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Confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2,\mathbf{R})$

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概要

この小論では、それぞれの球関数の明示公式を利用して、$Sp(2,\mathbf{R})$ 上の Siegel-Whittaker 関数から Whittaker functions への合流を論ずる。証明には、良い Whittaker (resp. Siegel-Whittaker) functions の、第 2 種の Whittaker (resp. Siegel-Whittaker) functions による展開定理を用いる.

Abstract

We discuss a confluence from Siegel-Whittaker functions to Whittaker functions on $Sp(2,\mathbf{R})$ by using their explicit formulae. In our proof, we use expansion theorems of the good Whittaker functions by the secondary Whittaker functions.

序

記号 π によって、ある半単純実 Lie 群 G の既約認容表現を表わす. G の閉部分群 R とその既約ユニタリ表現 η の対 (R,η) が与えられたとき、平滑に誘導された G-表現 $\mathrm{Ind}_R^G(\eta)$ に値を持つ絡空間 $\mathrm{Hom}_G(\pi,\mathrm{Ind}_R^G(\eta))$ あるいは $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi,\mathrm{Ind}_R^G(\eta))$ を考える. ここで K は G の極大コンパクト部分群で $\mathfrak{g}_{\mathbf{C}}$ は G Lie 環の複素化を表わす. この空間の非零元 I、即ち表現 π から $\mathrm{Ind}_R^G(\eta)$ への絡作用素 (intertwining operator) I を一つ選ぶとき、その像 $\mathrm{Im}(I)$ は π の $\mathrm{Ind}_R^G(\eta)$ の中での一般化されら球模型 (generalized spherical model) あるいは球実現 (generalzied spherical realization) である.

群 G が両側剰余類分分解 $G=RA_RK$ を有つような G の最大 $\mathbf{R}$-分裂カルタン部分群の分裂成分である閉部分群 A_R があるとし、それを一つ固定する. そのとき π の適切な K-type τ (通常は重複度 1 のもの) を選び、また絡空間 $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi,\mathrm{Ind}_R^G(\eta))$ は有限次元と仮定する. そのとき上の実現の τ-等質成分 $\mathrm{Im}(I)[\tau]$ of $\mathrm{Im}(I)$ の $U(\mathfrak{g}_{\mathbf{C}})$ 内の零化元 (annihilator) の A_R-動径成分からなるホロノミック系を計算する.

例えば $G=Sp(2,\mathbf{R})$ のとき、保型形式研究に用いるという動機から、我々はいろいろな種類の表現 π と対 (R,η) に対して、これらのホロノミック系とその明示的な解 (即ち G 上の一般化された球関数) を求めてきた (*cf.* [12], [13], [14], [16], [17], [22], [23], [26]).

一度同じ π に対して別の相異なる二つの対 (R_1,η_1) and (R_0,η_0) に対して、二つの異なる実現を得たとき、当然次はこれらの二つの模型の関係を知りたくなる. 例えば、直感的な説明をすると、G の球部分群 R_t とその表現 η_t の対の変形族 (R_t,η_t) で、$t\neq 0$ のときには各対 (R_t,η_t) は (R_1,η_1) は内部自己同型による捻りで、極限 $t\to 0$ においては「縮約」対 (R_0,η_0) になるものを考える. これは物理学者が考え始め、例えば Dooley [3] などによってきちんと取り扱われた Lie 群とその表現の contraction あるいは deformation という手法の変奏版である.

この論説では長いルートに付随する G のヤコビ極大部分群 P_J による parabolic induction で得られる一般化主系列表現の二つの模型間の合流を論ずる. 我々の設定では、(R_0,η_0) は極

大ユニポテント部分群 R_0 と R_0 の非退化指標 η_0 の対で、それは付随する球模型が通常の Whittaker 模型 π (*cf.* [18], [30], [19]) であることを意味する．そして対 (R_1, η_1) は π の Siegel-Whittaker 模型を導く．これは宮崎琢也 [21] や石井卓 [16] によって調べられた．Lie 群とその表現の縮約や変形族というアイデアを厳密に展開するのは、それによってここで得られる結果に比べると重い準備が必要となる．この高いコストを回避するため、ここでは単に関連する対の変形族 (R_t, η_t) の球関数の A_R-動径部分のみを直接に計算する．元の直感的なアイデアは第５節で、謂わば「後知恵」として記述される．

主結果は第４節にある．それぞれの球関数のホロノミック系の合流 (Theorem 4.1),「第２種」(secondary) 球関数の合流 (Theorem 4.2), そして最後に良い増大条件を有つ一意に定まる球関数の合流 (Theorem 4.3) という順で証明される．ここで言う secondary spherical functions とは、行列係数の漸近挙動を調べるために Harish-Chandra [7] によって考えられた、球関数のホロノミック系の無限遠における確定特異点でのべき級数解のことである（その後、Heckman-Opdam [11], [28] によってこの手法は一般化された）．さらに, この種の関数は、Miatello-Wallach [20] や Oda-Tsuzuki [27] の論文に見るように、Poincaré 級数の構成で基本的な役割を果たす．「良い球関数」の合流の証明で基本的なことは,「第２種」球関数合流の結果と、良い球関数の「第２種」球関数による展開定理をもちいることである (Theorems 2.4 and 3.4). 我々の結果は、それ自身興味深い点があると思うが（表現の変形は少なくとも最近はほとんど考えられていない）、さらに保型形式研究に、例えば無限素点での局所ゼータ関数の研究に、役に立つことを期待している (*cf.* [25]).

