(U)\cap\{|\xi|>d_1, |\eta|>d_2\}.$$

Hereafter we write (z,ξ,w,η) for coordinates $(z,w;\xi,\eta)$.

Let K be a compact subset of $S^*X\times S^*Y$.

Definition 2.1. $P(z,\xi,w,\eta)$ is said to be a symbol of product type on K if the following hold:

(1) There are a constant $d>0$ and an open set $U(\supset K)$ in $S^*X\times S^*Y$ such that $P(z,\xi,w,\eta)$ is holomorphic in $\gamma^{-1}(U;d,d)$.

(2) For each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|P(z,\xi,w,\eta)| \le C_\varepsilon e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U;d,d). \tag{2.1}$$

We denote by $S(K)$ the set of all such symbols on K. Then $S(K)$ becomes a commutative ring under the sum and product as holomorphic functions.

Definition 2.2. We denote by $R(K)$ the set of all $P(z,\xi,w,\eta) \in S(K)$ satisfying the following;
there are constants $d > 0, \delta > 0$, an open set $U \supset K$ in $S^*X \times S^*Y$, and a positive-valued locally bounded function $C(\cdot)$ on $(0,\infty)$ such that

$$|P(z,\xi,w,\eta)| \le C(|\xi|/|\eta|)e^{-\delta \min\{|\xi|,|\eta|\}}$$

on $\gamma^{-1}(U;d,d)$.

We call an element of $R(K)$ a symbol of 0-class.

Definition 2.3. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)$ is called a formal symbol of product type on K if the following hold.

(1) There are some constants $d > 0, 0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U;(j+1)d,(k+1)d)$ for each $j,k \ge 0$.

(2) For each $\varepsilon > 0$, there is some constant $C_\varepsilon > 0$ such that

$$|P_{j,k}(z,\xi,w,\eta)| \le C_\varepsilon A^{j+k} e^{\varepsilon(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U;(j+1)d,(k+1)d) \tag{2.2}$$

for each $j,k \ge 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K.
We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$ with indeterminates t_1 and t_2 instead of $\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)$.
We can easily obtain the following.

Proposition 2.4. *$\widehat{S}(K)$ becomes a commutative ring under the sum and the product as formal power series in t_1 and t_2.*

$S(K)$ is identified with a subring of $\widehat{S}(K)$ as follows:
$S(K) \simeq \widehat{S}(K)|_{\substack{t_1=0\\t_2=0}} = \{P = \sum t_1^j t_2^k P_{j,k};\, P_{j,k} \equiv 0 \text{ for all } (j,k) \ne (0,0)\}$.

Definition 2.5. We denote by $\widehat{R}(K)$ the set of all $P(t_1,t_2;z,\xi,w,\eta) := \sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$ in $\widehat{S}(K)$ such that there are some constants $d > 0, 0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following: For each $\varepsilon > 0$, there is some constant $C_\varepsilon > 0$ such that

$$\left|\sum_{\substack{0\le j\le s\\0\le k\le t}} P_{j,k}(z,\xi,w,\eta)\right| \le C_\varepsilon A^{\min\{s,t\}} e^{\varepsilon(|\xi|+|\eta|)}$$

on $\gamma^{-1}(U;(s+1)d,(t+1)d)$ for each $s,t \ge 0$.

We call an element of $\widehat{R}(K)$ a formal symbol of zero class.

Proposition 2.6. *Under the previous identification, we have the equality:* $S(K) \cap \widehat{R}(K) = R(K)$.

Proposition 2.7. $R(K)$ *is an ideal in* $S(K)$.

Proposition 2.8. $\widehat{R}(K)$ *is an ideal in* $\widehat{S}(K)$.

$\widehat{S}(K)/\widehat{R}(K)$ becomes a commutative ring by Proposition 2.8. By Propositions 2.6 and 2.7, the inclusion $S(K) \hookrightarrow \widehat{S}(K)$ induces the injective ring homomorphism

$$\iota_K : S(K)/R(K) \longrightarrow \widehat{S}(K)/\widehat{R}(K).$$

Definition 2.9. We call an element in the ring $\widehat{S}(K)/\widehat{R}(K)$ a pseudo-differential operator of product type on K. We write $: \sum P_{j,k} :$ for the associated pseudo-differential operator of product type on K using an element $\sum P_{j,k}$ in $\widehat{S}(K)$.

