数理解析研究所講究録 1425 巻 2005 年 113-121



# Mode generating effect of the solutions to nonlinear Schrödinger equations

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**Abstract**

We consider the initial value problem of the nonlinear Schrödinger equation with superposed $\delta$-functions as initial data. The speaker will treat this problem case by case, i.e., the cases in which the initial data consists of single and double $\delta$-functions, respectively. In particular, when the initial data consists of double $\delta$-functions, the solution receives the generation of new modes which is visible only in the nonlinear problem (see section 3).

## 1 Introduction

In this proceeding, we present several results on the initial value problem of the nonlinear Schrödinger equation like

$$\text{(NLS)} \qquad \begin{cases} i\partial_t u = -\partial_x^2 u + \lambda \mathcal{N}(u), \\ u(0,x) = (\text{superposition of } \delta\text{-functions}), \end{cases}$$

where $(t,x) \in \mathbf{R} \times \mathbf{R}$ and the unknown function $u = u(t,x)$ takes complex values. The nonlinearity $\mathcal{N}(u)$ is given by

$$\mathcal{N}(u) = |u|^{p-1}u \quad \text{with } 1 < p < 3.$$

The nonlinear coefficient $\lambda$ takes arbitrary complex number. The functional $\delta_a$ denotes the well-known point mass measure supported at $x = a \in \mathbf{R}$.

From the physical point of view, the cubic nonlinearity (i.e. $p = 3$ which is excluded in our assumption for mathematical reason) frequently appears. For example, (NLS) with $\lambda \in \mathbf{R}$ and $p = 3$ is said to govern the motion of vortex filament in the ideal fluid. In fact, letting $\kappa(t,x)$ be the curvature of the filament and $\tau(t,x)$ the tortion, we observe that $u(t,x) = \kappa(t,x)\exp(i\int_0^x \tau(t,y)\, dy)$ (which is called "Hasimoto transform" [3]) satisfies (NLS), where $x$ stands for the position parameter along the filament.

To our regret, our argument does not contain the cubic nonlinearity. However, if one allows us to treat the solution as a fine approximation of the physically important case, we can imagine the time evolution of vortex filament with the locally bended initial state (which is described as $\kappa(0,x) = \delta_a$).

The nonlinear evolution equations with measures as initial data are extensively sutudied for various kinds of initial value problem. As for the nonlinear parabolic equations like $\partial_t u - \partial_x^2 u + |u|^{p-1}u = 0$ with $u(0,x) = \delta_0$, Brezis-Friedman [2] give the critical power of nonlinearity concerning the solvability and unsolvability of the equation. They prove that, if $3 \leq p$, there exists no solution continuous at $t = 0$ in the distribution sense and that, if $1 < p < 3$, it is posibble to construct a solution with a general measure as the initial data. For the KdV equation, Tsutsumi [5] constructs a solution by making use of Miura transformation which deforms the original KdV equation into the modified one. Recently, Abe-Okazawa [1] have studied this kind of problem for the complex Ginzburg-Landau equation. The ideas of the proof for these known results are based on the strong smoothing effect of linear part or the nonlinear transformation of unknown functions into the suitably handled equation. In the present case, however, the nonlinear Schrödinger equation does not have the useful smoothing properties and the transformation into easily handeled equation. Therefore, it is still open whether we can construct a solution when the initial data is arbitrary measure.

We remark that Kenig-Ponce-Vega [4] studied the ill-posedness aspect of the nonlinear Schrödinger equation with $u(0,x) = \delta_0$ and $3 \leq p$. The situation is very similar to the nonlinear heat case introduced above. They proved that (NLS) possesses either no solution or more than one in $C([0,T];\mathcal{S}'(\mathbf{R}))$, where $\mathcal{S}'(\mathbf{R})$ denotes the tempered distribution. In this talk, we consider the construction of the solution to (NLS) for the subcritical nonlinearity. We prove that the solution is explicitly obtained when the initial data consists of single $\delta$-function (see section 2). Furthermore, we observe that, when the initial data consists of double (or more) $\delta$-functions, the superposition of infinitely many linear solutions immediately appers (see section 3). This aspect is called "the generalization of new modes". Throughout this note, the Lebesgue space $L^q_\theta$ denotes

$$L^q_\theta = \{f(\theta);\ \|f\|^q_{L^q_\theta} = \int_0^{2\pi} |f(\theta)|^q\, d\theta < \infty\}.$$

Let us state our main theorems case by case.