関連する周辺の問題はいろいろあるが、また別の機会に.

注意 用語 secondary spherical function の secondary であるが、「第２種」(of the second kind) の意味もあるが、元はボクシングの「セコンド」のような意味で使っている．Latin 語の語源は、secundus は元は、to follow の語から来ていると言う．当面は、素直に「補助的な」の意味で受け取って下さい.

記号 For $a \in \mathbf{C}$ and $n \in \mathbf{Z}$, $(a)_n = \Gamma(a+n)/\Gamma(a)$ the Pochhammer symbol. For complex numbers a_i $(1 \le i \le r)$ and b_j $(1 \le j \le s)$, set

$$\Gamma[a_1, \ldots, a_r] = \prod_{i=1}^{r} \Gamma(a_i), \quad \Gamma\left[\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right] = \prod_{i=1}^{r} \Gamma(a_i) \Big/ \prod_{i=1}^{s} \Gamma(b_i).$$

1 Preliminaries

1.1 Basic notions

Let G be the real symplectic group of degree two:

$$G = Sp(2, \mathbf{R}) = \left\{ g \in SL(4, \mathbf{R}) \,\middle|\, {}^t g J_2 g = J_2 = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \right\},$$

with 1_2 the unit matrix of degree two.

Fix a maximal compact subgroup K of G by

$$K = \left\{ k(A, B) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \,\middle|\, A, B \in M(2, \mathbf{R}) \right\}.$$

It is isomorphic to the unitary group $U(2)$ via the homomorphism

$$K \ni k(A, B) \longmapsto A + \sqrt{-1}B \in U(2).$$

Then the set of irreducible representations of K is parameterized by $\{(\lambda_1, \lambda_2) \in \mathbf{Z} \oplus \mathbf{Z} \mid \lambda_1 \geq \lambda_2\}$ and we denote by $\tau_{(\lambda_1,\lambda_2)} = \mathrm{Sym}^{\lambda_1-\lambda_2} \otimes \det^{\lambda_2}$ the representation corresponding to (λ_1, λ_2).

We define two spherical subgroups R_i of G and their representationes. The first one is a maximal unipotent radical of G given by

$$R_1 = \left\{ n(n_0,n_1,n_2,n_3) = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array}\right) \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \;\middle|\; n_i \in \mathbf{R} \right\}.$$

Any unitary character η_1 of R_1 can be written as

$$\eta_1(n(n_0,n_1,n_2,n_3)) = \exp(2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3))$$

with some $c_0, c_3 \in \mathbf{R}$. In this paper we assume η_1 is non-degenerate, that is, $c_0 c_3 \neq 0$. Taking a maximal split torus A of G by

$$A = \{a(a_1, a_2) = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0\},$$

we have the Iwasawa decomposition $G = R_1 A K$.

The second spherical subgroup R_2 is defined as follows. Let $P_S = L_S \ltimes N_S$ be the Siegel parabolic subgroup with the Levi part L_s and the abelian unipotent radical N_S given by

$$L_S = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mid A \in GL(2,\mathbf{R}) \right\},$$

$$N_S = \{n(0,n_1,n_2,n_3) \mid n_1, n_2, n_3 \in \mathbf{R}\}.$$

Fix a non-degenerate unitary character ξ of N_S by

$$\xi(n(0,n_1,n_2,n_3)) = \exp(2\pi\sqrt{-1}\,\mathrm{Tr}(H_\xi T))$$

with $T = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix}$, $H_\xi = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix} \in M(2,\mathbf{R})$ and $\det H_\xi \neq 0$. Consider the action of L_S on N_S by conjugation and the induced action on the character group $\widehat{N_S}$. Define $SO(\xi)$ to be the identity component of the subgroup of L_S which stabilize ξ:

$$SO(\xi) := \mathrm{Stab}_{L_S}(\xi)^\circ = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mid {}^tAH_\xi A = H_\xi \right\}.$$

Then $SO(\xi)$ is isomorphic to $SO(2)$ if $\det H_\xi > 0$ and to $SO_o(1,1)$ if $\det H_\xi < 0$. In this paper we treat the case that ξ is a 'definite' character, that is, $\det H_\xi > 0$. So we may assume $h_1, h_2 > 0$ and $h_3 = 0$ without loss of generality. We sometimes identify the element of $SO(\xi)$ with its upper left 2×2 component. Fix a unitary character χ_{m_0} ($m_0 \in \mathbf{Z}$) of $SO(\xi) \cong SO(2)$ by

$$\chi_{m_0}\left(\begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{h_1} & \\ & \sqrt{h_2} \end{pmatrix} \right) = \exp(\sqrt{-1}\, m_0 \theta).$$

We define $R_2 = SO(\xi) \ltimes N_S$ and $\eta_2 = \chi_{m_0} \boxtimes \xi$. Note that we also have the decomposition $G = R_2 A K$.

