Weak Kurepa trees and weak diamonds

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24th, December, 2004

Abstract

We consider combinatorial statements which fit between the Kurepa and the weak Kurepa hypotheses. We also formulate weak diamonds and consider their relations to these statements .

Introduction

Two weak forms of the diamond principle $\tilde{\diamondsuit}$ and $\tilde{\tilde{\diamondsuit}}$ are introduced in [W]. It is shown that (see p.110 of [W] for more information)

- $\diamondsuit$ implies $\tilde{\diamondsuit}$.
- The Kurepa hypothesis (KH) also implies $\tilde{\diamondsuit}$.
- $\tilde{\diamondsuit}$ in turn implies $\tilde{\tilde{\diamondsuit}}$.
- $\tilde{\tilde{\diamondsuit}}$ negates the saturation of the non-stationary ideal on ω_1.
- $\tilde{\tilde{\diamondsuit}}$ implies the weak Kurepa hypothesis (wKH), too.
- $\diamondsuit$ persists in the sense that if $\diamondsuit$ holds in a transitive model of ZFC which correctly computes ω_2, then $\tilde{\diamondsuit}$ holds in the universe.

The following are delt in this note.

(1) We give an equivalent statements to $\tilde{\diamondsuit}$ and $\tilde{\tilde{\diamondsuit}}$.

(2) Our equivalent to $\tilde{\tilde{\diamondsuit}}$ is seemingly more demanding than the original $\tilde{\tilde{\diamondsuit}}$. As a result, we get what we call stat-wKH which rather directly negates the saturation of the non-stationary ideal on ω_1.

(3) We formulate same types of weak Kurepa hypotheses as stat-wKH and consider weak diamonds to investigate the situation between KH and these wKH.

(4) We provide more information on these weak diamonds. For example, we get a new fragment of $\diamondsuit$ different from $\clubsuit$.

(5) We describe as many forcing constructions as we know of to separate these new combinatorial statements.

Though claims we make are within the reaches of established facts and forcing techniques, so-far-possibly-implicit points of view on KH, wKH and $\diamondsuit$ are examined.

§1. The KH, $\tilde{\diamondsuit}$, $\tilde{\tilde{\diamondsuit}}$ and the wKH

1.1 Definition. ([W]) $\tilde{\diamondsuit}$ holds, if there exist ω_2-many subsets $\langle A_\beta \mid \beta < \omega_2 \rangle$ of ω_1 and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ with each T_α countable and the following is stationary in ω_2

$$\{\beta_Y \mid Y \subset \mathcal{P}(\omega_1) \text{ is countable, } \langle T_\alpha \mid \alpha < \omega_1 \rangle \text{ guesses } Y\}$$

where,

$$\beta_Y = \sup\{\beta + 1 \mid A_\beta \in Y\}$$

and

$\langle T_\alpha \mid \alpha < \omega_1 \rangle$ guesses Y, if the following is cofinal in ω_1

$$\{\alpha < \omega_1 \mid E \cap \alpha \in T_\alpha \text{ for all } E \in Y\}$$

We record the following for the sake of clarity.

1.2 Proposition. (1) For $S \subseteq \{\beta < \omega_2 \mid \mathrm{cf}(\beta) = \omega\}$, the following are equivalent

- S is stationary in ω_2.
- $\{X \in [\omega_2]^\omega \mid \bigcup X \in S\}$ is stationary in $[\omega_2]^\omega$.

(2) For $S^* \subseteq [\omega_2]^\omega$, if S^* is stationary in $[\omega_2]^\omega$, then $\{\bigcup X \mid X \in S^*\}$ is stationary in ω_2. (The converse is false in some cases.)

In the manner we show the above on these two notions of stationary sets, we may show

1.3 Proposition. $\tilde{\diamondsuit}$ holds iff there exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- The following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \exists A \subseteq \omega_1 \ \exists B \subseteq X \text{ such that } \bigcup A = \omega_1, \bigcup B = \bigcup X,$$
$$\forall (\alpha, \beta) \in A \times B \ b_\beta \lceil \alpha \in S_\alpha\}$$

Proof. Let $\langle A_\beta \mid \beta < \omega_2 \rangle$ and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ satisfy $\tilde{\diamondsuit}$. For each $\beta < \omega_2$, let $b_\beta : \omega_1 \longrightarrow 2$ be the characteristic function of A_β. For each $\alpha < \omega_1$, let $S_\alpha = \{\chi_a \mid a \in T_\alpha \cap \mathcal{P}(\alpha)\}$, where $\chi_a : \alpha \longrightarrow 2$ is the characteristic function of a. Given $\varphi : {}^{<\omega}\omega_2 \longrightarrow \omega_2$, find $Y \subset \mathcal{P}(\omega_1)$ such that β_Y is a limit ordinal, β_Y is φ-closed and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ guesses Y. Let

$$A = \{\alpha < \omega_1 \mid \forall E \in Y \ E \cap \alpha \in T_\alpha\}$$

and

$$B = \{\beta < \omega_2 \mid A_\beta \in Y\}.$$

Let $X \in [\omega_2]^\omega$ be the φ-closure of B. Then X is φ-closed, $\bigcup A = \omega_1$, $\bigcup B = \bigcup X$ and for all $(\alpha, \beta) \in A \times B$, we have $b_\beta \lceil \alpha \in S_\alpha$.

Conversely, for each $\beta < \omega_2$, let $A_\beta = \{i < \omega_1 \mid b_\beta(i) = 1\}$. For each $\alpha < \omega_1$, let $T_\alpha = \{\{i < \alpha \mid \sigma(i) = 1\} \mid \sigma \in S_\alpha\}$. Let $C \subseteq \omega_2$ be a club. Take $X \in [\omega_2]^\omega$, $A \subseteq \omega_1$ and $B \subseteq X$ such that $\bigcup X \in C$, $\bigcup A = \omega_1$, $\bigcup B = \bigcup X$ and for all $(\alpha, \beta) \in A \times B$, we have $b_\beta \lceil \alpha \in S_\alpha$. We may assume $\bigcup X$ is a limit ordinal. Let $Y = \{A_\beta \mid \beta \in B\}$. Then $\beta_Y = \bigcup X \in C$ and $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ guesses this Y.

□

The following is almost verbatim from [W].

1.4 Definition. ([W]) $\tilde{\tilde{\Diamond}}$ holds, if there exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- The following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \exists \alpha \geq X \cap \omega_1 \ \exists B \subseteq X \text{ such that } \bigcup B = \bigcup X, \ \forall \beta \in B \ b_\beta \lceil \alpha \in S_\alpha\}$$

Here is our equivalent statement to $\tilde{\tilde{\Diamond}}$.

1.5 Proposition. $\tilde{\tilde{\Diamond}}$ holds iff there exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- The following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \exists \underline{\alpha = X \cap \omega_1} \ \exists B \subseteq X \text{ such that } \bigcup B = \bigcup X, \ \forall \beta \in B \ b_\beta \lceil \alpha \in S_\alpha\}$$

We record a well-known lemma, say, from [B] and [W].

1.6 Lemma. Let θ be a regular cardinal with $\theta \geq \omega_2$ and N be a countable elementary substructure of H_θ. By this we mean $(N, \in)$ is an elementary substructure of $(H_\theta, \in)$ with $|N| = \omega$ and may simply denote $N \prec H_\theta$. Define

$$N^* = \{f(N \cap \omega_1) \mid f \in N\}.$$

Then

- $(N^*, \in)$ is a countable elementary substructure of $(H_\theta, \in)$.
- $N \subset N^*$, $N \cap \omega_1 \in N^*$ and so $N \cap \omega_1 < N^* \cap \omega_1 < \omega_1$.
- However, $\sup(N \cap \omega_2) = \sup(N^* \cap \omega_2)$.

1.7 Corollary. Let θ be a regular cardinal with $\theta \geq \omega_2$. Then given any countable elementary substructure N of H_θ, we may automatically construct its canonical extensions $\langle N_i \mid i < \omega_1 \rangle$. By this we mean

- $N_0 = N$.
- Each N_i is a countable elementary substructure of H_θ.
- $N_{i+1} = N_i^*$.
- For limit i, we set $N_i = \bigcup\{N_k \mid k < i\}$.

Therefore,

- $\langle N_i \cap \omega_1 \mid i < \omega_1 \rangle$ forms a club in ω_1.
- However, $\sup(N_i \cap \omega_2) = \sup(N \cap \omega_2)$ constantly for all $i < \omega_1$.

Isomorphic-types of the canonical extensions are considered via φ_{AC} in [W].

Proof to the equivalence of $\tilde{\tilde{\Diamond}}$.

