Duality in Stochastic Optimal Control and Applications

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Abstract

We review a duality result and its applications for a stochastic control problem with fixed marginals obtained in [10]. This problem is the stochastic analog of the well known Monge and Monge-Kantorovich optimal transportation problems.

Keywords: optimal transportation problem, Legendre transform, duality theorem, stochastic control, forward-backward stochastic differential equation

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1 Introduction.

In the present paper we review a duality result and its applications for a stochastic control problem with fixed marginals published in [10]. For a few proofs we do not give all details, rather we preferred to focus on the arguments; details for these proofs can be found in [10].

The problem were are interested in is defined as follows: given $\epsilon > 0$,

\[ V_\epsilon(P_0, P_1) := \inf \left\{ E\left[ \int_0^1 L(t, X(t); \beta_X(t, X))dt \right] \mid PX(t)^{-1} = P(t = 0, 1), X \in A \right\}. \tag{1.1} \]

where $P_0$ and $P_1$ are Borel probability measures on $\mathbb{R}^d$ and $L(t, x; u) : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \mapsto [0, \infty)$ is measurable and convex w.r.t. $u$. The infimum is taken over the set $A$ of all $\mathbb{R}^d$-valued, continuous semimartingales $\{X(t)\}_{0 \leq t \leq 1}$ on a probability space $(\Omega_X, \mathcal{B}_X, P_X)$ such that there exists a Borel measurable $\beta_X : [0, 1] \times C([0, 1]) \mapsto \mathbb{R}^d$ for which

(i) $\omega \mapsto \beta_X(t, \omega)$ is $\mathcal{B}(C([0, t]))_{+}$-measurable for all $t \in [0, 1]$, where $\mathcal{B}(C([0, t]))$ denotes the Borel $\sigma$-field of $C([0, t])$,

(ii) $\{X(t) - X(0) - \int_0^t \beta_X(s, X)ds : \sqrt{\epsilon}W_X(t)\}_{0 \leq t \leq 1}$ where $W_X$ is a $\sigma[X(s) : 0 \leq s \leq t]$-Brownian motion (see [7]).

Remark It would appear more natural to consider semi martingales of the form

\[ X^u(t) = X_0 + \int_0^t u(s)ds + W(t) \quad (t \in [0, 1]). \tag{1.2} \]

with $\{u(t)\}_{0 \leq t \leq 1}$ a $(\mathcal{B}_t)$-progressively measurable stochastic process. However, if we set

\[ \beta_{X^u}(t, X^u) = E[u(t)|X^u(s), 0 \leq s \leq t], \tag{1.3} \]

then using conditional expectations Jensen inequality and convexity of $L$ one obtains,

\[ E\left[ \int_0^1 L(t, X^u(t); u(t))dt \right] \geq E\left[ \int_0^1 L(t, X^u(t); \beta_{X^u}(t, X^u))dt \right]. \tag{1.4} \]

and therefore it is sufficient to consider drifts of the form $\beta_X$ as long as one is interested in the minimizing problem $V_\epsilon(P_0, P_1)$. 
When \( L \) depends only on \( u \), problem \( \nu_\varepsilon \) has a counterpart in the deterministic setting, this counterpart has been intensively studied since it is the Monge-Kantorovich problem (for a complete list of references we refer the reader to [11] and [13])

\[
T(P_0, P_1) := \inf \left\{ E \left[ \int_0^1 \ell \left( \frac{d\phi(t)}{dt} \right) dt \right] \mid P\phi(t)^{-1} = P_t (t = 0, 1), \right. \\
\left. t \mapsto \phi(t) \text{ is absolutely continuous} \right\}. \tag{1.5}
\]