## 2 The case $u(0,x) = \mu_0\delta_0$

This case simply gives an explicit solution. Namely, the solution to (NLS) is given by

$$u(t,x) = A(t)\exp(it\partial_x^2)\delta_0, \tag{2.1}$$

where $\exp(it\partial_x^2)\delta_0 = (4\pi it)^{-1/2}\exp(ix^2/4t)$ and the modified amplitude $A(t)$ is

$$(2.2)\ A(t) = \begin{cases} \mu_0 \exp\left(\dfrac{2\lambda|\mu_0|^{p-1}}{i(3-p)}|4\pi t|^{-(p-1)/2}t\right) & \text{if } \mathrm{Im}\lambda = 0, \\ \mu_0\left(1 - \dfrac{2(p-1)\mathrm{Im}\lambda|\mu_0|^{p-1}}{3-p}|4\pi t|^{-(p-1)/2}t\right)^{\frac{i\lambda}{(p-1)\mathrm{Im}\lambda}} & \text{if } \mathrm{Im}\lambda \neq 0. \end{cases}$$

In fact, by substituting (2.1) into (NLS), we have the ordinary differential equation (ODE) of $A(t)$ :

$$\begin{cases} i\dfrac{dA}{dt} = \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(A), \\ A(0) = \mu_0. \end{cases}$$

This is easily solved and yields (2.2). Note that $\mathrm{Im}\lambda > 0$ implies blowing-up of $A(t)$ in positive finite time.

# 3 The case $u(0,x) = \mu_0\delta_0 + \mu_1\delta_a$

The superposition of $\delta$-functions causes "the mode generation" for $t \neq 0$. Before stating our results, let $\ell_\alpha^2$ be the weighted sequence space defined by

$$\ell_\alpha^2 = \{\{A_k\}_{k\in\mathbf{Z}};\ \|\{A_k\}_{k\in\mathbf{Z}}\|_{\ell_\alpha^2}^2 = \sum_{k\in\mathbf{Z}}(1+|k|^2)^\alpha|A_k|^2 < \infty\}.$$

For the simplicity of description, we often use the notation $\{A_k\}$ in place of $\{A_k\}_{k\in\mathbf{Z}}$. Then, our results are

**Theorem 3.1 (local result)** *For some $T > 0$, there exists a unique solution to (NLS) discribed as*

$$(3.1)\qquad u(t,x) = \sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial_x^2)\delta_{ka},$$

*where $\{A_k(t)\} \in C([0,T];\ell_1^2) \cap C^1((0,T];\ell_1^2)$ with $A_0(0) = \mu_0$, $A_1(0) = \mu_1$ and $\mu_k = 0$ ($k \neq 0,1$).*

**Remark 3.1.** Let us call $A_k(t)\exp(it\partial_x^2)\delta_{ka}$ the $k$-th mode. Then, (3.1) suggests that new modes away from 0-th and first ones appear in the solution while the initial data contains only the two modes. This special property is visible only in the nonlinear case.