Fix $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ so that $\tilde{\tilde{\Diamond}}$ is witnessed. We show

1.7.1 Claim. The following $N \in [H_{\omega_2}]^\omega$ are stationary in $[H_{\omega_2}]^\omega$.

- $N \prec H_{\omega_2}$,
- $\exists f \in N \cap {}^{\omega_1}\omega_1$ with $\forall \alpha < \omega_1\ f(\alpha) \geq \alpha$ such that
  $\exists B \subset N \cap \omega_2$ with $\bigcup B = \bigcup(N \cap \omega_2)$, $\forall \beta \in B\ b_\beta \lceil f(N \cap \omega_1) \in S_{f(N \cap \omega_1)}$.

Then by the Fodor's Lemma,

1.7.2 Claim. $\exists f_0 \in {}^{\omega_1}\omega_1\ \forall \alpha < \omega_1\ f_0(\alpha) \geq \alpha$ and the following is stationary in $[H_{\omega_2}]^\omega$.

$$\{N \in [H_{\omega_2}]^\omega \mid N \prec H_{\omega_2},\ \exists B \subset N \cap \omega_2 \text{ with } \bigcup B = \bigcup(N \cap \omega_2),$$
$$\forall \beta \in B\ b_\beta \lceil f_0(N \cap \omega_1) \in S_{f_0(N \cap \omega_1)}\}$$

Therefore, for each $\alpha < \omega_1$, may define S^*_α by

$$S^*_\alpha = S_{f_0(\alpha)} \lceil \alpha.$$

Then $S^*_\alpha \subset {}^\alpha 2$, S^*_α is countable and the following is stationary in $[H_{\omega_2}]^\omega$.

$$\{N \in [H_{\omega_2}]^\omega \mid \exists B \subset N \cap \omega_2 \text{ with } \bigcup B = \bigcup(N \cap \omega_2),\ \forall \beta \in B\ b_\beta \lceil (N \cap \omega_1) \in S^*_{N \cap \omega_1}\}$$

So we would be done, if we provide a proof to 1.7.1 Claim.

Proof of 1.7.1 Claim. (This part is based on [W])

Let $\varphi : {}^{<\omega} H_{\omega_2} \longrightarrow H_{\omega_2}$. Fix a sufficiently large regular cardinal θ and a countable elementary substructure M of H_θ with $\varphi \in M$. We may asssume $X = M \cap \omega_2$ has a cofinal subset $B \subseteq X$ and there exists $\alpha \geq X \cap \omega_1$ such that

$$\forall \beta \in B \; b_\beta \lceil \alpha \in S_\alpha.$$

Construct the canonical extensions $\langle M_i \mid i < \omega_1 \rangle$ of M. Since $\langle M_i \cap \omega_1 \mid i < \omega_1 \rangle$ forms a club in ω_1 with $\alpha \geq M_0 \cap \omega_1$, there exists $i < \omega_1$ such that

$$M_i \cap \omega_1 \leq \alpha < M_{i+1} \cap \omega_1.$$

By the definition of M_{i+1} from M_i, we have $f \in M_i$ such that

$$f(M_i \cap \omega_1) = \alpha \geq M_i \cap \omega_1.$$

We may assume that $f : \omega_1 \longrightarrow \omega_1$ and that for all $\overline{\alpha} < \omega_1$, $f(\overline{\alpha}) \geq \overline{\alpha}$.
Let $N = M_i \cap H_{\omega_2}$. Since $H_{\omega_2} \in M_i \prec H_\theta$,

- N is a countable elementary substructure of H_{ω_2}.
- $f \in N$, as ${}^{\omega_1}\omega_1 \subset H_{\omega_2}$.
- $B \subseteq N \cap \omega_2$ and $\bigcup B = \bigcup (N \cap \omega_2)$.
- $\forall \beta \in B \; b_\beta \lceil f(N \cap \omega_1) \in S_{f(N \cap \omega_1)}$.

Since N is φ-closed, this completes the proof.

□

We go on to make

1.8 Definition. Let us *stat-weak Kurepa hypothesis (stat-wKH)* denote the following: There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- For all $\beta < \omega_2$, $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ are stationary in ω_1.

We may view stat-wKH as a sort of $\diamondsuit$. Namely, stat-wKH guesses some ω_2-many subsets of ω_1, while $\diamondsuit$ does all subsets of ω_1. The weak diamond $\tilde{\tilde{\diamondsuit}}$ entails stat-wKH.

1.9 Proposition. $\tilde{\tilde{\diamondsuit}}$ implies stat-wKH.

Proof. It is just thinning. By our equivalent form of $\tilde{\tilde{\diamondsuit}}$, we get $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that the following is stationary in $[\omega_2]^\omega$.

$$\{X \in [\omega_2]^\omega \mid \exists \delta = X \cap \omega_1,\ \exists B \subseteq X \text{ with } \bigcup B = \bigcup X,\ \forall \beta \in B\ b_\beta \lceil \delta \in S_\delta\}$$

1.9.1 Claim. $\{\beta < \omega_2 \mid \{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ is stationary in $\omega_1\}$ is cofinal in ω_2.

Proof of Claim. Fix $\eta < \omega_2$. Take a sufficiently large regular cardinal θ and a countable elementary substructure M of H_θ such that $\langle b_\beta \mid \beta < \omega_2 \rangle, \langle S_\alpha \mid \alpha < \omega_1 \rangle, \eta \in M$. We may set $\delta = M \cap \omega_1$ and assume that there exists $B \subseteq M \cap \omega_2$ cofinal within $M \cap \omega_2$ such that

$$\forall \beta \in B\ b_\beta \lceil \delta \in S_\delta.$$

Therefore, we may fix some $\beta \in B$ such that $\eta < \beta$ and $b_\beta \lceil \delta \in S_\delta$.

1.9.1.1 Sub claim. $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ is stationary in ω_1.

Proof of sub claim. We make use of the elementarity of M. Fix a club $C \in M$. Then $\delta \in C$ and so

$$M \models \text{"}\forall C \subseteq \omega_1 \text{ club } \exists \alpha \in C\ b_\beta \lceil \alpha \in S_\alpha.\text{"}$$

Therefore $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ is really stationary in the universe.

□

1.10 Proposition. The stat-wKH implies that there exists a family $\mathcal{F}$ of almost disjoint stationary subsets of ω_1 with $|\mathcal{F}| = \omega_2$. And so the non-stationary ideal on ω_1 is not saturated.

Proof. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be as in stat-wKH.

Let $\langle \sigma_n^\alpha \mid n < \omega \rangle$ enumerate S_α. By thinning, say twice, we may assume that there exists $n < \omega$ such that for all $\beta < \omega_2$, the following T_β is stationary in ω_1.

$$T_\beta = \{\alpha < \omega_1 \mid b_\beta \lceil \alpha = \sigma_n^\alpha\}$$

Now consider $\mathcal{F} = \{T_\beta \mid \beta < \omega_2\}$. Then this $\mathcal{F}$ works.

□

The following is shown in [W] by generic ultra-power constructions over set models of set theory.

1.11 Corollary. ([W]) $\tilde{\tilde{\diamondsuit}}$ implies the non-stationary ideal on ω_1 is not saturated.

1.12 Definition. Let us *cof-weak Kurepa hypothesis (cof-wKH)* denote the following:

There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- For all $\beta < \omega_2$, $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ are cofinal in ω_1.

Therefore, stat-wKH implies cof-wKH. We return to this in the next section.

1.13 Proposition. The cof-wKH implies wKH. I.e, there exists a sub tree T of ${}^{<\omega_1}2$ such that $|T| = \omega_1$ and there are at least ω_2-many cofinal branches through T.

Proof. We argue as in the previous proposition. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be as in cof-wKH.

Let $\langle \sigma_n^\alpha \mid n < \omega \rangle$ enumerate S_α. By thinning, say twice, we may assume that there exists $n < \omega$ such that for all $\beta < \omega_2$, the following E_β is cofinal in ω_1.

$$E_\beta = \{\alpha < \omega_1 \mid b_\beta \lceil \alpha = \sigma_n^\alpha\}$$

Let $T = \{\sigma_n^\alpha \lceil \overline{\alpha} \mid \overline{\alpha} \leq \alpha < \omega_1\}$. Then this T works. The b_β provide cofinal branches through T.

□

1.14 Corollary. ([W]) $\tilde{\tilde{\diamondsuit}}$ implies wKH.

Since KH implies $\tilde{\diamondsuit}$ by [W], we conclude

1.15 Corollary. The following are all equiconsistent.

(1) There exists a strongly inaccessible cardinal.

(2) Either wKH, cof-wKH, stat-wKH, $\tilde{\tilde{\diamondsuit}}$, $\tilde{\diamondsuit}$ or KH gets negated.

§2. Weak Kurepa Trees

We recap stat-wKH and cof-wKH in this section and generalize them.

2.1 Definition. Let □ be either *cof, stat, club,* or *coint.* Let us *□-weak Kurepa hypothesis (□-wKH)* denote the following:

There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- For each $\beta < \omega_2$, either $\{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ is cofinal, stationary, contains a club, or is coinitial in ω_1, respectively.