Actually the most usual (and better known) form of the Monge-Kantorovich problem is

\[
T(P_0, P_1) := \inf \left\{ E(L(Y - X)) \mid X \sim P_0, Y \sim P_1 \right\}. \tag{1.6}
\]

where \( X \sim P_0 \) (resp. \( Y \sim P_1 \)) means that the law of \( X \) (resp. \( Y \)) is \( P_0 \) (resp. \( P_1 \)). It is not difficult to show that \( T(P_0, P_1) = T(P_0, P_1) \). In the quadratic case, that is when \( L(t, x, u) = \frac{1}{2}|u|^2 \), the Monge-Kantorovich problem has received much attention, in probability as well as in statistics, in particular because \( \sqrt{T(P_0, P_1)} \), called Wasserstein metric, metrizes convergence in distribution on the set of probability measures on \( \mathbb{R}^d \) with finite second moments. It is not difficult to show that \( T(P_0, P_1) = T(P_0, P_1) \). More recently the results obtained by Brenier (cf. [1], [2]) have revived the subject by enlightening its connection with fluid mechanics and geometry.

Duality results play a fundamental role in the study of Monge-Kantorovich problem. There are two duality results. For the sequel the most important for us is the duality result due to Evans ([5]):

\[
T(P_0, P_1) = \sup \left\{ \int_{\mathbb{R}^d} \psi(1, x)P_1(dx) - \int_{\mathbb{R}^d} \psi(0, x)P_0(dx) \right\}, \tag{1.7}
\]

where the supremum is taken over all continuous viscosity solutions \( \psi \) to the following Hamilton-Jacobi equation:

\[
\frac{\partial \psi(t, x)}{\partial t} + \ell^*(D_x \psi(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbb{R}^d) \tag{1.8}
\]

(see E Chap. 3). Here \( D_x := (\partial/\partial x_i)_{i=1}^d \) and for \( z \in \mathbb{R}^d \),
\ell^*(z) := \sup_{u \in \mathbb{R}^d} \{\langle z, u \rangle - \ell(u)\}

and \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(\mathbb{R}^d\).

The second duality result was chronologically proved before by Kantorovich and implies (1.7) (cf. for instance \(V\)):

\[
T(P_0, P_1) := \sup \left\{ \int_{\mathbb{R}^d} \psi(y)P_1(dy) + \int_{\mathbb{R}^d} \varphi(x)P_0(dx); \right. \\
\left. (\varphi, \psi) \in L^1(P_0) \times L^1(P_1), \varphi(x) + \psi(y) \leq L(y-x) \right\}
\tag{1.9}
\]

In the sequel we describe how it is possible to prove a duality theorem for \(W_t\) in the spirit of (1.7) and describe applications. We will not give all proofs in detail; for detailed proofs we refer the reader to [10].

## 2 Duality Theorem

For simplicity in what follows we restrict to the case when \(L(t, x, u) = L(u)\) (that is \(L\) depends only on \(u\)). However our main result (duality theorem) and its applications are valid even if \(L\) depends on \((t, x)\) (cf. [10]). Let us recall that \(P_0\) and \(P_1\) are given Borel probability measures on \(\mathbb{R}^d\), and \(L(u) : \mathbb{R}^d \to [0, \infty)\) is a measurable and convex function of \(u\). We moreover assume that

\[
W_e(P_0, P_1) < +\infty \tag{2.1}
\]

We will need assumptions on \(L\) which we denote as follows:

(A.1). \(L\) is superlinear: for some \(\delta > 1\),

\[
\liminf_{|u| \to \infty} \frac{L(u)}{|u|^{\delta}} > 0.
\]

(A.2). (i) \(L \in C^3(\mathbb{R}^d)\),
(ii) \(D^2_u L(u)\) is positive definite for all \(u \in \mathbb{R}^d\),

We will look for sufficient conditions for \(W_e\) to admit a minimizer, unique and/or Markovian and also for a characterization of minimizers. A duality theorem will provide such a characterization (the characterization itself will be obtained in the next section). As already mentioned we focus on the main steps and articulations of the argument.
2.1 Existence and uniqueness of a minimizer.

Results about existence and uniqueness are gathered in

Theorem 2.1  (i) $V_\epsilon(P_0, P_1)$ admits a minimizer.