**Remark 3.2**. Reading the proof of Theorem 3.1, we see that it is possible to generalize the initial data. Namely, we can construct a solution even when point masses are distributed on a line at equal intervals – more precisely, the initial data is given like

$$u(0,x) = \sum_{k\in\mathbf{Z}} \mu_k \delta_{ka}(x),$$

where $\{\mu_k\}_{k\in\mathbf{Z}} \in \ell_1^2$. In this case, the solution is described similarly to (3.1) but $\{A_k(0)\} = \{\mu_k\}$. The decay condition on the coefficients described in terms of $\ell_1^2$ is required to estimate the nonlinearity. This is because we will use the inequality like $\|\mathcal{N}(g)\|_{L^2_\theta} \le C\|g\|_{L^\infty_\theta}^{p-1}\|g\|_{L^2_\theta}$ where $g = g(t,\theta) = \sum_k A_k e^{-ik\theta} e^{i(ka)^2/4t}$ and $\theta \in [0,2\pi]$. Accordingly, to estimate $\|g\|_{L^\infty_\theta}$, we require the decay condition of $\{A_k\}$.

The sign of Im$\lambda$ determines the global solvability of (NLS).

**Theorem 3.2 (blowing up or global result)** *(1) Let Im$\lambda > 0$. Then, the solution as in Theorem 3.1 blows up in positive finite time. Precisely speaking, the $\ell_0^2$-norm of $\{A_k(t)\}$ tends to infinity at some positive time.*

*(2) Let Im$\lambda \le 0$. Then, there exists a unique global solution to (NLS) discribed as in Theorem 3.1 with $\{A_k(t)\} \in C([0,\infty);\ell_1^2) \cap C^1((0,\infty);\ell_1^2)$.*

In what follows, we present the rough sketch to prove Theorem 3.1 and 3.2. The idea is based on the reduction of (NLS) into the ODE system of $\{A_k\}_{k\in\mathbf{Z}}$. The next key lemma gives the representation formula of $\mathcal{N}(\sum_k A_k \exp(it\partial_x^2)\delta_{ka})$.

**Lemma 3.3** *Let $\{A_k\} \in C([-T,T];\ell_1^2)$. Then, we have*

$$\mathcal{N}(\sum_{k\in\mathbf{Z}} A_k(t)\exp(it\partial)\delta_{ka}) = |4\pi t|^{-n(p-1)/2} \sum_{k\in\mathbf{Z}} \tilde{A}_k(t)\exp(it\partial)\delta_{ka}, \tag{3.2}$$

*where*

$$\tilde{A}_k(t) = (2\pi)^{-1} e^{i(ka)^2/4t} \langle \mathcal{N}(\sum_j A_j e^{-ij\theta} e^{-i(ja)^2/4t}), e^{-ik\theta}\rangle_\theta,$$

*with $\langle f,g\rangle_\theta = \int_0^{2\pi} f(\theta)\overline{g(\theta)}d\theta$.*

**Proof of Lemma 3.3**. Note that the linear Schrödinger group is factorized as follows.

$$\begin{aligned}\exp(it\partial_x^2)f &= (4\pi i t)^{-1/2}\int \exp(i|x-y|^2/4t)f(y)dy \\ &= MD\mathcal{F}Mf,\end{aligned}$$

where

$$\begin{aligned} Mg(t,x) &= e^{ix^2/4t}g(x), \\ Dg(t,x) &= (2it)^{-1/2}g(x/2t), \\ \mathcal{F}g(\xi) &= (2\pi)^{-1/2}\int e^{-i\xi x}g(x)dx \quad \text{(Fourier transform of } g). \end{aligned}$$

Then, we see that

$$\begin{aligned} (3.3) \qquad & \mathcal{N}(\sum_k A_j(t)\exp(it\partial_x^2)\delta_{ja}) \\ &= \mathcal{N}((2\pi)^{-1/2}MD\sum_j A_j(t)e^{-ija\cdot x-i(ja)^2/4t}) \\ &= |4\pi t|^{-(p-1)/2}(2\pi)^{-1/2}MD\mathcal{N}(\sum_j A_j(t)e^{-ija\cdot x-i(ja)^2/4t}). \end{aligned}$$