The mapping $\gamma = \gamma_1 \circ \gamma_2$ is the composition of the following γ_1 and γ_2.

$$\overset{\circ\circ}{T^*}(X \times Y) \ni (z, w; \xi, \eta) \overset{\gamma_2}{\longmapsto} (z, w; \frac{\xi}{|(\xi,\eta)|}, \frac{\eta}{|(\xi,\eta)|}) \in \overset{\circ\circ}{S^*}(X \times Y),$$

$$\overset{\circ\circ}{S^*}(X \times Y) \ni (z, w; \frac{\xi}{|(\xi,\eta)|}, \frac{\eta}{|(\xi,\eta)|}) \overset{\gamma_1}{\longmapsto} (z, \frac{\xi}{|\xi|}) \times (w, \frac{\eta}{|\eta|}) \in S^*X \times S^*Y,$$

where

$$\overset{\circ\circ}{S^*}(X \times Y) := S^*(X \times Y) \backslash \{(S^*X \times Y) \cup (X \times S^*Y)\}.$$

Here we consider the relationship between our symbols and T. Aoki's symbols [A1]~[A8].

Proposition 2.10. *If* $P(z, \xi, w, \eta)$ *is a symbol of product type on* K, P *is a symbol on* $\gamma_1^{-1}(K)$ *in the sense of AOKI's symbol.*

Proposition 2.11. *If* $P(z, \xi, w, \eta)$ *is a symbol of product type of 0-class on* K, *that is,* $P \in R(K)$, *then* P *is a zero symbol on* $\gamma_1^{-1}(K)$ *in the sense of AOKI's symbol.*

Definition 2.12. The canonical mapping H_K is defined as follows;

$$S(K)/R(K) \ni : P : \overset{H_K}{\longmapsto} [P] \in \varinjlim_{U \supset \gamma_1^{-1}(K)} \mathcal{E}^{\mathbb{R}}(U).$$

Proposition 2.13. *Suppose that* K_1 *and* K_2 *are compact in* $S^*X \times S^*Y$, *respectively, and that* $K_1 \supset K_2$. *Then,*

$$H_{K_1}(: P :)|_{\gamma_1^{-1}(K_2)} = H_{K_2}(: P|_{K_2} :)$$

for all $P \in S(K)/R(K)$.

Definition 2.14. We define the product $*$ of two elements of $\widehat{S}(K)$ as follows:

$$(\sum_{j,k=0}^{\infty} P_{j,k}(z,\xi,w,\eta)) * (\sum_{j,k=0}^{\infty} Q_{j,k}(z,\xi,w,\eta)) = \sum_{j,k=0}^{\infty} R_{j,k}(z,\xi,w,\eta),$$

where

$$\sum_{j,k=0}^{\infty} t_1^j t_2^k R_{j,k}(z,\xi,w,\eta) := e^{t_1\langle\partial_\xi,\partial_{z^*}\rangle+t_2\langle\partial_\eta,\partial_{w^*}\rangle}\Big(\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z,\xi,w,\eta)$$
$$\times \sum_{j,k=0}^{\infty} t_1^j t_2^k Q_{j,k}(z^*,\xi^*,w^*,\eta^*)\Big)\Big|_{\substack{z^*=z,\xi^*=\xi\\ w^*=w,\eta^*=\eta}}.$$

That is,

$$R_{j,k}(z,\xi,w,\eta) := \sum_{\substack{j_1+j_2+|\alpha|=j\\ k_1+k_2+|\beta|=k}} \frac{1}{\alpha!\beta!}\partial_\xi^\alpha\partial_\eta^\beta P_{j_1,k_1}(z,\xi,w,\eta)$$
$$\times\, \partial_z^\alpha\partial_w^\beta Q_{j_2,k_2}(z,\xi,w,\eta).$$

Then we directly obtain the following by the Cauchy estimates.

Lemma 2.15. *If $\sum P_{j,k}$ and $\sum Q_{j,k}$ are formal symbols of product type on K, then $\sum R_{j,k}$ is also a formal symbol of product type on K.*

Proposition 2.16. *If $\sum P_{j,k}, \sum Q_{j,k} \in \widehat{S}(K)$ and at least one of them belongs to $\widehat{R}(K)$, then $\sum R_{j,k}$ is also in $\widehat{R}(K)$.*

By Lemma 2.15 and Proposition 2.16, the following composition of two elements in $\widehat{S}(K)/\widehat{R}(K)$ is well-defined;

$$:\sum P_{j,k}: \circ :\sum Q_{j,k}: := :(\sum P_{j,k}) * (\sum Q_{j,k}):.$$

We can directly verify the associativity about the operation $\circ$. That is, $\widehat{S}(K)/\widehat{R}(K)$ becomes an associative $\mathbb{C}$-algebra. Hence the mapping H_K is a ring homomorphism concerning the operation $\circ$, where

$$\mathcal{E}_{X\times Y}^{\mathbb{R},prod}(K) := \widehat{S}(K)/\widehat{R}(K) \xrightarrow{H_K} \mathcal{E}_{X\times Y}^{\mathbb{R}}(\gamma^{-1}(K)).$$

Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.17. A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following conditions hold;

(1) Λ is continuous,
(2) for each $\alpha > 1$, $\Lambda(\alpha t) \le \alpha\Lambda(t)$ on $(0,\infty)$,
(3) Λ is increasing,
(4) $\displaystyle\lim_{t\to\infty}\frac{\Lambda(t)}{t} = 0$.

Hereafter we fix two infra-linear functions Λ_1, Λ_2.

Definition 2.18. $P(z,\xi,w,\eta) \in S(K)$ is called a symbol of minimum type of growth order (Λ_1,Λ_2) on K if there exist some constants $C>0$, $d>0$, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following;

(1) $P(z,\xi,w,\eta)$ is holomorphic in $\gamma^{-1}(U;d,d)$,
(2) $|P(z,\xi,w,\eta)| \le C \cdot \min\{\Lambda_1(|\xi|),\Lambda_2(|\eta|)\}$ on $\gamma^{-1}(U;d,d)$.

Example 2.19. (by K. Kataoka[K3])
Let $\Omega = \Omega' := \{(z;\xi) \in T^*\mathbb{C}; |\arg\xi| < \delta, \quad \xi \ne 0\}$ with $(0<\delta<\pi/2)$. Let K be an arbitrary compact subset of $S^*\mathbb{C}_z \times S^*\mathbb{C}_w$ such that $\gamma^{-1}(K) \subset \Omega \times \Omega'$. We set $\Lambda_1(t) = \Lambda_2(t) := t^\sigma$ with $0<\sigma<1$. Then

$$P(z,\xi,w,\eta) := (\xi\eta)^{(1+\sigma)/2}/(\xi+\eta)$$

becomes a symbol of minimum type of growth order (Λ_1,Λ_2).

Remark 2.20. If P is a symbol of minimum type on K, e^P is a symbol of product type on K. Further such exponential of a symbol of minimum type plays a decisive role in Kataoka's microlocal energy method [K3].

Definition 2.21. $\sum P_{j,k}$ in $\widehat{S}(K)$ is called a formal symbol of minimum type of growth order (Λ_1,Λ_2) on K if there exist some constants $C>0$, $d>0$, $0<A<1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ satisfying the following;

(1) $P_{j,k}$ is holomorphic in $\gamma^{-1}(U;(j+1)d,(k+1)d)$ for each $j,k \ge 0$,
(2) the inequality

$$|P_{j,k}(z,\xi,w,\eta)| \le CA^{j+k}\min\{\Lambda_1(|\xi|),\Lambda_2(|\eta|)\}$$

holds on $\gamma^{-1}(U;(j+1)d,(k+1)d)$ for each $j,k \ge 0$.

Remark 2.22. If $\sum P_{j,k}$ is a formal symbol of minimum type on K, $e^{\sum P_{j,k}}$ is a formal symbol of product type on K.

Definition 2.23. For $(\lambda,\lambda^*) \in \mathbb{R}^2$, $a = \sum_{j,k\ge 0} t_1^j t_2^k a_{j,k}(z,\xi,w,\eta) \in S(K)$ is said to be of order at most (λ,λ^*) if there are some constants $C>0$, $d>0$, $0<A<1$ and an open set $U \supset K$ in $S^*X \times S^*Y$ such that
1) $a_{j,k}$ is holomorphic,
2) $|a_{j,k}(z,\xi,w,\eta)| \le CA^{j+k}|\xi|^\lambda|\eta|^{\lambda^*}$
on $\gamma^{-1}(U;(j+1)d,(k+1)d)$ for each $j,k \ge 0$.

From now on, we use the notation t,t^*,z^*,ξ^* instead of t_1,t_2,w,η. Further, $P\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix}$ stands for a formal symbol of product type

$$P(t,t^*;z,\xi,z^*,\xi^*) = \sum_{j,j^*\ge 0} t^j t^{*j^*} a_{j,j^*}(z,\xi,z^*,\xi^*)$$

for the sake of convenience.