1.2 Spherical functions

For the pair (R_i, η_i) defined as above, consider the space $C^\infty_{\eta_i}(R_i\backslash G)$ of complex valued C^∞ functions f on G satisfying

$$f(rg) = \eta_i(r)f(g) \quad \text{for all } (r,g) \in R_i \times G.$$

By the right translation, $C^\infty_{\eta_i}(R_i\backslash G)$ is a smooth G-module and we denote by the same symbol its underlying $(\mathfrak{g}_{\mathbf{C}}, K)$-module ($\mathfrak{g}_{\mathbf{C}}$ is the complexification of the Lie algebra of G). For an irreducible admissible representation (π, H_π) of G and the subspace $H_{\pi,K}$ of K-finite vectors, the intertwining space

$$\mathcal{I}_{\eta_i,\pi} = \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(H_{\pi,K}, C^\infty_{\eta_i}(R_i\backslash G))$$

between the $(\mathfrak{g}_{\mathbf{C}}, K)$-modules is called the space of *algebraic Whittaker functionals* for $i = 1$, or *algebraic Siegel-Whittaker functionals* for $i = 2$. For a finite-dimensional K-module (τ, V_τ), denote by $C^\infty_{\eta_i,\tau}(R_i\backslash G/K)$ the space

$$\{\phi : G \to V_\tau,\ C^\infty \mid \phi(rgk) = \eta_i(r)\tau(k^{-1})\phi(g), \quad \text{for all } (r,g,k) \in R_i \times G \times K\}.$$

Let (τ^*, V_{τ^*}) be a K-type of π and $\iota : V_{\tau^*} \to H_\pi$ be an injection. Here τ^* means the contragredient representation of τ. Then for $\Phi \in \mathcal{I}_{\eta_i,\pi}$, we can find an element ϕ_ι in

$$C^\infty_{\eta_i,\tau}(R_i\backslash G/K) = C^\infty_{\eta_i}(R_i\backslash G) \otimes V_{\tau^*} \cong \mathrm{Hom}_K(V_{\tau^*}, C^\infty_{\eta_i}(R_i\backslash G))$$

via $\Phi(\iota(v^*))(g) = \langle v^*, \phi_\iota(g)\rangle$ with $\langle\ ,\ \rangle$ the canonical paring on $V_{\tau^*} \times V_\tau$.

Since there is the decomposition $G = R_iAK$, our (generalized) spherical function ϕ_ι is determined by its restriction $\phi_\iota|_A$ to A, which we call the *radial part* of ϕ_ι. For a subspace X of $C^\infty_{\eta_i,\tau}(R_i\backslash G/K)$, we denote $X|_A = \{\phi|_A \in C^\infty(A) \mid \phi \in X\}$.

Let us define two spaces $\mathrm{Wh}(\pi, \eta_1, \tau)$ and $\mathrm{SW}(\pi, \eta_2, \tau)$ of spherical functions and their subspaces $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}$ and $\mathrm{SW}(\pi, \eta_2, \tau)^{\mathrm{rap}}$ as follows:

$$\mathrm{Wh}(\pi, \eta_1, \tau) = \bigcup_{\iota \in \mathrm{Hom}_K(\tau^*, \pi)} \{\phi_\iota \mid \Phi \in \mathcal{I}_{\eta_1,\pi}\},$$

$$\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}} = \{\phi_\iota \in \mathrm{Wh}(\pi, \eta_1, \tau) \mid \phi_\iota|_A \text{ is of moderate growth as } a_1, a_2 \to \infty\},$$

$$\mathrm{SW}(\pi, \eta_2, \tau) = \bigcup_{\iota \in \mathrm{Hom}_K(\tau^*, \pi)} \{\phi_\iota \mid \Phi \in \mathcal{I}_{\eta_2,\pi}\},$$

and

$$\mathrm{SW}(\pi, \eta_2, \tau)^{\mathrm{rap}} = \{\phi_\iota \in \mathrm{SW}(\pi, \eta_2, \tau) \mid \phi_\iota|_A \text{ decays rapidly as } a_1, a_2 \to \infty\}.$$

We call an element in $\mathrm{Wh}(\pi, \eta_1, \tau)$ (resp. $\mathrm{SW}(\pi, \eta_2, \tau)$) a *Whittaker function* (resp. *Siegel-Whittaker function*) for (π, η_i, τ).

As we shall see in the next two sections, radial parts of spherical functions satisfy certain holonomic systems of regular singular type. We call the power series solutions at the regular singularities of the systems *secondary spherical functions*, and the elements of $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}$ and $\mathrm{SW}(\pi, \eta_2, \tau)^{\mathrm{rap}}$ *good spherical functions*.

1.3 P_J-principal series representations

In this section we recall the generalized principal series representations of G associated with the Jacobi maximal parabolic subgroup P_J of G corresponding to the long root. A Langlands decomposition $P_J = M_J A_J N_J$ is given by

$$M_J = \left\{ \left(\begin{array}{cc|cc} \epsilon & & & \\ & a & & b \\ \hline & & \epsilon & \\ & c & & d \end{array} \right) \middle| \, \epsilon \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}) \right\},$$

$$A_J = \{a(a_1, 1) = \mathrm{diag}(a_1, 1, a_1^{-1}, 1) \in A \mid a_1 > 0\},$$
$$N_J = \{n(n_0, n_1, n_2, 0) \in N = R_1 \mid n_i \in \mathbf{R}\}.$$

A discrete series representation (σ, V_σ) of the semisimple part $M_J \cong \{\pm 1\} \times SL(2, \mathbf{R})$ of P_J is of the form $\sigma = \varepsilon \boxtimes D_k^{\pm}$ $(k \geq 2)$, where $\varepsilon : \{\pm 1\} \to \mathbf{C}^*$ is a character and D_k^+ (resp. D_k^-) is the discrete series representation of $SL(2, \mathbf{R})$ with Blattner parameter k (resp. $-k$). For $\nu \in \mathbf{C}$, define a quasi-character $\exp(\nu)$ of A_J by $\exp(\nu)(a(a_1, 1)) = a_1^\nu$. We call an induced representation

$$I(P_J; \sigma, \nu) = C^\infty\text{-}\mathrm{Ind}_{P_J}^G(\sigma \otimes \exp(\nu + 1) \otimes 1_{N_J})$$

the *P_J-principal series representation* of G.