Note that, to show the last equality in (3.3), we make use of the gauge invariance of the nonlinearity. Replacing $a\cdot x$ by $\theta$, we can regard $\mathcal{N}(\sum_j A_j(t)e^{-ij\theta-i(ja)^2/4t})$ as the $2\pi$-periodic function of $\theta$. Therefore, by the Fourier series expansion,

$$\begin{aligned} \mathcal{N}(\sum_j A_j(t)e^{-ij\theta-i(ja)^2/4t}) &= \sum_k \tilde{A}_k(t)e^{-i(ka)^2/4t}e^{-ik\theta} \\ &= (2\pi)^{n/2}\sum_k \tilde{A}_k(t)\mathcal{F}M\delta_{ka}. \end{aligned}$$

Plugging this into (3.3), we obtain Lemma 3.3. □

Our idea to solve the nonlinear equation is based on the reduction of (NLS) into the system of ODE's. By substituting $u=\sum_k A_k(t)\exp(it\partial_x^2)\delta_{ka}$ into (NLS) and noting that $i\partial_t\exp(it\partial_x^2)\delta_{ka}=-\partial_x^2\exp(it\partial_x^2)\delta_{ka}$, Lemma 3.3 yields

$$\sum_k i\frac{dA_k}{dt}\exp(it\partial_x^2)\delta_{ka} = |4\pi t|^{-(p-1)/2}\sum_k \tilde{A}_k\exp(it\partial_x^2)\delta_{ka}.$$

Equating the terms on both hand sides, we arrive at the desired ODE system:

$$(3.4) \qquad i\frac{dA_k}{dt} = |4\pi t|^{-(p-1)/2}\tilde{A}_k$$

with the initial condition $A_k(0)=\mu_k$. Now, showing the existence and uniqueness of (NLS) is equivalent to showing those of (3.4). To solve (3.4), let us consider the following integral equation.

$$\begin{aligned} A_k(t) &= \Phi_k(\{A_k(t)\}_{k\in\mathbf{Z}}) \\ (3.5) \qquad &\equiv \mu_k - i\int_0^t |4\pi\tau|^{-(p-1)/2}\tilde{A}_k(\tau)\,d\tau. \end{aligned}$$

Then, we want to see the contraction mapping property of $\{\Phi_k\}_{k\in\mathbf{Z}}$. The simple aplication of Parseval's identity derives the following.

**Lemma 3.4** *Let $I = [0,T]$ and $\{A_k\} = \{A_k\}_{k\in\mathbf{Z}}$. Then, we have*

$$\|\{\tilde{A}_k\}\|_{L^\infty(I;\ell_1^2)} \le C\|\{A_k\}\|_{L^\infty(I;\ell_1^2)}^p, \tag{3.6}$$

$$\begin{aligned}&\|\{\tilde{A}_k^{(1)}\} - \{\tilde{A}_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)}\\ &\le C(\max_{j=1,2}\|\{A_k^{(j)}\}\|_{L^\infty(I;\ell_1^2)})^{p-1}\|\{A_k^{(1)}\} - \{A_k^{(2)}\}\|_{L^\infty(I;\ell_0^2)}.\end{aligned} \tag{3.7}$$

**Proof of Lemma 3.4.** According to the description of $\tilde{A}_k$ as in Lemma 3.3 and the integration by parts, we see that

$$k\tilde{A}_k = (2\pi)^{-1} i e^{-i(ka)^2/4t}\langle \partial_\theta \mathcal{N}(\sum_j A_j e^{-ij\theta} e^{i(ja)^2/4t}), e^{-ik\theta}\rangle_\theta.$$

Then, Parseval's equality yields

$$\begin{aligned}\|\{k\tilde{A}_k\}\|_{\ell_0^2} &= (2\pi)^{-1/2}\|\partial_\theta\mathcal{N}(\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t})\|_{L_\theta^2}\\ &\le C\|\sum_j A_j e^{-ij\theta}e^{i(ja)^2/4t}\|_{L_\theta^\infty}^{p-1}\|\sum_j jA_j e^{-ij\theta}e^{i(ja)^2/4t}\|_{L_\theta^2}\\ &\le C\|\{A_j\}\|_{\ell_1^2}^p.\end{aligned}$$

Thus, we obtain (3.6). The proof for (3.7) follows similarly. Since there is a singularity at $u = 0$ of the nonlinearity $\mathcal{N}(u)$, we do not employ $\ell_1^2$-norm to measure $\{A_k^{(1)}\} - \{A_k^{(2)}\}$. □

We are now in the position to prove Theorem 3.1.