Let $p\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix}$ and $q\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix}$ be symbols of minimum type of growth order (Λ, Λ^*) on K. Further, let $a\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix}$ and $b\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix}$ be formal symbols of order $(\lambda_1, \lambda^*{}_1)$, $(\lambda_2, \lambda^*{}_2)$ respectively. Then our aim is the calculation of the following composition:

$$: ae^p :: be^q : .$$

For this aim, we use Aoki's idea on exponential calculus in [A1]~[A8]. By the definition of composition, we obtain the following.

$$: a \exp p :: b \exp q :=: \pi|_{\begin{pmatrix} t_2; & w, & \eta \\ t_2^*; & w^*, & \eta^* \end{pmatrix} = \begin{pmatrix} 1; & z, & \xi \\ 1; & z^*, & \xi^* \end{pmatrix}} :,$$

where

$$\pi = \exp\left(t_2 t \partial_\xi \cdot \partial_w + t_2^* t^* \partial_{\xi^*} \cdot \partial_{w^*}\right) a\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} b\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix}$$
$$\times \exp\left(p\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} + q\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix}\right).$$

We notice that π is the unique formal series solution to the following system of partial differential equations.

$$\partial_{t_2}\pi = t\partial_\xi \cdot \partial_w \pi, \quad \partial_{t_2^*}\pi = t^*\partial_{\xi^*} \cdot \partial_{w^*}\pi,$$
$$\pi|_{t_2=0, t_2^*=0} = a\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} b\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix}$$
$$\times \exp\left(p\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} + q\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix}\right).$$

We can find the solution of the above Cauchy problem in the following form:

$$\pi' = \sum_{\nu,\nu^*=0}^{\infty} t_2^\nu t_2^{*\nu^*} c_{\nu,\nu^*}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix}$$
$$\times \exp\left(\sum_{k,k^*=0}^{\infty} t_2^k t_2^{*k^*} r_{k,k^*}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix}\right).$$

We can verify that π' is the solution if $\{r_{k,k^*}\}$ and $\{c_{k,k^*}\}$ satisfy the following (2.3)-(2.8). That is, $\pi' = \pi$.

Define formal symbols $\{r_{k,k^*}\}$ and $\{c_{\nu,\nu^*}\}$ of product type by the following recursive relations:

$$\text{(2.3)} \qquad r_{0,0} = p\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} + q\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix},$$

$$(2.4) \qquad c_{0,0} = a\begin{pmatrix} t; & z, & \xi \\ t^*; & z^*, & \xi^* \end{pmatrix} \times b\begin{pmatrix} t; & w, & \eta \\ t^*; & w^*, & \eta^* \end{pmatrix},$$

$$(2.5) \qquad r_{k+1,k^*}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix} = \frac{t}{k+1}\Big\{\partial_\xi \cdot \partial_w r_{k,k^*} + \sum_{\substack{k'+k''=k \\ k'^*+k''^*=k^*}} \partial_\xi r_{k',k'^*} \cdot \partial_w r_{k'',k''^*}\Big\},$$

$$(2.6) \qquad r_{k,k^*+1}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix} = \frac{t^*}{k^*+1}\Big\{\partial_{\xi^*} \cdot \partial_{w^*} r_{k,k^*} + \sum_{\substack{k'+k''=k \\ k'^*+k''^*=k^*}} \partial_{\xi^*} r_{k',k'^*} \cdot \partial_{w^*} r_{k'',k''^*}\Big\},$$

$$(2.7) \qquad c_{\nu+1,\nu^*}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix} = \frac{t}{\nu+1}\Big\{\partial_\xi \partial_w c_{\nu,\nu^*} + \sum_{\substack{\nu'+\nu''=\nu \\ \nu'^*+\nu''^*=\nu^*}} (\partial_\xi c_{\nu',\nu'^*} \cdot \partial_w r_{\nu'',\nu''^*} + \partial_w c_{\nu',\nu'^*} \cdot \partial_\xi r_{\nu'',\nu''^*})\Big\},$$

$$(2.8) \qquad c_{\nu,\nu^*+1}\begin{pmatrix} t; & z, & \xi, & w, & \eta \\ t^*; & z^*, & \xi^*, & w^*, & \eta^* \end{pmatrix} = \frac{t^*}{\nu^*+1}\Big\{\partial_{\xi^*} \partial_{w^*} c_{\nu,\nu^*} + \sum_{\substack{\nu'+\nu''=\nu \\ \nu'^*+\nu''^*=\nu^*}} (\partial_{\xi^*} c_{\nu',\nu'^*} \cdot \partial_{w^*} r_{\nu'',\nu''^*} + \partial_{w^*} c_{\nu',\nu'^*} \cdot \partial_{\xi^*} r_{\nu'',\nu''^*})\Big\}.$$

If we put

$$r = \sum_{k,k^*=0}^{\infty} r_{k,k^*}\begin{pmatrix} t; & z, & \xi, & z^*, & \xi^* \\ t^*; & z^*, & \xi^*, & z^*, & \xi^* \end{pmatrix},$$

$$c = \sum_{k,k^*=0}^{\infty} c_{k,k^*}\begin{pmatrix} t; & z, & \xi, & z^*, & \xi^* \\ t^*; & z^*, & \xi^*, & z^*, & \xi^* \end{pmatrix},$$

we obtain the following theorem (C. H. Lee's first main theorem).