The K-types of $I(P_J; \sigma, \nu)$ is fully described in [23, Proposition 2.1] and [13, Proposition 2.3]. In particular, if $\pi = I(P_J; \varepsilon \boxtimes D_k^+, \nu)$ with $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = (-1)^k$ (*even P_J-*principal series), then the corner K-type $\tau^* = \tau_{(k,k)}$ occurs in π with multiplicity one.

2 Whittaker functions

2.1 Basic results

Let $\pi = I(P_J; \varepsilon \boxtimes D_k^+, \nu)$ be an irreducible even P_J-principal series representation of G with $\varepsilon(\mathrm{diag}(-1, 1, -1, 1)) = (-1)^k$, and $\tau^* = \tau_{(k,k)}$ is the corner K-type of π. We first prepare some basic facts on the Whittaker functions for (π, η_1, τ). Throughout this section we use a coordinate $x = (x_1, x_2)$ on A defined by

$$x_1 = \left(\pi c_0 \frac{a_1}{a_2}\right)^2, \quad x_2 = 4\pi c_3 a_2^2.$$

By combining the results of Kostant ([18, §6]), Wallach ([30, Theorem 8.8]), Matumoto ([19, Theorem 6.2.1]) and Miyazaki and Oda ([23, Proposition 7.1, Theorem 8.1]), we obtain

Proposition 2.1. *Let π and τ be as above. Then we have the following:*

(i) We have $\dim \mathcal{I}_{\eta_1, \pi} = \dim \mathrm{Wh}(\pi, \eta_1, \tau) = 4$, *and a function*

$$\phi_W(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi c_3 a_2^2) h_W(a)$$

on A is in the space $\mathrm{Wh}(\pi, \eta_1, \tau)|_A$ *if and only if $h_W(a) = h_W(x)$ is a smooth solution of the following holonomic system of rank* 4:

$$\left\{\partial_{x_1}\left(-\partial_{x_1} + \partial_{x_2} + \frac{1}{2}\right) + x_1\right\} h_W(x) = 0, \tag{2.1}$$

$$\left\{\left(\partial_{x_2}+\frac{k+\nu}{2}\right)\left(\partial_{x_2}+\frac{k-\nu}{2}\right)-x_2\left(-\partial_{x_1}+\partial_{x_2}+\frac{1}{2}\right)\right\}h_W(x)=0, \tag{2.2}$$

where $\partial_{x_i}=x_i(\partial/\partial x_i)$ $(i=1,2)$ is the Euler operator with respect to x_i.

(ii) $\dim \mathrm{Wh}(\pi,\eta_1,\tau)^{\mathrm{mod}}\le 1$. *Moreover this inequality is an equality if and only if $c_3>0$.*

Remark 1. Since [23] treated the case $\sigma=\varepsilon\boxtimes D_k^-$, we need a minor change by using the explicit formulas of 'shift operators' ([22, Proposition 8.3]).

2.2 Explicit formulas of secondary Whittaker functions

In this section we determine the space of smooth solutions of the holonomic system in Proposition 2.1, therefore the space of the Whittaker functions $\mathrm{Wh}(\pi,\eta_1,\tau)$, explicitly. Set

$$h_W(a)=h_W(x)=\sum_{m,n\ge 0}a_{m,n}x_1^{\sigma_1+m}x_2^{\sigma_2+n}$$

with $a_{0,0}\ne 0$. Then we have the following difference equations for $\{a_{m,n}\}$:

$$(\sigma_1+m)\left\{-(\sigma_1+m)+(\sigma_2+n)+\frac{1}{2}\right\}a_{m,n}+a_{m-1,n}=0, \tag{2.3}$$

$$\left(\sigma_2+n+\frac{k+\nu}{2}\right)\left(\sigma_2+n+\frac{k-\nu}{2}\right)a_{m,n}+\left\{(\sigma_1+m)-(\sigma_2+n)+\frac{1}{2}\right\}a_{m,n-1}=0. \tag{2.4}$$

Here we promise $a_{m,n}=0$ if $m<0$ or $n<0$. By putting $m=n=0$ in (2.3) and (2.4), we can find the characteristic indices

$$(\sigma_1,\sigma_2)=\left(0,\ \frac{-k\pm\nu}{2}\right),\left(\frac{-k\pm\nu+1}{2},\ \frac{-k\pm\nu}{2}\right).$$

If ν is not an integer, we can determine the coefficients $a_{m,n}$ inductively for each case and thus obtain

Proposition 2.2. *(cf. [29, Proposition 2.1]) For $\nu\notin \mathbf{Z}$, define the functions $h_W^i(\nu;a)=h_W^i(\nu;x)$ on A by*

$$h_W^1(\nu;x)=\sum_{m,n\ge 0}c_{m,n}^1x_1^m x_2^{n+(-k+\nu)/2},$$
$$h_W^2(\nu;x)=\sum_{m,n\ge 0}c_{m,n}^2x_1^{m+n+(-k+\nu+1)/2}x_2^{n+(-k+\nu)/2},$$

with

$$c_{m,n}^1=\Gamma\begin{bmatrix}-n-\nu, & -m+n+\frac{-k+\nu+1}{2}\\ -\nu, & \frac{-k+\nu+1}{2}\end{bmatrix}\frac{(-1)^{m+n}}{m!\,n!},$$
$$c_{m,n}^2=\Gamma\begin{bmatrix}-n-\nu, & -m-n+\frac{k-\nu-1}{2}\\ -\nu, & \frac{k-\nu-1}{2}\end{bmatrix}\frac{(-1)^{m+n}}{m!\,n!}.$$

Then the power series $h_W^i(\nu;x)$ converges for any $x\in\mathbf{C}^2$ and the set $\{h_W^i(\varepsilon\nu;x)\mid i=1,2,\ \varepsilon\in\{\pm1\}\}$ forms a basis of the space of solutions of the system in Proposition 2.1.