We view KH, $\tilde{\diamondsuit}$, $\tilde{\tilde{\diamondsuit}}$, stat-wKH, cof-wKH and wKH along this generalization and record the following.

2.2 Proposition. (1) KH iff coint-wKH.

(2)

- The coint-wKH implies club-wKH.
- The club-wKH implies stat-wKH.
- The stat-wKH implies cof-wKH.
- The cof-wKH implies wKH.

(3)

- The club-wKH implies $\tilde{\diamondsuit}$.
- ([W]) $\tilde{\diamondsuit}$ implies $\tilde{\tilde{\diamondsuit}}$.
- $\tilde{\tilde{\diamondsuit}}$ implies stat-wKH.

Proof. For (1): Suppose T is a Kurepa tree. We may assume $T \subset {}^{<\omega_1}2$. Let $\{b_\beta \mid \beta < \omega_2\} \subset {}^{\omega_1}2$ be one-to-one such that $b_\beta \lceil \alpha \in T_\alpha$ for all $\beta < \omega_2$ and $\alpha < \omega_1$. Let $S_\alpha = T_\alpha$ for all $\alpha < \omega_1$. Then S_α is countable and $b_\beta \lceil \alpha \in S_\alpha$ for every possible combination. Hence we certainly have coint-wKH.

Conversely, let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be witnesses to coint-wKH. By thinning, we may assume that there exists $\alpha_0 < \omega_1$ such that for all $\beta < \omega_2$ and all $\alpha \geq \alpha_0$, we have

$$b_\beta \lceil \alpha \in S_\alpha.$$

Let $T = \{b_\beta \lceil \alpha \mid \beta < \omega_2, \alpha < \omega_1\}$. If $\alpha \geq \alpha_0$, then $T_\alpha \subseteq S_\alpha$ which is countable. If $\alpha < \alpha_0$, then $T_\alpha \subset S_{\alpha_0} \lceil \alpha$ which is also countable. Each b_β provide different cofinal branch $\{b_\beta \lceil \alpha \mid \alpha < \omega_1\}$. Hence T is a Kurepa tree.

For (2): First three are trivial by definition and we have seen the fourth.

For (3): Since we have seen the last two items, we consider the first item. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be witnesses to club-wKH. Let $E_\beta = \{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$. Then for all $X \in [\omega_2]^\omega$, we set $A = \bigcap\{E_\beta \mid \beta \in X\} \subset \omega_1$ and $B = X$ so that $\bigcup A = \omega_1$, $\bigcup B = \bigcup X$ and for all $(\alpha, \beta) \in A \times B$, we have $b_\beta \lceil \alpha \in S_\alpha$. Hence we certainly have $\tilde{\diamondsuit}$.

□

2.3 Proposition. The club-wKH implies the transversal hypothesis (TH). Namely, there exists a family $\mathcal{F}$ of almost disjoint functions from ω_1 into ω with $|\mathcal{F}| = \omega_2$.

Proof. We must observe that there exist ω_2-many functions $g_\beta : \omega_1 \longrightarrow \omega$ such that if $\beta_1 \neq \beta_2$, then there exists $\alpha_{\beta_1\beta_2} < \omega_1$ such that for all α with $\alpha_{\beta_1\beta_2} \leq \alpha < \omega_1$, we have $g_{\beta_1}(\alpha) \neq g_{\beta_2}(\alpha)$.

To this end, let $\{\sigma_n^\alpha \mid n < \omega\}$ enumerate S_α. Then let $f_\beta(\alpha) =$ the least n such that $b_\beta \lceil \alpha = \sigma_n^\alpha$, if applicable. Then if $\beta_1 \neq \beta_2$, then $\{\alpha < \omega_1 \mid f_{\beta_1}(\alpha) \neq f_{\beta_2}(\alpha)\}$ contains a club. Now we may resort to a trick due to Jensen to produce g_β. See the proof of Lemma 1 on p. 72 of [D].

□

When I gave a talk on this at the Set Theory Seminar, Nagoya university, 17th, Dec. 2004, T. Sakai provided an idea for a direct proof on the spot. Accordingly, I record the following based on his idea.

Proof. Let us fix $\langle e_\alpha \mid \alpha < \omega_1 \rangle$ so that $e_\alpha : \omega \longrightarrow \alpha + 1$ onto. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ be as in club-wKH. Let $C_\beta \subset \{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ be a club and $\langle a_n^\alpha \mid n < \omega \rangle$ enumerate S_α.

For each β, let us define $g_\beta : \omega_1 \longrightarrow \omega \times \omega$ so that for any $\alpha \geq \min C_\beta$, if $\delta = \max\left(C_\beta \cap (\alpha + 1)\right)$, then $g_\beta(\alpha) = (n, m)$, where

$$n = \text{the least } n \text{ s.t. } e_\alpha(n) = \delta,$$

$$m = \text{the least } m \text{ s.t. } a_m^\delta = b_\beta \lceil \delta.$$

Let $\beta_1, \beta_2 < \omega_2$ with $\beta_1 \neq \beta_2$. Pick $\alpha^* < \omega_1$ so that $[\alpha_{\beta_1\beta_2}, \alpha^*] \cap (C_{\beta_1} \cap C_{\beta_2}) \neq \emptyset$, where if $\alpha' \geq \alpha_{\beta_1\beta_2}$, then $b_{\beta_1} \lceil \alpha' \neq b_{\beta_2} \lceil \alpha'$.

2.3.1 Claim. If $\alpha \geq \alpha^*$, then $g_{\beta_1}(\alpha) \neq g_{\beta_2}(\alpha)$.

Proof. Let $g_{\beta_1}(\alpha) = (n_1, m_1)$, $g_{\beta_2}(\alpha) = (n_2, m_2)$, $\delta_1 = e_\alpha(n_1)$ and $\delta_2 = e_\alpha(n_2)$.

Case 1. $n_1 \neq n_2$: Then $g_{\beta_1}(\alpha) \neq g_{\beta_2}(\alpha)$.

Case 2. $n_1 = n_2$: Then let $\delta = \delta_1 = \delta_2 \in C_{\beta_1} \cap C_{\beta_2}$. We have $b_{\beta_1} \lceil \delta = a_{m_1}^\delta$, $b_{\beta_2} \lceil \delta = a_{m_2}^\delta$ and $\delta \geq \alpha_{\beta_1\beta_2}$. Then $m_1 \neq m_2$ and so $g_{\beta_1}(\alpha) \neq g_{\beta_2}(\alpha)$.

□

We interpolated the following well-known.

2.4 Corollary. KH implies TH.

We provide a characterization of weak Kurepa trees along the line of □-wKH, where □ is either coint, club, stat, or cof.

2.5 Proposition. The following are equivalent.

(1) The wKH holds.

(2) There exist $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that

- Each b_β is a function from ω_1 into 2 and if $\beta_1 \neq \beta_2$, then $b_{\beta_1} \neq b_{\beta_2}$.
- Each S_α is countable and if $\sigma \in S_\alpha$, then $\sigma : \alpha \longrightarrow 2$.
- For all $\beta < \omega_2$, there exist $f_\beta : \omega_1 \longrightarrow \omega_1$ such that for all $\alpha < \omega_1$, we have $\alpha \leq f_\beta(\alpha)$ and $b_\beta \lceil \alpha \in S_{f_\beta(\alpha)} \lceil \alpha$.

Proof. (1) implies (2): Let T be a weak Kurepa tree. Let $\langle b_\beta \mid \beta < \omega_2 \rangle$ be a one-to-one enumeration of functions from ω_1 to 2 such that $b_\beta \lceil \alpha \in T_\alpha$ for all possible combinations of (α, β). Let $\langle \sigma_i \mid i < \omega_1 \rangle$ enumerate $\{b_\beta \lceil \alpha \mid \beta < \omega_2, \alpha < \omega_1\} \subseteq T$. For each $\alpha' < \omega_1$, let $S_{\alpha'} \subset {}^{\alpha'}2$ be countable so that for any $i \leq \alpha'$, if σ_i satisfies $|\sigma_i| \leq \alpha'$, then there exists $\tau \in S_{\alpha'}$ with $\sigma_i \subseteq \tau$. We claim these $\langle b_\beta \mid \beta < \omega_2 \rangle$ and $\langle S_{\alpha'} \mid \alpha' < \omega_1 \rangle$ work. To see this, let $\beta < \omega_2$ and $\alpha < \omega_1$. Let $\sigma_i = b_\beta \lceil \alpha$. Then take $\alpha' < \omega_1$ so large that $i, \alpha \leq \alpha'$. Since $i \leq \alpha'$ and $|\sigma_i| = \alpha \leq \alpha'$, we have $\tau \in S_{\alpha'}$ with $\sigma_i \subseteq \tau$ and so $b_\beta \lceil \alpha \in S_{\alpha'} \lceil \alpha$. Let $f_\beta(\alpha) = \alpha'$.