(ii) If assumption (A.I) holds with $\delta = 2$, $V_\epsilon(P_0, P_1)$ admits a Markovian minimizer

(iii) If $L$ is strictly convex and assumption (A.I) holds with $\delta = 2$, then $V_\epsilon(P_0, P_1)$ admits a unique minimizer (which is Markovian from (ii)).

Our tool for the proof of (ii) and (iii) in Theorem 2.1 is the following minimization problem with fixed marginals

$$V_\epsilon(P_0, P_1) := \inf \int_0^1 \int_{\mathbb{R}^d} L(b(t, x)) P(t, dx) dt,$$

where the infimum is taken over all $(b(t, x), P(t, dx))$ for which $P(t, dx)$ $0 \leq t \leq 1$ are Borel probability measures, on $\mathbb{R}^d$, such that $p(t, x) := P(t, dx)/dx$ exists for all $t \in (0, 1]$, $P(t, dx) = P_t$ ($t = 0, 1$) and the following Fokker-Planck pde

$$\frac{\partial P(t, dx)}{\partial t} = \frac{\epsilon}{2} \Delta P(t, dx) - div(b(t, x)P(t, dx))$$

is satisfied. Let us notice that $V_\epsilon$ is a stochastic analog of the problem considered by Benamou and Brenier in [3]. Then

Proposition 2.1  (cf. [10] Lemma 3.5). Assume (A.I) with $\delta = 2$ holds. Then $V_\epsilon(P_0, P_1) = \underline{V}_\epsilon(P_0, P_1)$.

Proof of Theorem 2.1. Proof of (i): Let $(X_n)$ denote a minimizing sequence of processes in the set $A$; this means that

$$\lim_{n \to \infty} E \left[ \int_0^1 L(\beta_{X_n}(t, X_n)) dt \right] = V_\epsilon(P_0, P_1)$$

(2.4)

Since $X_n \in A$ for all $n$ and assumption (A.I) holds ($L$ is superlinear), it follows that the sequence $(X_n)$ is tight: the sufficient condition for tightness of [14] is satisfied. In particular (A.I) implies that

$$E \left[ \int_0^1 |\beta_{X_n}(t, X_n)|^\delta dt \right] < +\infty$$

(2.5)
(with $\delta > 1$). Hence there exists a subsequence $(X_{n_k})$ weakly converges weakly; let us denote its limit by $(X(t))$. The process $X$ belongs to $A$; from [14], Theorem 5, we obtain that $\frac{1}{\sqrt{t}}(X(t) - X(0) - A(t))_{t \in [0,1]}$ is a standard Brownian motion and $\{A(t)\}_{t \in [0,1]}$ is absolutely continuous. Moreover $(X(t))$ satisfies

$$
\lim_{k \to \infty} \mathbb{E}\left[ \int_0^1 L(\beta_{X_{n_k}}(t, X_{n_k})) dt \right] \geq \mathbb{E}\left[ \int_0^1 \frac{dA(t)}{dt} dt \right].
$$

which implies that it is a minimizer of $V_{\epsilon}$. Inequality (2.6) may be proved following the argument of [9] in the proof of Theorem 1, which is here simplified since $L$ depends on $u$ only.

Proof of (ii): we now assume that (A.1) holds with $\delta = 2$. Using the same argument as in the proof of (i) one can show that $\mathcal{V}_{\epsilon}(P_0, P_1)$ admits a minimizer. From Proposition 2.1 this minimizer also is a minimizer of $V_{\epsilon}$ (here it is actually sufficient that $V_{\epsilon} \geq \mathcal{V}_{\epsilon}$).

Proof of (ii): we moreover assume that $L$ is strictly convex. From Proposition (actually it is sufficient that $V_{\epsilon} \leq \mathcal{V}_{\epsilon}$) it is enough to show uniqueness for $\mathcal{V}_{\epsilon}$ (cf. [10] proof of Proposition 2.2 where we use the strict convexity of $L$ and the linearity of Fokker-Planck pde). Q.E.D.