2.3 Explicit formulas of good Whittaker functions

When $c_3 < 0$, Proposition 2.1 tells us that there is no non-zero moderate growth Whittaker function. Therefore let us assume $c_3 > 0$ in the following discussion. The integral expression for the Whittaker functions of moderate growth was obtained by Miyazaki and Oda.

Proposition 2.3. *([23, Theorem 8.1]) Let π and τ be as before. Define*

$$g_W(a) = g_W(x) := x_2^{-1/2} \int_0^\infty t^{-k+1/2} W_{0,\nu}(t) \exp\Big(-\frac{t^2}{16x_2} - \frac{16x_1x_2}{t^2}\Big)\frac{dt}{t},$$

with $W_{\kappa,\mu}$ the classical Whittaker function. Then the function

$$\phi_W(a) = a_1^{k+1}a_2^{k+1}\exp(-2\pi c_3 a_2^2) g_W(a)$$

gives a non-zero element in $\mathrm{Wh}(\pi, \eta_1, \tau)^{\mathrm{mod}}|_A$ which is unique up to constant multiple.

2.4 Expansion theorem for Whittaker functions

Now we express the moderate growth Whittaker function g_W as a linear combination of h_W^i.

Theorem 2.4. *For $\nu \notin \boldsymbol{Z}$, let $h_W^i(\nu; a)$ and $g_W(a)$ be the function defined in Proposition 2.2 and 2.3, respectively. Then*

$$g_W(a) = c_W \sum_{\varepsilon\in\{\pm 1\}} \Big(\Gamma\Big[-\varepsilon\nu,\ \frac{-k+\varepsilon\nu+1}{2}\Big] h_W^1(\varepsilon\nu; a) + \Gamma\Big[-\varepsilon\nu,\ \frac{k-\varepsilon\nu-1}{2}\Big] h_W^2(\varepsilon\nu; a)\Big)$$

with $c_W = 2^{1-2k}\pi^{-1/2}$.

3 Siegel-Whittaker functions

3.1 Basic results

Miyazaki ([21]) studied the Siegel-Whittaker functions for P_J-principal series and obtained the multiplicity one property and the explicit integral representation for rapidly decreasing function. As in the previous section, we introduce the coordinate $y = (y_1, y_2)$ on A by

$$y_1 = \frac{h_1 a_1^2}{h_2 a_2^2}, \quad y_2 = 4\pi h_2 a_2^2.$$

We remark on a compatibility condition. For a non-zero element ϕ of $C^\infty_{\eta_i, \tau_{(-k,-k)}}(R_i\backslash G/K)$, we have

$$\phi(a) = \phi(mam^{-1}) = (\chi_{m_0} \boxtimes \xi)(m)\tau_{(-k,-k)}(m)\phi(a),$$

where $a \in A$ and $m \in SO(\xi) \cap Z_K(A) = \{\pm 1_4\}$. If we take $m = -1_4$, $(\chi_{m_0} \boxtimes \xi)(m) = \chi_{m_0}(m) = \exp(\pi\sqrt{-1}m_0)$ and $\tau_{(-k,-k)}(m) = 1$ imply that m_0 is an even integer.

Proposition 3.1. *([21, Proposition 7.2]) Let π and τ be as in §2.1. Then we have the following:*

(i) We have $\dim \mathcal{I}_{\eta_2,\pi} = \dim \mathrm{SW}(\pi,\eta_2,\tau) \leq 4$ *and a function*

$$\phi_{SW}(a) = a_1^{k+1} a_2^{k+1} \exp(-2\pi(h_1 a_1^2 + h_2 a_2^2)) h_{SW}(a)$$

is in the space $\mathrm{SW}(\pi,\eta_2,\tau)|_A$ *if and only if* $h_{SW}(a) = h_{SW}(y)$ *is a smooth solution of following system:*

$$(3.1)\quad \Big\{\partial_{y_1}\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big) + \frac{y_1}{y_1 - 1}\Big(-\partial_{y_1} + \frac{1}{2}\partial_{y_2}\Big) + \frac{m_0^2}{4}\frac{y_1}{(y_1-1)^2}\Big\} h_{SW}(y) = 0,$$

$$(3.2)\quad \Big\{\Big(\partial_{y_2} + \frac{k+\nu}{2}\Big)\Big(\partial_{y_2} + \frac{k-\nu}{2}\Big) - y_1 y_2\Big(\partial_{y_1} + \frac{1}{2}\Big) - y_2\Big(-\partial_{y_1} + \partial_{y_2} + \frac{1}{2}\Big)\Big\} h_{SW}(y) = 0,$$

with $\partial_{y_i} = y_i(\partial/\partial y_i)$.

(ii) $\dim \mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}} \leq 1$.

3.2 Explicit formulas of secondary Siegel-Whittaker functions

We consider the power series solution of the system in Proposition 3.1 around $(y_1, y_2) = (0,0)$. In the notation in [16], this is the solution at Q_∞.