(2) implies (1): Let $T = \{b_\beta \lceil \alpha \mid \beta < \omega_2, \alpha < \omega_1\}$. Then for each $\beta < \omega_2$, $\{b_\beta \lceil \alpha \mid \alpha < \omega_1\}$ is a cofinal branch through T. For each $\alpha < \omega_1$, we have $T_\alpha \subseteq \bigcup \{S_{\alpha'} \lceil \alpha \mid \alpha \leq \alpha', \alpha' < \omega_1\}$ which is at most of size ω_1. Hence T is a weak Kurepa tree.

□

The following is also from the Set Theory Seminar, Nagoya university, and due to S. Fuchino and T. Sakai.

2.6 Note. The following are equivalent.

(1) The CH holds.

(2) There exists $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that $S_\alpha \subseteq {}^\alpha 2$, $|S_\alpha| \leq \omega$ and for all $b \in {}^{\omega_1}2$ and $\alpha < \omega_1$, there exist $\alpha' < \omega_1$ such that $\alpha \leq \alpha'$ and $b \lceil \alpha \in S_{\alpha'} \lceil \alpha$.

(3) Same as above with $|S_\alpha| = 1$.

Along the lines of guessing all subsets of ω_1, we have the three principles $\diamondsuit$, $\diamondsuit^*$ and $\diamondsuit^+$. Now we are tempted to consider the following $\diamondsuit$(coint).

2.7 Note. However, $\diamondsuit$(coint) is false, where $\diamondsuit$(coint) denotes that there exists $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ such that $S_\alpha \subseteq {}^\alpha 2$, $|S_\alpha| \leq \omega$ and for all $b \in {}^{\omega_1}2$, $\{\alpha < \omega_1 \mid b \lceil \alpha \in S_\alpha\}$ are coinitial in ω_1.

§3. Weak Diamonds

We formulate weak diamonds and investigate their impacts on the situation between wKH and KH.

3.1 Definition. Let □ denote either cof, stat, club or coint. We denote $\overline{\Phi}(\square)$, if for any $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ and any $\langle b_\beta \mid \beta < \omega_2 \rangle$ (no need to be one-to-one) such that each b_β is a member of ${}^{\omega_1}2$, there exists $g : \omega_1 \longrightarrow \omega_1$ such that for each $\beta < \omega_2$, we have either $\{\alpha < \omega_1 \mid F(b_\beta \lceil \alpha) < g(\alpha)\}$ is cofinal, stationary, contains a club, or is coinitial in ω_1, respectively.

So for example, $\overline{\Phi}$(stat) claims that given any coloring of the nodes of the tree ${}^{<\omega_1}2$ by countable ordinals, if we fix at most ω_2-many cofinal branches and concentrate on the nodes in $\{b_\beta \lceil \alpha \mid \beta < \omega_2, \alpha < \omega_1\}$, then there exists a uniform coloring $g : \omega_1 \longrightarrow \omega_1$ such that g correctly bounds each $\langle \alpha \mapsto F(b_\beta \lceil \alpha) \mid \alpha < \omega_1 \rangle$ stationary often.

We also formulate a stronger diamond along the line of $\overline{\Phi}(\square)$.

3.2 Definition. Let $\square$ denote either cof, stat, club or coint. We denote $\Phi(\square)$, if for any $F : {}^{<\omega_1}2 \longrightarrow \omega_1$, there exists $g : \omega_1 \longrightarrow \omega_1$ such that for any $b : \omega_1 \longrightarrow 2$, we have either $\{\alpha < \omega_1 \mid F(b_\beta \lceil \alpha) < g(\alpha)\}$ is cofinal, stationary, contains a club, or is coinitial in ω_1, respectively.

Therefore, given any coloring of ${}^{<\omega_1}2$ with countable ordinals, the principle $\Phi(\text{stat})$ provides a uniform coloring g which correctly bounds every possible cofinal branch's coloring as often as a stationary subset of ω_1.

3.3 Definition. We denote $(<^*)$, if for any $\langle f_\beta \mid \beta < \omega_2\rangle$ such that for each β, f_β is a function from ω_1 into ω_1, there exists $f : \omega_1 \longrightarrow \omega_1$ such that for every $\beta < \omega_2$, we have $f_\beta <^* f$. By this we mean that $\{\alpha < \omega_1 \mid f_\beta(\alpha) < f(\alpha)\}$ is coinitial in ω_1.

3.4 Proposition. Let $\square$ denote either cof, stat, club or coint.

(1) The wKH combined with $\overline{\Phi}(\square)$ implies $\square$-wKH.

(2) $(<^*)$ implies $\overline{\Phi}(\square)$.

Proof. For (1): Let T be a weak Kurepa tree. Then T has at least ω_2-many cofinal branches. So let $\langle b_\beta \mid \beta < \omega_2\rangle$ be a one-to-one enumeration such that for all $(\alpha, \beta) \in \omega_1 \times \omega_2$, $b_\beta \lceil \alpha \in T_\alpha$. Now let us fix $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ so that $F \lceil T$ is one-to-one. Then by $\overline{\Phi}(\square)$, get $g : \omega_1 \longrightarrow \omega_1$ such that for all $\beta < \omega_2$, we have $\{\alpha < \omega_1 \mid F(b_\beta \lceil \alpha) < g(\alpha)\}$ are $\square$ in ω_1. Define $\langle S_\alpha \mid \alpha < \omega_1\rangle$ by

$$S_\alpha = \{\sigma \in {}^{\alpha}2 \cap T \mid F(\sigma) < g(\alpha)\}.$$

Since $F\lceil T$ is one-to-one, S_α is countable. If $F(b_\beta\lceil\alpha) < g(\alpha)$, then $b_\beta\lceil\alpha \in S_\alpha$ holds. Hence these b_β and S_α work.

For (2): Let $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ and $\langle b_\beta \mid \beta < \omega_2\rangle$ be given. Define $\langle f_\beta \mid \beta < \omega_2\rangle$ by

$$f_\beta(\alpha) = F(b_\beta\lceil\alpha).$$

Then get $f : \omega_1 \longrightarrow \omega_1$ such that for all $\beta < \omega_2$,

$$\{\alpha < \omega_1 \mid f_\beta(\alpha) < f(\alpha)\}$$

are coinitial. Hence $\{\alpha < \omega_1 \mid F(b_\beta\lceil\alpha) < f(\alpha)\}$ is $\square$ in ω_1.

$\square$

The following is a rendition from [We].

3.5 Corollary. If CH, $2^{\omega_1} = \omega_3$ and GMA(σ-closed, $\aleph_1$-linked, well-met) hold, then KH holds.

Proof. Suppose CH, $2^{\omega_1} = \omega_3$ and GMA(σ-closed, $\aleph_1$-linked, well-met). Then we get ($<^*$). But CH implies wKH. Hence wHK and $\overline{\Phi}$(coint) hold. So coint-wKH holds. Namely, KH holds.

□

3.6 Proposition. Let □ denote either cof, stat, club or coint.

(1) $\Phi(\square)$ implies $\overline{\Phi}(\square)$.

(2) $\Phi(\mathrm{cof})$ implies $2^\omega < 2^{\omega_1}$.

(3) CH + Φ(stat) iff $\diamondsuit$.

(4) CH + Φ(club) iff $\diamondsuit^*$.

Proof. For (1): Fix $F : {}^{<\omega_1}2 \longrightarrow \omega_1$. Then $\Phi(\square)$ provides a uniform coloring $g : \omega_1 \longrightarrow \omega_1$ which works for all $b : \omega_1 \longrightarrow 2$. Hence g works for any prefixed $\langle b_\beta \mid \beta < \omega_2\rangle$ with each $b_\beta : \omega_1 \longrightarrow 2$.

For (2): We follow [MHD]. Suppose not and let $H : {}^\omega 2 \longrightarrow {}^{\omega_1}\omega_1$ be a bijection. Define $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ by

$$F(\sigma) = H(\sigma\lceil\omega)(|\sigma|), \text{ if } |\sigma| \geq \omega.$$

Then get $g : \omega_1 \longrightarrow \omega_1$ such that for all $b : \omega_1 \longrightarrow 2$, $\{\alpha < \omega_1 \mid F(b\lceil\alpha) < g(\alpha)\}$ are cofinal in ω_1.

Take $b \in {}^{\omega_1}2$ with $H(b\lceil\omega) = g$. Then for each $\alpha \geq \omega$, we have

$$F(b\lceil\alpha) = H(b\lceil\omega)(\alpha) = g(\alpha).$$

Hence $\{\alpha < \omega_1 \mid F(b\lceil\alpha) = g(\alpha)\}$ is cointial in ω_1. This is a contradiction.

For (3) and (4): We show (3), since (4) has a similar proof. Suppose CH and Φ(stat). Let $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ be a bijection via CH. Apply, Φ(stat). We have $g : \omega_1 \longrightarrow \omega_1$ such that for all $b \in {}^{\omega_1}2$, $\{\alpha < \omega_1 \mid F(b\lceil\alpha) < g(\alpha)\}$ are stationary in ω_1.