Theorem 2.24. *There exist a formal symbol r of minimum type of growth order (Λ, Λ^*), and a formal symbol c of product type of order at most $(\lambda_1 + \lambda_2, \lambda_1^* + \lambda_2^*)$ respectively on K such that*

$$: a \exp p :: b \exp q :=: c \exp r : .$$

Next, we consider the exponentials of minimum type operators. Suppose

$$p = p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix} = \sum_{j,j^* \geq 0} t^j t^{*j^*} p_{j,j^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}$$

is a formal symbol of $\widetilde{\Lambda}$-type on K, where

$$\widetilde{\Lambda}(\xi,\xi^*) := \min\{\Lambda(|\xi|), \Lambda^*(|\xi^*|)\}.$$

We define the operator $\exp(s :p:)$ $(s \in \mathbb{C})$ as follows by introducing copies t_1, t_1^*, t_2, t_2^* of t, t^*.

$$(2.9)\quad p^{(0)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := 1,$$

$$(2.10)\quad p^{(l+1)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := \exp(t_2\langle\partial_\xi, \partial_y\rangle + t_2^*\langle\partial_{\xi^*}, \partial_{y^*}\rangle)$$
$$\times\, p\begin{pmatrix} t_1; & x, & \xi \\ t_1^*; & x^*, & \xi^* \end{pmatrix} p^{(l)}\begin{pmatrix} t_1, & t_2; & y, & \eta \\ t_1^*, & t_2^*; & y^*, & \eta^* \end{pmatrix}\Bigg|_{\begin{smallmatrix} y, & \eta \\ y^*, & \eta^* \end{smallmatrix} = \begin{smallmatrix} x, & \xi \\ x^*, & \xi^* \end{smallmatrix}}.$$

By the definition of $p^{(l)}$, we obtain

$$:p^{(l)}: = (:p:)^l.$$

Therefore we have the following expression of $\exp(s : p :)$.

$$E\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix} = \sum_{l=0}^{\infty} \frac{s^l}{l!} p^{(l)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} \quad (s \in \mathbb{C}).$$

Firstly, we must prove that $E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}$ is a formal symbol of $e^{\widetilde{\Lambda}}$ type on K.

Indeed $E\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix}$ formally satisfies the following differential equation.

$$\begin{cases} \partial_s E = \exp(t_2\langle\partial_\xi, \partial_y\rangle + t_2^*\langle\partial_{\xi^*}, \partial_{y^*}\rangle) p\begin{pmatrix} t_1; & x, & \xi \\ t_1^*; & x^*, & \xi^* \end{pmatrix} \\ \times\, E\begin{pmatrix} t_1, & t_2; & s, & y, & \eta \\ t_1^*, & t_2^*; & & y^*, & \eta^* \end{pmatrix}\Bigg|_{\left(\begin{smallmatrix} y, & \eta \\ y^*, & \eta^* \end{smallmatrix}\right) = \left(\begin{smallmatrix} x, & \xi \\ x^*, & \xi^* \end{smallmatrix}\right)}, \\ E\begin{pmatrix} t_1, & t_2; & 0, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix} = 1. \end{cases}$$

Hence $E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}$ defines an operator

$$:E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}:$$

which satisfies

$$\partial_s :E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}:$$

$$= {}^{\bullet}_{\bullet}p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}{}^{\bullet}_{\bullet}\ {}^{\bullet}_{\bullet}E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}{}^{\bullet}_{\bullet}$$

$${}^{\bullet}_{\bullet}E\begin{pmatrix} t, & t; & 0, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}{}^{\bullet}_{\bullet} = 1.$$

Therefore we define

$$\exp(s\,{}^{\bullet}_{\bullet}p{}^{\bullet}_{\bullet}) := {}^{\bullet}_{\bullet}E\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}{}^{\bullet}_{\bullet}.$$