Proposition 3.2. *For $\nu \notin \boldsymbol{Z}$, set $h^i_{SW}(\nu;a) = h^i_{SW}(\nu;y)$ by*

$$h^1_{SW}(\nu;y) = (1-y_1)^{|m_0|/2} \sum_{m,n\geq 0} c^1_{m,n}\, \Gamma\begin{bmatrix} m - n + \frac{k+|m_0|-\nu}{2}, & m + \frac{|m_0|+1}{2} \\ -n + \frac{k+|m_0|-\nu}{2}, & \frac{|m_0|+1}{2} \end{bmatrix} y_1^m y_2^{n+(-k+\nu)/2},$$

$$h^2_{SW}(\nu;y) = (1-y_1)^{|m_0|/2} \sum_{m,n\geq 0} c^2_{m,n}\, \Gamma\begin{bmatrix} m + n + \frac{-k+|m_0|+\nu}{2} + 1, & m + \frac{|m_0|+1}{2} & \frac{k+|m_0|-\nu}{2} \\ \frac{-k+|m_0|+\nu}{2} + 1, & \frac{|m_0|+1}{2} & n + \frac{k+|m_0|-\nu}{2} \end{bmatrix}$$
$$\cdot\, y_1^{m+n+(-k+\nu+1)/2} y_2^{n+(-k+\nu)/2}.$$

Here $c^1_{m,n}$ and $c^2_{m,n}$ are the coefficients defined in Proposition 2.2. Then the power series $h^i_{SW}(\nu;y)$ converges $|y_1| < 1$ and $y_2 \in \mathbf{C}$ and the set $\{h^i_{SW}(\varepsilon\nu;y) \mid i = 1,2,\ \varepsilon\{\pm 1\}\}$ forms a basis of the space of solutions of the system in Proposition 3.1.

3.3 Explicit formulas of good Siegel-Whittaker functions

The integral representation of the unique element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ is given by Miyazaki ([21, Theorem 7.5]). For our purpose, however, we need another integral expression for this function. Inspired by the work of Debiard and Gaveau ([1],[2]), we obtain the following Euler type integral. See also Iida ([15]) and Gon ([6]).

Proposition 3.3. *Define*

$$g_{SW}(a) = g_{SW}(y) := (1-y_1)^{|m_0|/2} y_2^{|m_0|/2}$$
$$\cdot \int_0^1 t^{(|m_0|-1)/2}(1-t)^{(|m_0|-1)/2} F\Big(\frac{y_2}{2}\{1 - t(1-y_1)\}\Big)\,dt,$$

with

$$F(z)=e^{z}(2z)^{(-k-|m_0|-1)/2}W_{(k-|m_0|-1)/2,\nu/2}(2z).$$

Then the function

$$\phi_{SW}(a)=a_1^{k+1}a_2^{k+1}\exp(-2\pi(h_1a_1^2+h_2a_2^2))g_{SW}(a)$$

gives a non-zero element in $\mathrm{SW}(\pi,\eta_2,\tau)^{\mathrm{rap}}|_A$ *which is unique up to constant multiple.*

Proof. See [6, 8.4]. □

3.4 Expansion theorem for Siegel-Whittaker functions

Theorem 3.4. *For $\nu\notin\boldsymbol{Z}$, let $h^i_{SW}(\nu;a)$ and $g_{SW}(a)$ be the function defined in Proposition 3.2 and 3.3, respectively. Then*

$$g_{SW}(a)=c_{SW}\sum_{\varepsilon\in\{\pm1\}}\left(\Gamma\left[-\varepsilon\nu,\ \frac{-k+\varepsilon\nu+1}{2}\right]h^1_{SW}(\varepsilon\nu;a)\right.$$
$$\left.+\Gamma\left[\begin{matrix}-\varepsilon\nu,\ \frac{k-\varepsilon\nu-1}{2},\ \frac{-k+|m_0|+\varepsilon\nu}{2}+1\\ \frac{k+|m_0|-\varepsilon\nu}{2}\end{matrix}\right]h^2_{SW}(\varepsilon\nu;a)\right)$$

with

$$c_{SW}=\Gamma\left[\begin{matrix}\frac{|m_0|+1}{2}\\ \frac{-k+|m_0|-\nu}{2}+1,\ \frac{-k+|m_0|+\nu}{2}+1\end{matrix}\right].$$

4 Confluences

4.1 Confluence of the differential equations

Theorem 4.1. *If we substitute*

$$h_1=t^2c_3,\quad h_2=c_3,\quad m_0=\frac{2\pi c_0}{t}\tag{4.1}$$

in the system in Proposition 3.1 and take the limit $t\to0$, then we obtain the system in Proposition 2.1.

4.2 Confluence of the secondary spherical functions

Theorem 4.2. *For $\nu\notin\boldsymbol{Z}$, define the functions $h^i_{SW}(\nu,t;a)$ $(i=1,2)$ by substituting (4.1) in $h^i_{SW}(\nu;a)$. Then*

$$\lim_{t\to0}h^1_{SW}(\nu,t;a)=h^1_W(\nu;a),$$
$$\lim_{t\to0}t^{k-\nu-1}h^2_{SW}(\nu,t;a)=h^2_W(\nu;a).$$

4.3 Confluence of the spherical functions

Theorem 4.3. *Define the function $g_{SW}(t;a)$ by substituting (4.1) in $g_{SW}(a)$. Then*

$$\lim_{t\to 0}\frac{g_{SW}(t;a)}{c_{SW}}=\frac{g_W(a)}{c_W}.$$

5 Deformation from (R_2,η_2) to (R_1,η_1) and the confluence

In this section we explain the main results in the previous section from the points of view of deformations and contractions of Lie groups (cf. [3]). This is to supply a heuristic background for the computations in the previous sections.