For each $\alpha < \omega_1$, let

$$S_\alpha = \{\sigma \in {}^\alpha 2 \mid F(\sigma) < g(\alpha)\}.$$

Then S_α is countable and for any $b \in {}^{\omega_1}2$, it holds that $\{\alpha < \omega_1 \mid b\lceil\alpha \in S_\alpha\}$ is stationary in ω_1. Hence $\diamondsuit$ holds.

Conversely, suppose $\diamondsuit$. We know CH holds. To show Φ(stat), let $\langle S_\alpha \mid \alpha < \omega_1\rangle$ be a diamond sequence such that for any $b \in {}^{\omega_1}2$, it holds that $\{\alpha < \omega_1 \mid b\lceil\alpha \in S_\alpha\}$ is stationary in ω_1.

Given $F : {}^{<\omega_1}2 \longrightarrow \omega_1$, let $g : \omega_1 \longrightarrow \omega_1$ be such that for all $\alpha < \omega_1$ and all $\sigma \in S_\alpha$, $F(\sigma) < g(\alpha)$. This is possible, as $|S_\alpha| \leq \omega$. Then for any $g : \omega_1 \longrightarrow 2$, it certainly holds that $\{\alpha < \omega_1 \mid F(b\lceil\alpha) < g(\alpha)\}$ is stationary in ω_1. Hence Φ(stat) holds.

□

It is known that $\diamondsuit$ negates the following CB.

3.7 Definition. The complete bounding (CB) holds, if for each $f \in {}^{\omega_1}\omega_1$ there exists $\gamma \in (\omega_1, \omega_2)$ and $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ such that X_α are continuously increasing countable subsets of γ with $\bigcup\{X_\alpha \mid \alpha < \omega_1\} = \gamma$ and for all $\alpha < \omega_1$, we have $f(\alpha) < \text{o.t.}(X_\alpha)$.

3.8 Proposition. $\overline{\Phi}$(stat) negates CB.

Proof. Define $F : {}^{<\omega_1}2 \longrightarrow \omega_1$ so that $F(\sigma) = \alpha$, if σ codes a countable ordinal α. And consider $\langle b_\gamma \mid \omega_1 < \gamma < \omega_2 \rangle$ such that $b_\gamma : \omega_1 \longrightarrow 2$ codes γ. We show the contrapositive.

Suppose CB. Fix any possible $g : \omega_1 \longrightarrow \omega_1$. Then we have γ and X_α with $g(\alpha) < \text{o.t.}(X_\alpha)$. Let $b = b_\gamma$. Take a sufficiently large regular cardinal θ and any countable elementary substructure N of H_θ with $b \in N$. Let $\delta = N \cap \omega_1$. Now we transitive collapse N. Then

$$b\lceil\delta \text{ codes o.t.}(N \cap \gamma).$$

Since $X_\delta = N \cap \gamma$, we have

$$F(b\lceil\delta) = \text{o.t.}(N \cap \gamma) = \text{o.t.}(X_\delta) > g(\delta).$$

Hence $\{\alpha < \omega_1 \mid F(b\lceil\alpha) \leq g(\alpha)\}$ is non-stationary.

□

3.9 Corollary. $\diamondsuit$ negates CB.

Proof. $\diamondsuit$ implies Φ(stat). And Φ(stat) implies $\overline{\Phi}$(stat).

□

We know that $\diamondsuit$ iff CH + ♣.

3.10 Question. (1) It is known, say by [W] and [F], that ♣ negates the saturation of the non-stationary ideal on ω_1. Is it ever holds that Con(♣ + CB) ?

(2) We know $\diamondsuit$(coint) iff CH + Φ(coint) but $\diamondsuit$(coint) is always false. Is it simply that Φ(coint) is false ?

§4. Not Club-wKH + Stat-wKH

We look at the standard model of set theory in which KH gets negated ([Si] and [K]).

4.1 Theorem. Let κ be a strongly inaccessible cardinal and Lv(κ, ω_1) denote the Levy collapse which turns κ into ω_2. Then $\neg$club-wKH holds in the generic extensions $V[\text{Lv}(\kappa, \omega_1)]$.

Since $\diamondsuit$ holds in $V[\text{Lv}(\kappa, \omega_1)]$, we have

4.2 Corollary. The following are all equiconsistent.

(1) Con(There exists a strongly inaccessible cardinal).
(2) Con($\neg$club-wKH + $\diamondsuit$).
(3) Con($\neg$club-wKH + $\tilde{\diamondsuit}$).
(4) Con($\neg$club-wKH + stat-wKH).
(5) Con($\neg$KH).

Proof of theorem. We repeat the standard proof, due to Silver, for showing $\neg$KH. Then we notice that it actually shows $\neg$club-wKH.

Here are some details. We first provide

4.2.1 Claim. Let $S_\alpha \subset {}^{\alpha}2$ be countable for all $\alpha < \omega_1$. Let $\dot{b}$ and $\dot{C}$ be $\mathrm{Lv}(\kappa, \omega_1)$-names. Then $\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "if $\dot{C}$ is a club in ω_1 and $\dot{b} : \omega_1 \longrightarrow 2$ such that $\dot{b}\lceil\alpha \in S_\alpha$ for all $\alpha \in \dot{C}$, then $\dot{b} \in V$" holds.

Proof. By contradiction. Suppose $p \Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\dot{C}$ is a club in ω_1 and $\dot{b} : \omega_1 \longrightarrow 2$ such that $\dot{b}\lceil\alpha \in S_\alpha$ for all $\alpha \in \dot{C}$" and $p \Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\dot{b} \notin V$". We derive a contradiction.

To this end, let N be a countable elementary substructure of H_{κ^+} with $p, \kappa, \dot{b}, \dot{C} \in N$. Denote $\delta = N \cap \omega_1$.

Construct $\langle (p_s, b_s) \mid s \in {}^{<\omega}2\rangle$ by recursion on $|s|$ such that for each $s \in {}^{<\omega}2$,

- $p_\emptyset = p$ and $b_\emptyset = \emptyset$.
- $p_s \in \mathrm{Lv}(\kappa, \omega_1) \cap N$ and $b_s \in S_{|b_s|} \cup \{\emptyset\}$.
- $p_s \Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ " $|b_s| \in \dot{C} \cup \{0\}$ and $b_s \subset \dot{b}$".
- $p_{s^\frown\langle i\rangle} \leq p_s$, $b_{s^\frown\langle i\rangle} \supset b_s$ for $i = 0, 1$ and $b_{s^\frown\langle 0\rangle}$, $b_{s^\frown\langle 1\rangle}$ are incomparable. I.e, $b_{s^\frown\langle 0\rangle} \not\subseteq b_{s^\frown\langle 1\rangle}$ and $b_{s^\frown\langle 1\rangle} \not\subseteq b_{s^\frown\langle 0\rangle}$.
- $\langle p_{f\lceil n} \mid n < \omega\rangle$ is a $(\mathrm{Lv}(\kappa, \omega_1), N)$-generic sequence for all $f \in {}^{\omega}2$.

Let $p_f = \bigcup\{p_{f\lceil n} \mid n < \omega\}$ and $b_f = \bigcup\{b_{f\lceil n} \mid n < \omega\}$ for each $f \in {}^{\omega}2$. Then $p_f \Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\delta = N[\dot{G}] \cap \omega_1 \in \dot{C}$ and $\dot{b}\lceil\delta = b_f : \delta \longrightarrow 2$" for all $f \in {}^{\omega}2$, where $\dot{G}$ denotes the canonical $\mathrm{Lv}(\kappa, \omega_1)$-name of the generic filters. Hence $p_f \Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\dot{b}\lceil\delta \in S_\delta$" and so $\{b_f \mid f \in {}^{\omega}2\} \subset S_\delta$. Since $|\{b_f \mid f \in {}^{\omega}2\}| = 2^\omega$ and S_δ is countable, this is a contradiction.

□

Now back to the proof of theorem, we proceed by contradiction. Suppose $\langle b_\beta \mid \beta < \kappa\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ satisfy club-wKH in $V[\mathrm{Lv}(\kappa, \omega_1)]$. Since $\mathrm{Lv}(\kappa, \omega_1)$ has the κ-c.c, we may assume $\langle S_\alpha \mid \alpha < \omega_1\rangle \in V$. Then by claim, we know that $b_\beta \in V$ for all $\beta < \kappa$. Hence $2^{\omega_1} \geq \kappa$. But κ is a strongly inacceccible cardinal. This is a contradiction.

□

The following is a later half of the exercise (J6) on p.300 in [K] .

4.3 Corollary. $\neg\diamondsuit^*$ holds in $V[\mathrm{Lv}(\kappa, \omega_1)]$.

Proof. $\diamondsuit^*$ iff CH + Φ(club). It in turn implies wKH + $\overline{\Phi}$(club). And so $\diamondsuit^*$ implies club-wKH.