2.2 Duality Theorem.

Theorem 2.2 Suppose that (A.1) and (A.2) are satisfied. Then

$$
\mathcal{V}_{\epsilon}(P_0, P_1) = \sup \left\{ \int_{\mathbb{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbb{R}^d} \varphi(0, x) P_0(dx) \right\},
$$

where the supremum is taken over all classical solutions $\varphi$, to the following HJB equation, for which $\varphi(1, \cdot) \in C_b^\infty(\mathbb{R}^d)$:

$$
\frac{\partial \varphi(t, x)}{\partial t} + \frac{\epsilon}{2} \Delta \varphi(t, x) + H(D_x \varphi(t, x)) = 0 \quad ((t, x) \in (0, 1) \times \mathbb{R}^d)
$$

Proof of 2.2 The two main arguments of the proof are:

1. A property of the Legendre transform: on a Banach space if $f$ is a lower semi continuous function not identically equal to $+\infty$, then $f^{**} = f$ where $*$ denotes Legendre transform.
2. A representation of the value function of a stochastic control problem (with sufficiently regular terminal cost) by a solution of an Hamilton-Jacobi-Bellman pde.

For point 1., we rely on results of [4] (namely Theorem 2.2.15 and Lemma 3.2.3). To apply these results, one has to prove first that $P \mapsto V(P_0, P)$ is lower semicontinuous and convex. This is proved in detail in [10] Lemmas 3.1 and 3.2. It follows that

$$V(P_0, P_1) = \sup_{f \in C_b(R^d)} \left\{ \int_{R^d} f(x) P_1(dx) - V^*_0(f) \right\},$$

(2.9)

where for $f \in C_b(R^d)$,

$$V^*_0(f) := \sup_{P \in \mathcal{M}_1(R^d)} \left\{ \int_{R^d} f(x) P(dx) - V(P_0, P) \right\},$$

and $\mathcal{M}_1(R^d)$ denotes the complete separable metric space, with a weak topology, of Borel probability measures on $R^d$.

For point 2., we refer the reader to [6]: for $f \in C^\infty_b(R^d)$,

$$V^*_0(f) = \sup \left\{ E[f(X(1))] - E\left[ \int_0^1 L(t, X(t); \beta_X(t, X)) dt \right] : X \in A, PX(0)^{-1} = P_0 \right\} = \int_{R^d} \varphi_f(0, x) P_0(dx),$$

(2.10)

where $\varphi_f$ denotes the unique classical solution to the HJB equation (2.3) with $\varphi(1, \cdot) = f(\cdot)$. Using both identities (2.9) and (2.10), we obtain

$$V_\epsilon(P_0, P_1) \geq \sup_{f \in C^\infty_b(R^d)} \int_{R^d} \varphi(1, y) P_1(dy) - \int_{R^d} \varphi(0, x) P_0(dx),$$

(2.11)

To prove the converse inequality we have to pass from $C_b(R^d)$ to $C^\infty_b(R^d)$ with the help of a mollifier sequence. Take $\Phi \in C^\infty([-1, 1]^d, [0, \infty))$ for which $\int_{R^d} \Phi(x) dx = 1$, and for $\delta > 0$, and define

$$\Phi_\delta(x) := \delta^{-d} \Phi(x/\delta).$$

For $f \in C_b(R^d)$, we set
\[ f_\delta(x) := \int_{\mathbb{R}^d} f(y) \Phi_\delta(x - y) dy. \] (2.12)

Then \( f_\delta \in C^\infty_b(\mathbb{R}^d) \) and

\[
\sup_{f \in C^\infty_b(\mathbb{R}^d)} \int_{\mathbb{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbb{R}^d} \varphi(0, x) P_0(dx) \\
\geq \int_{\mathbb{R}^d} f_\delta(x) P_1(dx) - V_{P_0}(f_\delta) \\
\geq \int_{\mathbb{R}^d} f(x) \Phi_\delta * P_1(dx) - V_{\Phi_\delta * P_0}(f).
\]