**Proof of Theorem 3.1.** The proof relies on the contraction mapping principle of $\{\Phi_k(\{A_j\})\}$. Let $\|\{\mu_k\}\|_{\ell_1^2} \le \rho_0$ and

$$\overline{B}_{2\rho_0} = \{\{A_k\} \in L^\infty([0,T];\ell_1^2);\ \|\{A_k\}\|_{L^\infty([0,T];\ell_1^2)} \le 2\rho_0\}$$

endowed with the metric in $L^\infty([0,T];\ell_0^2)$. Then, in virture of Lemma 3.4, we see that $\{\Phi_k(\{A_j\})\}$ is the contraction map on $\overline{B}_{2\rho_0}$ if $T$ is sufficiently small. Thus, Theorem 3.1 is obtained. □

To prove Theorem 3.2, we apply the a priori estimates described in the following.

**Lemma 3.5** *Let $\{A_k(t)\}$ be the solution to (3.4) in $C([0,T];\ell_1^2) \cap C^1((0,T];\ell_1^2)$.*

*(1) Then, we have*

$$\frac{d\|\{A_k(t)\}\|_{\ell_0^2}}{dt} = \frac{Im\lambda}{\pi}(4\pi t)^{-(p-1)/2}\|v(t)\|_{L_\theta^{p+1}}^{p+1}, \tag{3.8}$$

*where* $v(t,\theta) = \sum_k A_k(t)e^{-k\theta}e^{i(ka)^2/4t}$.

*(2) In addition, if* $Im\lambda < 0$, *then we have*

$$\|\{kA_k(t)\}\|_{\ell_0^2} \leq Ce^{t/2}, \tag{3.9}$$

*where the positive constant* $C$ *does not depend on* $T$.

**Remark 3.3** The a priori bound in (3.9) may be refined by sophisticating the estimates in the proof.

**Formal Proof of Lemma 3.5.** Note that $v(t,\theta)(=v)$ satisfies the nonlinear equation like

$$i\partial_t v = -\frac{a^2}{4t^2}\partial_\theta^2 v + \lambda|4\pi t|^{-(p-1)/2}\mathcal{N}(v). \tag{3.10}$$

Also, let us remark that $\|\{A_k(t)\}\|_{\ell_0^2} = \|v(t)\|_{L_\theta^2}$ and $\|\{kA_k(t)\}\|_{\ell_0^2} = \|\partial_\theta v(t)\|_{L_\theta^2}$. Then, multiplying $\overline{v}$ and taking the imaginary part of integration, we obtain (3.8). On the other hand, multiplying $\overline{\partial_t v}$ and taking the real part of integration, we have

$$\begin{aligned} 0 &= -\frac{a^2}{4t^2}\frac{d}{dt}\|\partial_\theta v\|_{L_\theta^2}^2 + \frac{2\mathrm{Re}\lambda}{p+1}|4\pi t|^{-(p-1)/2}\frac{d}{dt}\|v\|_{L_\theta^{p+1}}^{p+1} \\ &\quad -2(\mathrm{Im}\lambda)|4\pi t|^{-(p-1)/2}\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta. \end{aligned} \tag{3.11}$$