We put

$$p^{(l)}\begin{pmatrix} t_1, & t_2; & x, & \xi \\ t_1^*, & t_2^*; & x^*, & \xi^* \end{pmatrix} := \sum_{\substack{j,k\geq 0\\ j^*,k^*\geq 0}} t_1^j t_2^k t_1^{*j^*} t_2^{*k^*} p^{(l)}_{(j,\,k)(j^*,\,k^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

Then by (2.9) and (2.10), we obtain the following recursive formulas.

$$(2.11)\quad p^{(0)}_{(j,\,k)(j^*,\,k^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \begin{cases} 1, & \text{if } (j,\ k, j^*,\ k^*) = (0,0,0,0) \\ 0, & \text{otherwise} \end{cases}$$

$$(2.12)\quad p^{(l+1)}_{(j,\,k)(j^*,\,k^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}$$

$$:= \sum_{\substack{i+\mu=j,\ |\alpha|+\nu=k\\ i^*+\mu^*=j^*, |\alpha^*|+\nu^*=k^*}} \frac{1}{\alpha!}\frac{1}{\alpha^*!}\partial_\xi^\alpha\partial_{\xi^*}^{\alpha^*} p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}$$

$$\times\ \partial_x^\alpha\partial_{x^*}^{\alpha^*} p^{(l)}_{(\mu,\nu)(\mu^*,\nu^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

It is clear that $p^{(l)}_{(j,k)(j^*,k^*)}$ is holomorphic on $\gamma^{-1}(U; (j+1)d, (j^*+1)d)$ for $j, j^* \geq 0$.

Theorem 2.25. *For each $s \in \mathbb{C}$, the formal series E is a formal symbol of product type of growth order $e^{\tilde{\Lambda}}$.*

Then the second main theorem of C. H. Lee is concerning the relationship between exponentials of minimum type operators and operators whose symbols are given by exponentials of minimum type formal symbols.

That is, let $p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}$ be a formal symbol of $\tilde{\Lambda}$-type on K. We see that $\exp(s\,{}^{\bullet}_{\bullet}p{}^{\bullet}_{\bullet})$ $(s \in \mathbb{C})$ defined by

$$\exp(s\,{}^{\bullet}_{\bullet}p{}^{\bullet}_{\bullet}) =: E\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix}\Bigg|_{\substack{t_1 = t_2 = t\\ t_1^* = t_2^* = t^*}} :$$

is a formal symbol of product type of growth order $e^{\tilde{\Lambda}}$.

Therefore our aim here is to construct $q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}$ as a formal symbol of $\tilde{\Lambda}$-type such that for each $s \in \mathbb{C}$

$$\exp(s\,{:}p{:}\,) = {:}\exp\left(q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}\right){:}\,.$$

We put

$$\begin{aligned} q &:= \sum_{i,i^*\geq 0} t^i t^{*i^*} q_{i,i^*}\begin{pmatrix} s, & x, & \xi \\ & x^*, & \xi^* \end{pmatrix} \\ &= \sum_{i,i^*\geq 0} t^i t^{*i^*} \sum_{l=1}^{\infty} s^l q^{(l)}_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}. \end{aligned}$$

Due to T. Aoki's idea [A1], we construct q as the solution of the following equation:

$$\begin{cases} \partial_s {:}\exp(q){:} = {:}p{:}\ {:}\exp(q){:}\,, \\ {:}\exp q\begin{pmatrix} t; & 0, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}{:} = 1. \end{cases}$$

Here,

$$\begin{aligned} {:}p{:}\ {:}\exp(q){:} = {:}\exp\left(t\langle\partial_\xi,\partial_y\rangle + t^*\langle\partial_{\xi^*},\partial_{y^*}\rangle\right)\left[p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}\right. \\ \left.\times \exp\left(q\begin{pmatrix} t; & s, & y, & \eta \\ t^*; & & y^*, & \eta^* \end{pmatrix}\right)\right]\Bigg|_{\begin{pmatrix} y & \eta \\ y^* & \eta^* \end{pmatrix}=\begin{pmatrix} x & \xi \\ x^* & \xi^* \end{pmatrix}}{:}\,. \end{aligned}$$