Indeed, for any \( X \in A \)

\[ E[f_\delta(X(1))] = \int_{\mathbb{R}^d} \Phi(z) dz E[f(X(1) - \delta z)] \] (2.13)

Then identity (2.9) implies that

\[
\sup_{f \in C^\infty_b(\mathbb{R}^d)} \int_{\mathbb{R}^d} \varphi(1, y) P_1(dy) - \int_{\mathbb{R}^d} \varphi(0, x) P_0(dx) \\
\geq V(\Phi_\delta * P_0, \Phi_\delta * P_1)
\]

It remains to let \( \delta \) go to 0 and use the lower semi-continuity of \((P, Q) \mapsto V(P, Q)\) proved in [10]. Q.E.D.

3 Applications.

3.1 Characterization.

We first recall the following property of Legendre transform which we will use repeatedly: if \( L \) is strictly convex, superlinear (i.e. satisfies (A.1)) and smooth (for instance belongs to \( C^2(\mathbb{R}^d) \)) then \( L^{**} = L; \nabla L : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a bijection from \( \mathbb{R}^d \) onto itself and \( \nabla H = \nabla L^{-1} \) where \( H = L^* \). If moreover \( D^2 L \) is positive definite, \( H \) is twice differentiable and

\[ D^2 H(\nabla L(u)) = D^2 L(u)^{-1} \] (3.1)
Theorem 3.1 Suppose that (A.1) and (A.2) hold. Then for any minimizer \( \{X(t)\}_{0 \leq t \leq 1} \) of \( V_\epsilon(P_0, P_1) \), there exists a sequence of classical solutions \( \{\varphi_n\}_{n \geq 1} \) to the HJB equation (2.8), such that \( \varphi_n(1, \cdot) \in C^\infty_b(\mathbb{R}^d) \) \( (n \geq 1) \) and that the following holds:

\[
\beta_X(t, X) = b_X(t, X(t)) := E[\beta_X(t, X)|t, X(t)]
\]

\[
= \lim_{\eta \rightarrow \infty} D_z H(t, X(t); D_x \varphi_n(t, X(t))) \quad dtdP(\cdot)^{-1} - a.e..
\]

Proof of Theorem 3.1 From Theorem 2.2 here exists a sequence of classical solutions \( \{\varphi_n\}_{n \geq 1} \) to the HJB equation (2.8), such that \( \varphi_n(1, \cdot) \in C^\infty_b(\mathbb{R}^d) \) \( (n \geq 1) \) and

\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_n(1, y)P_1(dy) - \int_{\mathbb{R}^d} \varphi_n(0, x)P_0(dx) = V_\epsilon(P_0, P_1)
\]

Therefore, for \( X \) a minimizer of \( V_\epsilon \), it holds

\[
\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi_n(1, y)P_1(dy) - \int_{\mathbb{R}^d} \varphi_n(0, x)P_0(dx) = E\left[ \int_0^1 L(\beta_X(t, X)) dt \right] = 0
\]

Since \( X(0) \sim P_0 \) (resp. \( X(1) \sim P_1 \)) and \( \{\varphi_n\}_{n \geq 1} \) solves the HJB pde (2.8), Ito formula yields

\[
\lim_{n \rightarrow \infty} E \int_0^1 \langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle > -L(\beta_X(t, X)) - H(\nabla \varphi_n(t, X(t))) dt = 0
\]