To estimate $\mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta$ in (3.11), let us multiply $\overline{\mathcal{N}(v)}$ on both hand sides of (3.10). Then, we see that

$$\begin{aligned} \mathrm{Im}\langle\mathcal{N}(v),\partial_t v\rangle_\theta &= -\frac{a^2}{4t^2}\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta + (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L_\theta^{2p}}^{2p} \\ &\geq (\mathrm{Re}\lambda)|4\pi t|^{-(p-1)/2}\|v\|_{L_\theta^{2p}}^{2p}, \end{aligned} \tag{3.12}$$

since $\mathrm{Re}\langle\partial_\theta^2 v,\mathcal{N}(v)\rangle_\theta \leq 0$. Combining (3.11) and (3.12), we have

$$\frac{d}{dt}\|\partial_\theta v\|_{L_\theta^2}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\frac{d}{dt}\|v\|_{L_\theta^{p+1}}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)t^{3-p}\|v\|_{L_\theta^{2p}}^{2p} \leq 0, \tag{3.13}$$

where $K_1 = \dfrac{8}{(p+1)a^2(4\pi)^{(p-1)/2}}$ and $K_2 = \dfrac{8}{a^2(4\pi)^{p-1}}$. This is equivalent to

$$\frac{d}{dt}E(t) \leq \frac{(5-p)K_1\mathrm{Re}\lambda}{2}t^{(3-p)/2}\|v\|_{L_\theta^{p+1}}^{p+1}, \tag{3.14}$$

where

$$E(t) = \|\partial_\theta v\|_{L^2_\theta}^2 + K_1(\mathrm{Re}\lambda)t^{(5-p)/2}\|v\|_{L^{p+1}_\theta}^{p+1} - K_2(\mathrm{Im}\lambda)(\mathrm{Re}\lambda)\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}_\theta}^{2p}\, d\tau.$$

In this proof, we only consider the most complicated case that $\mathrm{Im}\lambda$ and $\mathrm{Re}\lambda < 0$. The other case follows more easily. By (3.14), we have $E(t) \leq$ (const.) for $t > t_0$, i.e.,

$$\|\partial_\theta v\|_{L^2_\theta}^2 \leq C_1 + C_2 t^{(5-p)/2}\|v\|_{L^{p+1}_\theta}^{p+1} + C_3\int_{t_0}^t \tau^{3-p}\|v(\tau)\|_{L^{2p}_\theta}^{2p}\, d\tau \tag{3.15}$$

for some positive constants $C_1, C_2$ and $C_3$. Applying the Gagliardo-Nirenberg inequalities:

$$\begin{aligned} \|v\|_{L^{p+1}_\theta}^{p+1} &\leq C\|v\|_{H^1_\theta}^{(p+1)\beta}\|v\|_{L^2_\theta}^{(p+1)(1-\beta)}, \\ \|v\|_{L^{2p}_\theta}^{2p} &\leq C\|v\|_{H^1_\theta}^{2p\gamma}\|v\|_{L^2_\theta}^{2p(1-\gamma)}, \end{aligned}$$

where $1/(p+1) = \beta(1/2-1) + (1-\beta)2$ and $1/2p = \gamma(1/2-1) + (1-\gamma)/2$, and using Young's inequality, we have

$$\|v(t)\|_{H^1_\theta}^2 \leq C\langle t\rangle^3 + \int_{t_0}^t \|v(\tau)\|_{H^1_\theta}^2\, d\tau. \tag{3.16}$$

We here note that, since $\|v(t)\|_{L^2}$ has a finite bound in virture of (3.8), it is included in the positive constant $C$. Applying Gronwall's inequality to (3.16), we obtain (3.9). □

**Proof of Theorem 3.2.** If $\mathrm{Im}\lambda > 0$, then, Lemma 3.5 (3.8) and Hölder's inequality $\|v\|_{L^{p+1}_\theta}^{p+1} \geq (2\pi)^{-(p-1)/2}\|v\|_{L^2_\theta}^{p+1}$ give

$$\frac{d}{dt}\|v\|_{L^2_\theta}^2 \geq C\|v\|_{L^2_\theta}^{p+1}.$$

This implies that $\|v(t)\|_{L^2_\theta} = \|\{A_k(t)\}\|_{\ell^2_0}$ blows up in positive finite time. On the other hand, if $\mathrm{Im}\lambda \leq 0$, then, Lemma 3.5 gives the a priori bound of $\|\{A_k(t)\}\|_{\ell^2_1}$ for any positive $t$. Hence, the local solution to (3.4) is continuated to the global one. □

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