□

§5. Not KH + Club-wKH

5.1 Theorem. Con(There exists a strongly inaccessible cardinal) implies Con($\neg$KH + club-wKH).

Proof. We first out-line. Then provide some details.

(Out-line) Let κ be a strongly inaccessible cardinal in the ground model V. We first Levy collapse κ over V so that κ becomes new ω_2, while ω_1 remains the same. In this generic extension $V[\mathrm{Lv}(\kappa,\omega_1)]$, we have $\neg$KH due to Silver. We prepare some $\langle b_\beta \mid \beta < \kappa\rangle$ and $\langle S_\alpha \mid \alpha < \omega_1\rangle$ in $V[\mathrm{Lv}(\kappa,\omega_1)]$ such that

- $b_\beta \in {}^{\omega_1}2$ for all $\beta < \kappa$,
- $S_\alpha \subset {}^{\alpha}2$ and S_α are countable for all $\alpha < \omega_1$,
- If we denote $E_\beta = \{\alpha < \omega_1 \mid b_\beta \lceil \alpha \in S_\alpha\}$ and $E = \{X \in [\kappa]^\omega \mid \forall \beta \in X\ X \cap \omega_1 \in E_\beta\}$, then the E_β are stationary in ω_1 and so is E in $[\kappa]^\omega$.

We next side-by-side force over $V[\mathrm{Lv}(\kappa,\omega_1)]$ so that clubs C_β are added with $C_\beta \subset E_\beta$ for all $\beta < \kappa$. Let us denote this notion of forcing by $R \in V[\mathrm{Lv}(\kappa,\omega_1)]$. We show that R has the κ-c.c. and is E-complete in the sense of [S] whose meaning explained later. In particular, R is σ-Baire and so preserves both ω_1 and ω_2. Hence club-wKH holds in the final extension $V[\mathrm{Lv}(\kappa,\omega_1)][R]$.

We claim $\neg$KH is preserved into $V[\mathrm{Lv}(\kappa,\omega_1)][R]$. To this end, fix any possible Kurepa tree T in $V[\mathrm{Lv}(\kappa,\omega_1)][R]$. We clarify the following among others.

- We factor $V[\mathrm{Lv}(\kappa,\omega_1)][R]$ into

$$V[\mathrm{Lv}(\kappa,\omega_1)][R(\beta^*)][R([\beta^*,\kappa))]$$

so that $T \in V[\mathrm{Lv}(\kappa,\omega_1)][R(\beta^*)]$ for some $\beta^* < \kappa$.

According to [J-S],

- $\neg$KH gets preserved over $V[\mathrm{Lv}(\kappa,\omega_1)]$ by any notion of forcing which is σ-Baire and of size at most ω_1.

Hence T has at most ω_1-many cofinal branches in the intermidiate $V[\mathrm{Lv}(\kappa,\omega_1)][R(\beta^*)]$.

- We show no new cofinal branches are added through T over $V[\mathrm{Lv}(\kappa,\omega_1)][R(\beta^*)]$.

To this, we observe the quotient $R([\beta^*,\kappa))$ is E-complete in $V[\mathrm{Lv}(\kappa,\omega_1)][R(\beta^*)]$. We then modify Silver's construction for σ-closed notion of forcing to obverve the last item.

Therefore T fails to be a Kurepa tree in $V[\mathrm{Lv}(\kappa,\omega_1)][R]$.

Some details follow.

(Step 1) Let κ be a strongly inaccessible cardinal. We force with the Levy collapse $\mathrm{Lv}(\kappa,\omega_1)$ over the ground model V. To save symbols, let us write $V[\mathrm{Lv}(\kappa,\omega_1)]$ for the generic extensions.

Argue in $V[\mathrm{Lv}(\kappa,\omega_1)]$. For each $(1<)\,\beta<\kappa$, Let us write $g_\beta:\omega_1\longrightarrow\beta$ for the β-th generic function added via $\mathrm{Lv}(\kappa,\omega_1)$.

We prepare $\langle b_\beta \mid \beta<\kappa\rangle$ and $\langle S_\alpha \mid \alpha<\omega_1\rangle$. To define $b_\beta:\omega_1\longrightarrow 2$, we make use of $g_{\omega_1+\beta}$. To define S_α, say, for limit α, we make use of $g_i\lceil\omega$ $(\alpha\le i<\alpha+\alpha)$. More precisely,

$$b_\beta(\alpha)=1 \text{ iff } g_{\omega_1+\beta}(\alpha) \text{ is odd.}$$

$$S_\alpha=\{\sigma^\alpha_n \mid n<\omega\},\ \sigma^\alpha_n:\alpha\longrightarrow 2.$$

$$\sigma^\alpha_n(i)=1 \text{ iff } g_{\alpha+i}(n) \text{ is odd.}$$

We know how to construct conditions via generic sequences with respect $\mathrm{Lv}(\kappa,\omega_1)$ upon fixing countable elementary substructures. In such constructions, we know which parts of what g_β are decided and what g_β are left open. Hence it is not hard to show that $E=\{X\in[\kappa]^\omega \mid \forall\beta\in X\ X\cap\omega_1\in E_\beta\}$ is stationary in $[\kappa]^\omega$. It then follows that each $E_\beta=\{\alpha<\omega_1 \mid b_\beta\lceil\alpha\in S_\alpha\}$ must be stationary in ω_1.

For an explicit proof, we show E is stationary in $[\kappa]^\omega$. Suppose $p\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\dot\varphi: {}^{<\omega}\kappa\longrightarrow\kappa$". We want to find $q^*\le p$ and $X\in[\kappa]^\omega$ such that $q^*\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$X\in\dot E$ and X is $\dot\varphi$-closed", where $\dot E$ denotes the canonical name of E. To this end let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $p,\dot\varphi\in N$. Let $\delta=N\cap\omega_1$ and $X=N\cap\kappa$. Take a $(\mathrm{Lv}(\kappa,\omega_1),N)$-generic sequence $\langle p_n \mid n<\omega\rangle$ with $p_0=p$. Let $q=\bigcup\{p_n \mid n<\omega\}$. Then $q\in\mathrm{Lv}(\kappa,\omega_1)$ is $(\mathrm{Lv}(\kappa,\omega_1),N)$-generic and $\mathrm{dom}(q)=N\cap(\kappa\times\omega_1)=X\times\delta$. Hence q decides $g_{\omega_1+\beta}\lceil\delta$ for all $\beta\in X$ and $q\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$X=N[\dot G]\cap\kappa$ is $\dot\varphi$-closed".

We may place the countable set $\{g_{\omega_1+\beta}\lceil\delta \mid \beta\in X\}$ on $[\delta,\delta+\delta)\times\omega$. Namely, we may extend q to q^* so that $q^*\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$\dot b_\beta\lceil\delta\in\dot S_\delta$ for all $\beta\in X$". Hence $q^*\Vdash_{\mathrm{Lv}(\kappa,\omega_1)}$ "$X\in\dot E$".

(Step 2) We side-by-side force clubs for all E_β over $V[\mathrm{Lv}(\kappa,\omega_1)]$. Let $X\subseteq\kappa$. Define $p\in R(X)$, if $p=\langle C^p_\beta \mid \beta\in X^p\rangle$ such that

- $X^p\in[X]^{\le\omega}$,
- C^p_β is a countable closed subset of E_β for all $\beta\in X^p$.

For $p,q\in R(X)$, set $q\le_{R(X)}p$, if

- $X^q\supseteq X^p$,
- C^q_β end-extends C^p_β at each $\beta\in X^p$.

Notice that we do not require $\max C^p_{\beta_1}=\max C^p_{\beta_2}$ for $\beta_1,\beta_2\in X^p$.

5.1.1 Lemma. (1) $R(X)$ has the ω_2-c.c.

(2) $R(X)$ is E-complete. I.e, for all sufficiently large regular cardinals θ and all countable elementary substructures N of H_θ such that $R(X) \in N$ and $N \cap \kappa \in E$, if $\langle r_n \mid n < \omega \rangle$ is a $(R(X), N)$-generic sequence, then there exists $r \in R(X)$ such that for all $n < \omega$, $r \leq_{R(X)} r_n$.

Proof. For (1): In $V[\mathrm{Lv}(\kappa, \omega_1)]$, we have $\diamondsuit$ and so CH holds. By a standard Δ-system lemma, we may conclude $R(X)$ has the ω_2-c.c.

For (2): Let us fix any regular cardinal θ with $\theta > \kappa$. Let N be any countable elementary substructure of H_θ such that $R(X) \in N$ and $N \cap \kappa \in E$. Hence we have

$$\forall \beta \in N \cap \kappa \; N \cap \omega_1 \in E_\beta.$$

Let $\langle r_n \mid n < \omega \rangle$ be any $(R(X), N)$-generic sequence. Then by genericity, we have $N \cap X = \bigcup \{X^{r_n} \mid n < \omega\}$. For each $\beta \in N \cap X$, let $C_\beta = \bigcup \{C_\beta^{r_n} \mid \beta \in X^{r_n}, \; n < \omega\} \cup \{N \cap \omega_1\}$ and $r = \langle C_\beta \mid \beta \in N \cap X \rangle$. Then $C_\beta \subset E_\beta$ are clubs. Hence $r \in R(X)$ such that for all $n < \omega$, we have $r \leq r_n$.