Moreover by definition of \( H \) as the Legendre transform of \( L \), the integrand in (3.5) is positive. Hence the sequence

\[
\langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle > -L(\beta_X(t, X)) - H(\nabla \varphi_n(t, X(t)))
\]

converges to 0 in \( L^1(dtdP) \) and admits a subsequence which converges a.s. For simplicity we still denote this subsequence by \( (\varphi_n) \). Let \( (t, \omega) \) be such that the sequence \( \langle \beta_X(t, X), \nabla \varphi_n(t, X(t)) \rangle > -H(\nabla \varphi_n(t, X(t))) \) converges to \( L(\beta_X) = H^*(\beta_X) \). The supremum in the definition of

\[
H^*(u) = \sup(p, u > -H(p))
\]

is attained at \( p^* = \nabla L(u) \). We therefore obtain that

\[
\lim \nabla \varphi_n(t, X(t)) = \nabla L(\beta_X(t, X))
\]
or equivalently \( \beta_X(t, X) = \lim \nabla H(\nabla \varphi_n(t, X(t)) \). Q.E.D.

We would like to show now that a minimizer solves a stochastic equation. We were able to prove such a result under the additional assumption: (A.3). \( D^2L(u) \) is bounded.

The following lemma will be useful below:

**Lemma 3.1** Let \( L \in C^2(\mathbb{R}^d) \) be strictly convex and superlinear such that

\[
C := \sup\{<D^2L(u)z, z>: (u, z) \in \mathbb{R}^d \times \mathbb{R}^d, |z| = 1\} < +\infty \quad (3.9)
\]

Then

\[
\forall (u, z) \in \mathbb{R}^d \times \mathbb{R}^d \quad ||z - \nabla L(u)||^2 \leq C|L(u) - (<u, z>-H(z))| \quad (3.10)
\]

**Proof of Lemma 3.1.** By definition of \( H = L^* \), for all \((u, z), L(u) - (<u, z>-H(z)) \geq 0\). The assumptions of the lemma ensure that for all \( u, u = \nabla H(\nabla L(u)) \) and \( H(p) = <p, \nabla H(p)> - L(\nabla H(p)) \) for all \( p \). We therefore have

\[
L(u) - (<u, z>-H(z)) = H(z) - H(\nabla L(u)) - <\nabla H(\nabla L(u)), z - \nabla L(u)> \quad (3.11)
\]

The conclusion follows from identity (3.1). Q.E.D.

**Theorem 3.2** Suppose that (A.1) holds with \( \delta = 2 \) as well as (A.2) and (A.3). Then for the unique minimizer \( \{X(t)\}_{0 \leq t \leq 1} \) of \( V_\epsilon(P_0, P_1) \), (1) there exist \( f(\cdot) \in L^1(\mathbb{R}^d, P_1(dx)) \) and a \( \sigma[X(s) : 0 \leq s \leq t] \)-continuous semimartingale \( \{Y(t)\}_{0 \leq t \leq 1} \) such that

\[
\{(X(t), Y(t), Z(t) := D_uL(b(x, t, X(t))))\}_{0 \leq t \leq 1}
\]

satisfies the following FBSDE in a weak sense: for \( t \in [0, 1] \),

\[
X(t) = X(0) + \int_0^t D_xH(Z(s))ds + \sqrt{\epsilon}W(t), \quad (3.12)
\]

\[
Y(t) = f(X(1)) - \int_t^1 L(D_xH(Z(s)))ds \]

\[
- \int_t^1 <Z(s), dW(s)> .
\]
(2) there exist $f_0(\cdot) \in L^1(\mathbb{R}^d, P_0(dx))$ and $\varphi(\cdot, \cdot) \in L^1([0,1] \times \mathbb{R}^d, P((t, X(t)) \in dt dx))$ such that $Y(0) = f_0(X(0))$ and such that

$$Y(t) - Y(0) = \varphi(t, X(t)) - \varphi(0, X(0)) \quad dtdP X(\cdot)^{-1} - a.e., \quad (3.13)$$

that is, $Y(t)$ is a continuous version of $\varphi(t, X(t)) - \varphi(0, X(0)) + f_0(X(0))$.