5.1 From $SO(2)$ to N_0

We first consider the deformation of two subgroups of $SL(2,\mathbf{R})$:

$$SO(2)=\left\{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\,\Big|\,\theta\in\mathbf{R}\right\},$$
$$N_0=\left\{\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\,\Big|\,c\in\mathbf{R}\right\}.$$

Under the usual action of $SL(2,\mathbf{R})$ to the upper half plane $\mathfrak{h}=\{z\in\mathbf{C}\mid \mathrm{Im}(z)>0\}$, $SO(2)$ is the stabilizer subgroup of $\sqrt{-1}$ and N_0 fixes $\sqrt{-1}\infty$. Set $z_t=\sqrt{-1}/t$ for $t>0$. Then $\lim_{t\to\infty} z_t=\sqrt{-1}\infty$ and the stabilizer subgroup $\mathrm{Stab}_{SL(2,\mathbf{R})}(z_t)$ of z_t in $SL(2,\mathbf{R})$ is

$$\begin{aligned}\mathrm{Stab}_{SL(2,\mathbf{R})}(z_t)&=\left\{r_\theta(t)=\begin{pmatrix}\cos\theta & \sin\theta/t\\ -t\sin\theta & \cos\theta\end{pmatrix}\,\Big|\,\theta\in\mathbf{R}\right\}\\ &=\begin{pmatrix}t^{-1/2} & 0\\ 0 & t^{1/2}\end{pmatrix}SO(2)\begin{pmatrix}t^{1/2} & 0\\ 0 & t^{-1/2}\end{pmatrix}.\end{aligned}$$

For our purpose, we have to move $\theta=\theta(t)$ such as $\sin\theta(t)/t\to c$ as $t\to 0$. Let $\theta(t)=ct$. Then

$$r_{\theta(t)}(t)=\begin{pmatrix}\cos\theta(t) & \sin\theta(t)/t\\ -t\sin\theta(t) & \cos\theta(t)\end{pmatrix}\to\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}\tag{5.1}$$

as desired.

5.2 From R_2 to R_1

Let ξ_t $(t\neq 0)$ be the definite character of N_S associated with $H_{\xi_t}=\begin{pmatrix}ht^2 & 0\\ 0 & h\end{pmatrix}$ $(h>0)$:

$$\xi_t(n(0,n_1,n_2,n_3))=\exp\bigl(2\pi\sqrt{-1}(ht^2n_1+hn_3)\bigr).$$

Then the stabilizer subgroup $SO(\xi_t)$ is identified with $\{r_\theta(t)\mid t>0\}$. Therefore if we take $\theta=\theta(t)=tn_0$ as in the previous subsection, (5.1) implies

$$\lim_{t\to 0}SO(\xi_t)=\{n(n_0,0,0,0)\mid n_0\in\mathbf{R}\}\tag{5.2}$$

in L_S makes up $R_1=N$ together with N_S, as we expected.

5.3 From η_2 to η_1

Define the character $\chi_{m_0(t)}$ of $SO(\xi_t)$ by

$$\chi_{m_0(t)}(r_\theta(t)) = \exp\sqrt{-1}(m_0(t)\theta).$$

and put $\eta_{2,t} = \chi_{m_0(t)} \boxtimes \xi_t$.

$$\eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \exp 2\pi\sqrt{-1}(ht^2 n_1 + hn_3) \cdot \exp\sqrt{-1}(m_0(t)\theta(t)).$$

Since $\theta(t) = n_0 t$, we should take

$$m_0(t) = \frac{2\pi c_0}{t}.$$

Then the right hand side goes to $\exp 2\pi\sqrt{-1}(c_3 n_3 + c_0 n_0)$ (after the replacement $h = c_3$), and thus combined with (5.2), we obtain

$$\lim_{t\to 0} \eta_{2,t}(r_{\theta(t)}(t) \cdot n(0, n_1, n_2, n_3)) = \eta_1(n(n_0, n_1, n_2, n_3)).$$

Remark 2. Our result should be regarded as the investigation of the intertwining spaces:

$$\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(H_{\pi,K}, C^\infty_{\eta_{2,t}}(R_t\backslash G))$$

with

$$R_t = SO(\xi_t) \ltimes N_s \quad (t > 0).$$

6 Further comments

We only treat the even P_J-principal series, however, we also have the same results for the odd case, that is, $\varepsilon(\mathrm{diag}(-1,1,-1,1)) = -(-1)^k$.

In the case of the principal series (induced from minimal parabolic subgroup of G), the holonomic systems of rank 8 for the radial part of Whittaker functions (resp. Siegel-Whittaker functions) are obtained in [22] (resp. [21], [16]) and we can prove the same assertion as Theorem 4.1. However, explicit formulas for secondary spherical functions are known only for Whittaker functions ([17]), we can not say any more.

The other kinds of spherical functions on $Sp(2,\mathbf{R})$ are studied by Moriyama ([24]) and by Hirano ([12], [13], [14]). The spherical subgroup of [24] is $SL(2,\mathbf{C})$ and of [12], [13] and [14] is $SL(2,\mathbf{R}) \ltimes H_3$, with H_3 the 3-dimensional Heisenberg group. We hope that similar results hold between the two spherical functions.