□

Let $R = R(\kappa)$. Since R adds clubs C_β with $C_\beta \subset E_\beta$ for all $\beta < \kappa$, we have club-wKH in the extensions $\mathrm{V}[\mathrm{Lv}(\kappa, \omega_1)][R]$.

(Step 3) We want to show $V[\mathrm{Lv}(\kappa, \omega_1)][R] \models$ "$\neg$ KH". To this end let T be a possible Kurepa tree in $V[\mathrm{Lv}(\kappa, \omega_1)][R]$. Then by the κ-c.c. of R, we have $\beta^* < \kappa$ such that $T \in V[\mathrm{Lv}(\kappa, \omega_1)][R(\beta^*)]$. Let $V_1 = V[\mathrm{Lv}(\kappa, \omega_1)]$ for short. Then

- R and $R(\beta^*) \times R([\beta^*, \kappa))$ are isomorphic in V_1.
- $V_1 \models$ "$R(\beta^*)$ is E-complete and so σ-Baire".

  Hence,

- $V_1[R(\beta^*)] \models$ "E remains stationary in $[\kappa]^\omega$".

  Since $R(\beta^*)$ is σ-Baire and so by absoluteness,

- $V_1[R(\beta^*)] \models$ "$R([\beta^*, \kappa))$ is E-complete".

  Since $R(\beta^*)$ is of size ω_1 in V_1, we have $\overline{\kappa} < \kappa$ such that

- $R(\beta^*) \in V[\mathrm{Lv}(\overline{\kappa}, \omega_1)]$.

Since $R(\beta^*)$ is σ-Baire in $V[\mathrm{Lv}(\overline{\kappa}, \omega_1)] \subset V[\mathrm{Lv}(\kappa, \omega_1)]$, the p.o. set $\mathrm{Lv}([\overline{\kappa}, \kappa), \omega_1))$ has the same meaning in both $V[\mathrm{Lv}(\overline{\kappa}, \omega_1)]$ and $V[\mathrm{Lv}(\overline{\kappa}, \omega_1)][R(\beta^*)]$. Now we apply the Product Lemma in $V[\mathrm{Lv}(\overline{\kappa}, \omega_1)]$ so that

- We have
$$V_1[R(\beta^*)] = V[\mathrm{Lv}(\overline{\kappa}, \omega_1)][R(\beta^*)][\mathrm{Lv}([\overline{\kappa}, \kappa), \omega_1)]$$
  and so $V_1[R(\beta^*)] \models$ "$\neg$KH" holds.

Therefore T has at most ω_1-many cofinal branches in $V_1[R(\beta^*)]$. We know

$$V_1[R] = V_1[R(\beta^*)][R([\beta^*, \kappa))]$$

and $R([\beta^*, \kappa))$ is E-complete in $V_1[R(\beta^*)]$. Hence it suffices to show the following.

5.1.2 Lemma. Let P be a p.o. set which is E-complete for some stationary $E \subset [\kappa]^\omega$ and T be a tree of height ω_1 whose levels are all of size countable. Then T gets now new cofinal branches in the generic extensions $V[P]$.

Proof. Suppose $p \Vdash_P$ "$\dot{b}$ is a cofinal branch through T with $\dot{b} \notin V$". We derive a contradiction. To this end, let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $p, P, T, \dot{b} \in N$ and $N \cap \kappa \in E$. This is possible, as E is stationary. Denote $\delta = N \cap \omega_1$.

Construct $\langle (p_s, b_s) \mid s \in {}^{<\omega}2 \rangle$ by recursion on $|s|$ such that for each $s \in {}^{<\omega}2$,

- $p_\emptyset = p$ and we may assume $\{b_\emptyset\} = T_0$.
- $p_s \in P \cap N$ and $b_s \in T \cap N$.
- $p_s \Vdash_P$ "$b_s \in \dot{b}$".
- $p_{s^\frown\langle i \rangle} \leq p_s$, $b_s <_T b_{s^\frown\langle i \rangle}$ for $i = 0, 1$ and $b_{s^\frown\langle 0 \rangle}$, $b_{s^\frown\langle 1 \rangle}$ are incomparable. I.e, $b_{s^\frown\langle 0 \rangle} \not<_T b_{s^\frown\langle 1 \rangle}$ and $b_{s^\frown\langle 1 \rangle} \not<_T b_{s^\frown\langle 0 \rangle}$.
- $\langle p_{f\lceil n} \mid n < \omega \rangle$ is a (P, N)-generic sequence for all $f \in {}^\omega 2$.

Since P is E-complete, we may fix $p_f \in P$ such that $p_f \leq_P p_{f\lceil n}$ for all $n < \omega$. We may assume, by extending p_f further, there exists $b_f \in T_\delta$ such that $p_f \Vdash_P$ "$b_f \in \dot{b}$". Since $|\{b_f \mid f \in {}^\omega 2\}| = 2^\omega$ and T_δ is countable, this is a contradiction.

□

§6. ♣ and Φ(stat) are different

We separate Φ(stat) and ♣.

6.1 Theorem. $\mathrm{Con}(\mathrm{MA}_{\omega_1}(\mathrm{Fn}(\omega_1, 2)) + \Phi(\mathrm{stat}))$.

6.2 Corollary. $\mathrm{Con}(\neg ♣ + \Phi(\mathrm{stat}))$.

Proof. $\mathrm{MA}_{\omega_1}(\mathrm{Fn}(\omega_1, 2))$ implies $\neg$♣.

□

Proof of theorem. We first out-line. Then provide some details.

(Out-line) Since Φ(stat) entails Φ(cof), we must have $2^\omega < 2^{\omega_1}$. Suppose CH and $2^{\omega_1} = \omega_2$. Add ω_3-many functions from ω_1 into ω_1. Then we have

- $\mathrm{CH} + 2^{\omega_1} = \omega_3$.

- $\forall F : \underline{{}^{<\omega_1}\omega_2} \longrightarrow \omega_1 \ \exists g : \omega_1 \longrightarrow \omega_1 \ \forall b \in \underline{{}^{\omega_1}\omega_2} \ \{\alpha < \omega_1 \mid \underline{F(b\lceil\alpha) = g(\alpha)}\}$ is stationary.

Next, we add ω_2-many subsets of ω . Since we can capture relevant names, we have

- $2^\omega = \omega_2 + \mathrm{MA}_{\omega_1}(\mathrm{Fn}(\omega_1, 2)) + 2^{\omega_1} = \omega_3$.
- $\forall F : \underline{{}^{<\omega_1}2} \longrightarrow \omega_1 \ \exists g : \omega_1 \longrightarrow \omega_1 \ \forall b \in \underline{{}^{\omega_1}2} \ \{\alpha < \omega_1 \mid \underline{F(b\lceil\alpha) < g(\alpha)}\}$ is stationary.

Here are some details.

(Step 1) Let $P = \mathrm{Fn}(\omega_3 \times \omega_1, \omega_1, \omega_1)$. Then P is σ-closed. By CH, P has the ω_2-c.c.

Let $\langle g_\xi \mid \xi < \omega_3\rangle$ denote the canonical objects added by P. In particular, $g_\xi : \omega_1 \longrightarrow \omega_1$. By counting the number of P-names, we have

$$V[\langle g_\xi \mid \xi < \omega_3\rangle] \models \text{“}\mathrm{CH} + 2^{\omega_1} = \omega_3\text{”}.$$

Let $F : {}^{<\omega_1}\omega_2 \longrightarrow \omega_1$ in $V[\langle g_\xi \mid \xi < \omega_3\rangle]$. Since P has the ω_2-c.c, we have $\xi^* < \omega_3$ such that $F \in V[\langle g_\xi \mid \xi < \xi^*\rangle]$. Notice

$$V[\langle g_\xi \mid \xi < \omega_3\rangle] = V[\langle g_\xi \mid \xi < \xi^*\rangle][g_{\xi^*}][\langle g_\xi \mid \xi^* < \xi < \omega_3\rangle].$$

Let $V_1 = V[\langle g_\xi \mid \xi < \xi^*\rangle]$ and $Q = \mathrm{Fn}([\xi^*, \omega_3) \times \omega_1, \omega_1, \omega_1)$. Then the following suffices.

6.2.1 Claim. $\Vdash_Q^{V_1}$ “$\forall \dot{b} : \omega_1 \longrightarrow \omega_2 \ \{\alpha < \omega_1 \mid F(\dot{b}\lceil\alpha) = \dot{g}_{\xi^*}(\alpha)\}$ is stationary.”