**Proof of Theorem 3.2** Let $\varphi_n$ be a sequence satisfying the same conditions as in the proof of Theorem 3.1 and $X$ a minimizer of $V_\epsilon$. From Ito formula,

$$\varphi_n(t, X(t)) - \varphi_n(0, X(0)) = \int_0^t \{<b_X(s, X(s)), D_x\varphi_n(s, X(s))> - H(D_x\varphi_n(s, X(s)))\} ds$$

$$+ \int_0^t <D_x\varphi_n(s, X(s)), \sqrt{\epsilon}dW(s)>.$$

We first consider convergence of the martingale part. By Doob's inequality

$$E(\sup_{0 \leq t \leq 1} |\int_0^t <D_x\varphi_n(s, X(s)) - D_u(b_X(s, X(s))), dW(s)>|^2) \leq 4E(\int_0^1 |D_x\varphi_n(s, X(s)) - D_u(b_X(s, X(s)))|^2 ds) \quad (3.15)$$

By Lemma 3.1 it follows that

$$E(\sup_{0 \leq t \leq 1} |\int_0^t <D_x\varphi_n(s, X(s)) - D_u(b_X(s, X(s))), dW(s)>|^2) \leq 4CE(\int_0^1 |L(b_X(s, X(s))) - (b_X(s, X(s)), D_x\varphi_n(s, X(s))> - H(D_x\varphi_n(s, X(s)))| ds)$$

which converges to 0 by Theorem 3.1. This theorem also implies that

$$\int_0^t \{<b_X(s, X(s)), D_x\varphi_n(s, X(s))> - H(D_x\varphi_n(s, X(s)))\} ds \quad (3.16)$$

converges in $L^1$ to $\int_0^1 L(b_X(s, X(s))) ds$. We therefore obtain that $\varphi_n(1, y) - \varphi_n(0, x)$ and $\varphi_n(t, y) - \varphi_n(0, x)$ are convergent in $L^1(\mathbb{R}^d \times \mathbb{R}^d, P((X(0), X(1)) \in dt dx))$. 


$dxdy$) and $L^1(\mathbb{R}^d \times [0,1] \times \mathbb{R}^d, P((X(0), (t, X(t))) \in dxdtdy))$, respectively. The question is whether the limit is still of the separable form $\psi(1, y) - \psi(0, x)$ and $\psi(t, y) = \psi(0, x)$ respectively. From [12] this is indeed the case provided that the law of $(X(0), X(1))$ (resp. $(X(0), X(t))$) is absolutely continuous with respect to $P_0(dx)P_1(dy)$ (resp. $P_0(dx)P_t(dy)$) where $P_t$ denotes the law of $X_t$. These conditions are satisfied here since (A.1) holds with $\delta = 2$ and consequently the process $X$ has finite entropy w.r.t. the Wiener measure on $C(\mathbb{R}^d)$ with initial law $P_0$. Hence, from [12], Prop. 2, there exist $f \in L^1(\mathbb{R}^d, P_1(dx))$, $f_0 \in L^1(\mathbb{R}^d, P_0(dx))$, $\varphi_0 \in L^1(\mathbb{R}^d, P_0(dx))$ and $\varphi \in L^1([0, 1] \times \mathbb{R}^d, P((f, X(t)) \in dtdy))$ such that

$$\lim_{n \to \infty} E[|\varphi_n(1, X(1)) - \varphi_n(0, X(0)) - \{f(X(1)) - f_0(X(0))\}|] = 0, \quad (3.17)$$

and

$$\lim_{n \to \infty} E\left[\int_0^1 |\varphi_n(t, X(t)) - \varphi_n(0, X(0)) - \{\varphi(t, X(t)) - \varphi_0(X(0))\}| dt\right] = 0. \quad (3.18)$$

It is easy to check that $(Y(t))$ defined by

$$Y(t) := f_0(X(0)) + \int_0^t L(s, X(s); b_X(s, X(s))) ds$$

$$+ \int_0^t \langle D_u L(s, X(s); b_X(s, X(s))), dW(s) \rangle,$$ \hspace{1cm} (3.19)

satisfies the statement of Theorem 3.2. Q.E.D.

**References**