Hence we introduce another formal symbol ψ by the equation:

$$\begin{aligned} \text{(2.13)}\quad & \exp\left(t_3t_2\langle\partial_\xi,\partial_y\rangle + t_3^*t_2^*\langle\partial_{\xi^*},\partial_{y^*}\rangle\right)\left[p\begin{pmatrix} t_1; & x, & \xi \\ t_1^*; & x^*, & \xi^* \end{pmatrix}\right. \\ &\qquad\qquad \left.\times \exp\left(q\begin{pmatrix} t_1, & t_2; & s, & y, & \eta \\ t_1^*, & t_2^*; & & y^*, & \eta^* \end{pmatrix}\right)\right] \\ &= \psi\begin{pmatrix} t_1, & t_2, & t_3; & s, & x, & \xi, & y, & \eta \\ t_1^*, & t_2^*, & t_3^*; & & x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \\ &\qquad\qquad \times \exp\left(q\begin{pmatrix} t_1, & t_2; & s, & y, & \eta \\ t_1^*, & t_2^*; & & y^*, & \eta^* \end{pmatrix}\right). \end{aligned}$$

Here,

$$q\begin{pmatrix} t, & t; & s, & y, & \eta \\ t^*, & t^*; & & y^*, & \eta^* \end{pmatrix} = q\begin{pmatrix} t; & s, & y, & \eta \\ t^*; & & y^*, & \eta^* \end{pmatrix},$$

$$\psi = \sum_{l,i,j,k,i^*,j^*,k^*} s^l t_1^i t_2^j t_3^k t_1^{*i^*} t_2^{*j^*} t_3^{*k^*} \psi^{(l)}_{(ijk)(i^*j^*k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}.$$

Then, it is easy to see that $\psi \exp q$ in the right side of (2.13) satisfies the equations

$$\begin{cases} \partial_{t_3}\psi = & e^{-q}\partial_{t_3}(\psi e^q) = e^{-q}t_2\partial_\xi \cdot \partial_y(\psi e^q) \\ \quad\;\; = & t_2(\partial_\xi \cdot \partial_y \psi + \partial_y q \cdot \partial_\xi \psi), \\ \partial_{t_3^*}\psi = & e^{-q}\partial_{t_3^*}(\psi e^q) = e^{-q}t_2^*\partial_{\xi^*} \cdot \partial_{y^*}(\psi e^q) \\ \quad\;\; = & t_2^*(\partial_{\xi^*} \cdot \partial_{y^*}\psi + \partial_{y^*}q \cdot \partial_{\xi^*}\psi), \end{cases}$$

and that $q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}$ satisfies the equations

$$\vdots e^q \partial_s q\begin{pmatrix} t; & s, & x, & \xi \\ t^*; & & x^*, & \xi^* \end{pmatrix}\vdots = \partial_s(\vdots e^q \vdots) = \vdots p\vdots \, \vdots e^q \vdots$$
$$= \vdots e^q \psi\begin{pmatrix} t, & t, & 1; & s, & x, & \xi, & x, & \xi \\ t^*, & t^*, & 1; & & x^*, & \xi^*, & x^*, & \xi^* \end{pmatrix}\vdots .$$

Hence by putting

(2.14)
$$q\begin{pmatrix} t_1, & t_2; & s, & x, & \xi \\ t_1^*, & t_2^*; & & x^*, & \xi^* \end{pmatrix} = \sum_{l,i,j,i^*,j^*} s^l t_1^i t_2^j t_1^{*i^*} t_2^{*j^*} q^{(l)}_{(i,j)(i^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix},$$

(2.15)
$$q^{(l)}_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \sum_{\substack{0\le j\le i \\ 0\le j^*\le i^*}} q^{(l)}_{(i-j,j)(i^*-j^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix},$$

we have the following recursive formulas:

(2.16)
$$q^{(l+1)}_{(ij)(i^*j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix} = \frac{1}{l+1}\sum_{\substack{0\le k\le j \\ 0\le k^*\le j^*}} \psi^{(l)}_{(ijk)(i^*j^*k^*)}\begin{pmatrix} x, & \xi, & x, & \xi \\ x^*, & \xi^*, & x^*, & \xi^* \end{pmatrix},$$

(2.17)
$$\begin{aligned} &(k+1)\psi^{(l)}_{(i,j,k+1)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \\ &\quad = \langle \partial_\xi, \partial_y \rangle \psi^{(l)}_{(i,j-1,k)(i^*,j^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} \\ &\quad + \sum{}'' \sum_{l'+l''=l} \langle \partial_\xi \psi^{(l')}_{(i',j',k)(i'^*,j'^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}, \\ &\qquad\qquad \partial_y q^{(l'')}_{(i'',j'')(i''^*,j''^*)}\begin{pmatrix} y, & \eta \\ y^*, & \eta^* \end{pmatrix}\rangle, \end{aligned}$$