We finally remark on the case of the special unitary group $SU(2,2)$, which has the same restricted root system as $Sp(2,\mathbf{R})$. (Siegel-) Whittaker functions on $SU(2,2)$ are studied by Hayata and Oda ([10], [8], [9]) and by Gon ([6]). Since their differential equations are compatible to those of $Sp(2,\mathbf{R})$, analogous argument seems to be possible.

参考文献

[1] A. Debiard and B. Gaveau, Représentations intégrales de certaines séries de fonctions sphériques d'un système de racines BC, J. Funct. Anal. **96** (1991), 256-296.

[2] A. Debiard and B. Gaveau, Integral formulas for the spherical polynomials of a root system of type BC_2, J. Funct. Anal. **119** (1994), 401-454.

[3] A.H. Dooley, Contractions of Lie groups and applications to analysis, Topics in modern harmonic analysis, Vol. I, II, Ist. Naz. Alta Mat. Francesco Severi, Rome, (1983), 483-515.

[4] A. Erdelyi et. al., Higher Transcendental Functions, vol.1, McGraw-Hill, (1953).

[5] A. Erdelyi et. al., Higher Transcendental Functions, vol.2, McGraw-Hill, (1953).

[6] Y. Gon, Generalized Whittaker functions on $SU(2,2)$ with respect to the Siegel parabolic subgroup, Memoirs of the AMS **738** (2002).

[7] Harish-Chandra, Spherical functions on a semi-simple Lie group I, II, Amer. J. Math. **80** (1958), 241-310, 553-613.

[8] T. Hayata, Differential equations for principal series Whittaker functions on $SU(2,2)$, Indag. Math., N.S. **8** (4), (1997), 493-528.

[9] T. Hayata, Whittaker functions of generalized principal series on $SU(2,2)$, J. Math. Kyoto Univ. **37** (1997), 531-546.

[10] T. Hayata and T. Oda, An explicit integral representation of Whittaker functions for the reprensetations of the discrete series –the case of $SU(2,2)$–, J. Math. Kyoto Univ. **37** (1997), 519-530.

[11] G.J. Heckman and E.M. Opdam, Root system and hypergeometric functions I, II, Compositio Math. **64** (1987), 329-352, 353-373.

[12] M. Hirano, Fourier-Jacobi type spherical functions for discrete series representations of $Sp(2,\mathbf{R})$, Compositio Math. **128** (2001), 177-216.

[13] M. Hirano, Fourier-Jacobi type spherical functions for P_J-principal series representations of $Sp(2,\mathbf{R})$, J. London Math. Soc. (2) **65** (2002), 524-546.

[14] M. Hirano, Fourier-Jacobi type spherical functions for principal series representations of $Sp(2,\mathbf{R})$, Indag. Math., N.S. **15** (2004), 43-54.

[15] M. Iida, Spherical functions of the principal series representations of $Sp(2,\mathbf{R})$ as hypergeometric functions of C_2-type, Publ. RIMS, Kyoto Univ. **32** (1996), 689-727.

[16] T. Ishii, Siegel-Whittaker functions on $Sp(2,\mathbf{R})$ for principal series representations, J. Math. Sci. Univ. of Tokyo **9** (2002), 303-346.

[17] T. Ishii, On principal series Whittaker functions on $Sp(2,\mathbf{R})$, preprint.

[18] B. Kostant, On Whittaker vectors and representation theory, Invent. Math. **48** (1978), 101-184.

[19] H. Matumoto. Whittaker vectors and the Goodman-Wallach operators, Acta Math. **161** (1988), 183-241.

[20] R. Miatello, N. Wallach, Automorphic forms constructed from Whittaker vectors, J. Funct. Anal. **86** (1989), 411–487.

[21] T. Miyazaki, The Generalized Whittaker Functions for $Sp(2,\mathbf{R})$ and the Gamma Factor of the Andrianov's L-function, J. Math. Sci. Univ. of Tokyo **7** (2000), 241-295.

[22] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2,\mathbf{R})$, – Explicit formulae of differential equations–, Proc. of the 1993 Workshop, Automorphic Forms and Related Topics, The Pyungsan Institute for Math. Sci., 59-92.

[23] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2,\mathbf{R})$ II, Tôhoku Math. J. **50** (1998), 243-260; Errata, ibid. **54** (2002), 161-162.

[24] T. Moriyama, Spherical functions for the semisimple symmetric pair $(Sp(2,\mathbf{R}), SL(2,\mathbf{C}))$, Canad. J. of Math. **54** (2002), 828-865.

[25] T. Moriyama, Entireness of the spinor L-functions for certain generic cusp forms on $GSp(2)$, Amer. J. Math. **126** (2004), 899–920.

[26] T. Oda, An explicit integral representation of Whittaker functions on $Sp(2,\mathbf{R})$ for large discrete series representations, Tôhoku Math. J. **46** (1994), 261-279.

[27] T. Oda, and M. Tsuzuki, Automorphic Green functions associated with the secondary spherical functions, Publ. Res. Inst. Math. Sci. **39** (2003), 451–533.

[28] E.M. Opdam, Root system and hypergeometric functions III, IV, Compositio Math. **67** (1988), 21-49, 191-209.

[29] H. Sakuno, A Basis on the Space of Whittaker Functions for the Representations of the Discrete Series –the Case of $Sp(2,\mathbf{R})$ and $SU(2,2)$–, J. Math. Sci. Univ. of Tokyo **6** (1999), 757-791.

[30] N. Wallach, Asymptotic expansions of generalized matrix entries of representations of real reductive groups, Lect. Note in Math. **1024** (1984), 287-369.