Proof. Argue in V_1. Suppose $r \Vdash_Q^{V_1}$ “$\dot{b} : \omega_1 \longrightarrow \omega_2$ and $\dot{C} \subseteq \omega_1$ is a club”. Let θ be a sufficiently large regular cardinal and N be a countable elementary substructure of H_θ with $r, Q, \dot{b}, \dot{C} \in N$. Let $\langle r_n \mid n < \omega\rangle$ be a (Q, N)-generic sequence with $r_0 = r$. Let $r' = \bigcup\{r_n \mid n < \omega\}$ and $\delta = N \cap \omega_1$. Then there is $\sigma \in {}^\delta\omega_2$ such that $r' \Vdash_Q^{V_1}$ “$\dot{b}\lceil\delta = \sigma$”. Let $r^* = r' \cup \{((\xi^*, \delta), F(\sigma))\}$. Then $r^* \leq r'$ and $r^* \Vdash_Q^{V_1}$ “$F(\dot{b}\lceil\delta) = \dot{g}_{\xi^*}(\delta)$ and $\delta \in \dot{C}$”. □

(Step 2) For notational simplicity, suppose the following in V.

- $\mathrm{CH} + 2^{\omega_1} = \omega_3$.
- $\forall F : {}^{<\omega_1}\omega_2 \longrightarrow \omega_1 \ \exists g : \omega_1 \longrightarrow \omega_1 \ \forall b \in {}^{\omega_1}\omega_2 \ \{\alpha < \omega_1 \mid F(b\lceil\alpha) = g(\alpha)\}$ is stationary.

We force with $Q = \mathrm{Fn}(\omega_2 \times \omega, 2)$ over V. Then in $V[Q]$,

6.2.2 Claim. $\forall F : {}^{<\omega_1}2 \longrightarrow \omega_1 \ \exists g : \omega_1 \longrightarrow \omega_1 \ \forall b \in {}^{\omega_1}2 \ \{\alpha < \omega_1 \mid F(b\lceil\alpha) < g(\alpha)\}$ is stationary.

Proof. Let $\Vdash_Q$ “$\dot{F} : {}^{<\omega_1}2 \longrightarrow \omega_1$”. Let $\mathcal{A} = \{A \subset Q \mid A$ is an antichain of $Q\}$. Then $|\mathcal{A}| = \omega_2$. Define $F_0 : {}^{<\omega_1}\mathcal{A} \longrightarrow \omega_1$ so that for any $\sigma \in {}^\alpha\mathcal{A}$, we have $\Vdash_Q$“$\dot{F}(\dot{s}(\sigma)) <$

$F_0(\sigma)$", where $s(\sigma)$ is a member of ${}^{\alpha}2$ naturally defined from σ in $V[Q]$. This is possible, as Q has the c.c.c.

Now by assumption, we have $g_0 : \omega_1 \longrightarrow \omega_1$ such that

$$\forall b \in {}^{\omega_1}\mathcal{A}\ \{\alpha < \omega_1 \mid F_0(b\lceil\alpha) = g_0(\alpha)\} \text{ is stationary.}$$

6.2.2.1 Sub claim. $\Vdash_Q$"$\forall \dot{b} \in {}^{\omega_1}2\ \{\alpha < \omega_1 \mid \dot{F}(\dot{b}\lceil\alpha) < g_0(\alpha)\}$ is stationary".

Proof. By the Maximal Principle of the Q-names, we may take $b : \omega_1 \longrightarrow \mathcal{A}$ such that for all $\alpha < \omega_1$, $\Vdash_Q$"$\dot{b}\lceil\alpha = s(b\lceil\alpha)$". By the choice of g_0, we have

$$\{\alpha < \omega_1 \mid F_0(b\lceil\alpha) = g_0(\alpha)\} \text{ is stationary.}$$

Notice $F_0(b\lceil\alpha) = g_0(\alpha)$ implies $\Vdash_Q$"$\dot{F}(\dot{b}\lceil\alpha) = \dot{F}(s(b\lceil\alpha)) < F_0(b\lceil\alpha) = g_0(\alpha)$". Since the stationary subsets of ω_1 remain stationary in $V[Q]$, we conclude

$$\{\alpha < \omega_1 \mid \dot{F}(\dot{b}\lceil\alpha) < g_0(\alpha)\} \text{ is stationary.}$$

□

6.2.3 Claim. $\mathrm{MA}_{\omega_1}(\mathrm{Fn}(\omega_1, 2))$ holds in $V[Q]$.

Proof. Given $\mathcal{D} = \langle D_i \mid i < \omega_1\rangle$ dense subsets of $\mathrm{Fn}(\omega_1, 2)$, there exists $\beta < \omega_2$ such that $\mathcal{D} \in V[Q\lceil\beta]$. Hence the next ω_1-many coordinates provide a $\mathcal{D}$-generic filter.

□

We may separate ♣ and Φ(stat) the other way round, too.

6.3 Theorem. Con(♣ + $\neg\Phi$(stat)).

Proof. Since $2^{\omega} = 2^{\omega_1}$ negates Φ(stat), we look for this property. We consider a model in [S], where Con(♣ + $\neg$CH) is shown.

Let $2^{\omega} = \omega_1$, $2^{\omega_1} = \omega_2$, $2^{\omega_2} = \omega_3$ and $\diamondsuit(S^2_0)$ in V. First add ω_3-many new subsets of ω_1. Then collpase ω_1 to countable. Let

$$V^* = V[\mathrm{Fn}(\omega_3, 2, \omega_1)][\mathrm{Fn}(\omega, \omega_1)].$$

Then we have $2^{\omega} = 2^{\omega_1} = \omega_2$ and ♣ in V^*.

We record:

- $V[\mathrm{Fn}(\omega_3, 2, \omega_1)] \models$ "$2^{\omega} = \omega_1 + 2^{\omega_1} = 2^{\omega_2} = \omega_3 + ♣(S^2_0)$".
- $V^* \models$ "$2^{\omega} = 2^{\dot{\omega}_1} = \dot{\omega}_2 + ♣$".

□

§7. A summary of implications, the chart

(A)

$$\begin{array}{ccccccccccccc}
\Diamond^+ & & \Diamond^* & & \Diamond & & & & & & & & \text{CH}\\
\Downarrow & & \Downarrow & & \Downarrow & & & & & & & & \Downarrow\\
\text{coint-wKH} & \Rightarrow & \text{club-wKH} & \Rightarrow & \tilde{\Diamond} & \Rightarrow & \tilde{\tilde{\Diamond}} & \Rightarrow & \text{stat-wKH} & \Rightarrow & \text{cof-wKH} & \Rightarrow & \text{wKH}\\
\Updownarrow & & \Downarrow & & & & & & \Downarrow & & & & \\
\text{KH} & & \text{TH} & & & & & & \neg\text{SAT} & & & &
\end{array}$$

(B)

$$\begin{array}{ccccc}
\Phi(\text{club}) & \Rightarrow & \Phi(\text{stat}) & \Rightarrow & \Phi(\text{cof})\\
\Downarrow & & \Downarrow & & \Downarrow\\
\overline{\Phi}(\text{club}) & & \overline{\Phi}(\text{stat}) & & 2^{\omega} < 2^{\omega_1} + \overline{\Phi}(\text{cof})
\end{array}$$

(C)

$$\begin{array}{ccccccccc}
(<^*) & \Rightarrow & \overline{\Phi}(\text{coint}) & \Rightarrow & \overline{\Phi}(\text{club}) & \Rightarrow & \overline{\Phi}(\text{stat}) & \Rightarrow & \overline{\Phi}(\text{cof})\\
 & & & & & & \Downarrow & & \\
 & & & & & & \neg\,\text{CB} & &
\end{array}$$

(D)

$$\begin{array}{ccccccc}
\text{False} & & \Diamond^* & & \Diamond & & \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \\
\text{CH} + \Phi(\text{coint}) & \Rightarrow & \text{CH} + \Phi(\text{club}) & \Rightarrow & \text{CH} + \Phi(\text{stat}) & \Leftrightarrow & \text{CH} + \Phi(\exists\,\alpha \geq \omega)\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
\text{wKH} + \overline{\Phi}(\text{coint}) & \Rightarrow & \text{wKH} + \overline{\Phi}(\text{club}) & \Rightarrow & \text{wKH} + \overline{\Phi}(\text{stat}) & \Rightarrow & \text{wKH} + \overline{\Phi}(\text{cof})\\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow\\
\text{coint-wKH} & & \text{club-wKH} & & \text{stat-wKH} & & \text{cof-wKH}
\end{array}$$

(E)

$$\text{CH} + 2^{\omega_1} = \omega_3 + \text{GMA}_{\omega_2} \Rightarrow \text{CH} + (<^*) \Rightarrow \text{wKH} + \overline{\Phi}(\text{coint})$$

7.1 Note. ([W]) Con(NS_{ω_1} is ω_1-dense and wKH).

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