(2.18)

$$(k^*+1)\psi^{(l)}_{(i,j,k)(i^*,j^*,k^*+1)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$= \langle \partial_{\xi^*}, \partial_{y^*}\rangle \psi^{(l)}_{(i,j,k)(i^*,j^*-1,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix}$$
$$+ \sum\nolimits''^{*} \sum_{l'+l''=l} \langle \partial_{\xi^*}\psi^{(l')}_{(i',j',k)(i'^*,j'^*,k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix},$$
$$\cdot \partial_{y^*} q^{(l'')}_{(i'',j'')(i''^*,j''^*)}\begin{pmatrix} y, & \eta \\ y^*, & \eta^* \end{pmatrix}\rangle.$$

Here $\sum''$ is the sum over

$$i'+i''=i,\ j'+j''=j-1,\ i'^*+i''^*=i^*,\ j'^*+j''^*=j^*$$

and $\sum''^*$ is the sum over

$$i'+i''=i,\ j'+j''=j,\quad i'^*+i''^*=i^*,\ j'^*+j''^*=j^*-1.$$

Lemma 2.26. *We have the following initial and boundary conditions:*

$$\text{(2.19)}\quad \psi^{(l)}_{(ij0)(i^*j^*0)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} = \delta_{l,0}\delta_{j,0}\delta_{j^*,0}\cdot p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix},$$

$$\text{(2.20)}\quad \psi^{(l)}_{(ijk)(i^*j^*k^*)}\begin{pmatrix} x, & \xi, & y, & \eta \\ x^*, & \xi^*, & y^*, & \eta^* \end{pmatrix} = 0$$
$$(j<l \text{ or } j<k \text{ or } j^*<l \text{ or } j^*<k^*),$$

$$\text{(2.21)}\quad q^{(l)}_{(i,j)(i^*,j^*)} = 0 \quad \text{if } l>j+1 \text{ or } l>j^*+1.$$

Proof. The first equation directly follows from (2.13). Further, the second equation is also clear from (2.13) when $j<k$ or $j^*<k^*$. On the other hand, using the recursive formulas above, we can prove the second equation for $j<l$ or $j^*<l$, and the third equation for $j+1<l$ or $j^*+1<l$ simultaneously by double mathematical induction on l and $k+k^*$. That is, the main induction on l and the supplementary induction on $k+k^*$ hold. □

In particular, we have the following equation:

$$\text{(2.22)}\quad q_{i,i^*}\begin{pmatrix} s, & x, & \xi \\ & x^*, & \xi^* \end{pmatrix} = p_{i,i^*}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}$$
$$+ \sum_{\substack{0\le j\le i \\ 0\le j^*\le i^* \\ (j,j^*)\neq(0,0)}} \sum_{l=1}^{\min\{j,j^*\}+1} s^l q^{(l)}_{(i-j,j)(i^*-j^*,j^*)}\begin{pmatrix} x, & \xi \\ x^*, & \xi^* \end{pmatrix}.$$

The followings are the second and the third main theorems of C. H. Lee.

Theorem 2.27. *For each $s \in \mathbb{C}$, the formal series q is a formal symbol of $\tilde{\Lambda}$-type on K satisfying the following.*

$$:\exp q\begin{pmatrix} t, & t; & s, & x, & \xi \\ t^*, & t^*; & & x^*, & \xi^* \end{pmatrix}: = \exp\left(s :p\begin{pmatrix} t; & x, & \xi \\ t^*; & x^*, & \xi^* \end{pmatrix}:\right).$$

Further we have the inverse version of the above theorem.

Theorem 2.28. *Let q be of $\tilde{\Lambda}$-type. Then there exists p as a formal symbol of $\tilde{\Lambda}$-type satisfying the relation $e^{:p:} = :e^q:$.*

In fact, if we have such a solution $p = \sum_{i,i^*} p_{i,i^*}(x, \xi, x^*, \xi^*)$, we can construct a formal symbol $q(s) = \sum_{i,i^*} q_{i,i^*}(s, x, \xi, x^*, \xi^*)$ with a holomorphic parameter s satisfying

$$e^{s:p:} = :e^{q(s)}:$$

as seen in the preceding theorem. Hence, we construct p from the equation

$$q(1) = q,$$

where $q = \sum_{i,i^*} q_{i,i^*}(x, \xi, x^*, \xi^*)$ is the given formal symbol of $\tilde{\Lambda}$-type. Indeed, we have some recursive formulas obtained by (2.22), (2.16), (2.17), (2.18) (we omit the details).

Finally, we obtain the following theorem as a direct corollary of the three main theorems above.

Theorem 2.29. *The space of pseudodifferential operators of the form of $:e^p:$, where p is of minimum type, forms a group under the composition.*

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