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On formal solution and genuine solution for some partial differential equations

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1 Introduction

Let $\mathbb{C}$ be the complex plane or the set of all complex numbers, t be the variable in $\mathbb{C}_t$, and $x=(x_1,\ldots,x_n)$ be the variable in $\mathbb{C}_x^n=\mathbb{C}_{x_1}\times\cdots\times\mathbb{C}_{x_n}$. We use the notations: $\mathbb{N}=\{0,1,2,\ldots\}$, $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, and $(\partial/\partial x)^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. Let $|x|=\max_{1\le i\le n}\{|x_i|\}$, $D_R=\{x\in\mathbb{C}_x^n;\ |x|<R\}$ and $S_\theta(T)=\{t\in\mathbb{C}_t;\ 0<|t|<T$ and $|\arg t|<\theta\}$. $\mathcal{O}(D_R)$ $(\mathcal{O}(S_\theta(T)\times D_R))$ is the set of all holomorphic function on D_R (resp. $S_\theta(T)\times D_R$). $S_{\theta_0}(T_0)\subset\subset S_\theta(T)$ means $\theta_0<\theta$ and $T_0<T$.

We consider the following nonlinear partial differential operator $D(u)$:

$$D(u(t,x)):=\sum_{|q|\ge 0}t^{\sigma_q}c_q(t,x)\prod_{j+|\alpha|\le m}\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\right\}^{q_{j,\alpha}}. \tag{1.1}$$

Let $Z^q=\prod_{j+|\alpha|\le m}Z_{j,\alpha}{}^{q_{j,\alpha}}$ and $|q|=\sum_{j+|\alpha|\le m}q_{j,\alpha}$. We assume that $\sum_{|q|\ge 0}t^{\sigma_q}c_q(t,x)Z^q$ is a convergent power series in Z and the coefficients $c_q(t,x)$ are holomorphic in $S_\theta(T)\times D_R$ and satisfy $c_q(0,x)\not\equiv 0$.

Ōuchi studied an equation $D(u)=0$ in the case that $\sum_{|q|\ge 0}t^{\sigma_q}c_q(t,x)Z^q$ is a polynomial in Z with degree M. Let us introduce a function class and some conditions in Ōuchi[3]. Put $l_q=\max\{j+|\alpha|;\ q_{j,\alpha}\ne 0\}$.

Let us define Newton polygon for an operator $D(u)$. We put

$$\Pi(a,b):=\{(x,y)\in\mathbb{R}^2;\ x\le a \text{ and } y\ge b\}.$$

Then we define Newton polygon $NP(D)$ and $NP(\mathcal{D})$ by

$$NP(D)=CH\Big\{\bigcup_{|q|\ge 1}\Pi(l_q,\sigma_q);\ c_q(t,x)\not\equiv 0\Big\}\text{ and }NP(\mathcal{D})=CH\Big\{\bigcup_{|q|=1}\Pi(l_q,\sigma_q);\ c_q(t,x)\not\equiv 0\Big\},$$

where $CH\{\cdot\}$ is the convex hull of a set $\{\cdot\}$.

The boundary of Newton polygon $NP(D)$ consists of a vertical half line $\Sigma^*_{D,p}$, a horizontal half line $\Sigma^*_{D,0}$ and segments $\Sigma^*_{D,i}$ $(1\le i\le p-1)$. Let $\gamma^*_{D,i}$ be the slope of $\Sigma^*_{D,i}$ for $i=0,\ldots,p$. Then we have $0=\gamma^*_{D,0}<\gamma^*_{D,1}<\cdots<\gamma^*_{D,p}=\infty$. Further Newton polygon $NP(D)$ have p point vertices, we denote them by (l_i,σ_i) with $l_0<l_1<\cdots<l_{p-1}=m$.

For $NP_L(\mathcal{D})$, we define $\Sigma^*_{\mathcal{D},i}$, $\gamma^*_{\mathcal{D},i}$ for $i=0,\ldots,p_{\mathcal{D}}$ and $(l_{\mathcal{D},i},\sigma_{\mathcal{D},i})$ for $i=0,\ldots,p_{\mathcal{D}}-1$ by same rules.

Further we treat an equation that is obtained by substituting $t^\nu u(t,x)$ instead of $u(t,x)$ in the operator (1.1). For the operator obtained by this, we define the former and denote by $NP(D;\nu)$,

$NP(\mathcal{D};\nu)$, $\Sigma^*_{D,i}(\nu)$, $\Sigma^*_{\mathcal{D},i}(\nu)$ and so on. Then $\gamma^*_{\mathcal{D},i} = \gamma^*_{\mathcal{D},i}(\nu)$ holds for $i = 0,\dots,p_{\mathcal{D}}-1$.

At next let us define an operator $\mathcal{L}_i$ with respect to $\Sigma^*_{\mathcal{D},i}$ for $i = 1,\cdots,p_{\mathcal{D}}-1$. Let $I_i = \{q;\ \sigma_{\mathcal{D},i} - \sigma_q = \gamma^*_{\mathcal{D},i}(l_{\mathcal{D},i} - l_q) \text{ and } |q| = 1\}$. Then we define

$$\begin{aligned}\mathcal{L}_i u(t,x) &= \sum_{q\in I_i} t^{\sigma_q} c_q(t,x) \prod_{j+|\alpha|\le m} \left\{\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\right\}^{q_{j,\alpha}} \\ &= \sum_{(j,\alpha)\in J_i} t^{\sigma_{j,\alpha}} c_{j,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial x}\right)^\alpha u(t,x)\end{aligned}$$

where $J_i = \{(j,\alpha) \in \mathbb{N}\times\mathbb{N}^n;\ j + |\alpha| \le m \text{ and } \sigma_{\mathcal{D},i} - \sigma_{j,\alpha} = \gamma^*_i(l_{\mathcal{D},i} - j - |\alpha|)\}$. Let m_i be a differential order with respect to x of $\mathcal{L}_i$.

Ōuchi's three conditions are as follows;

Condition 1. *$\sum_{|q|\ge 0} t^{\sigma_q} c_q(t,x) Z^q$ is a polynomial in Z with degree M.*

Condition 2. *The equation $D(u) = 0$ has a linear part with order m.*

Condition 3. *For $\mathcal{L}_i$ the followings hold;*
(1) If $j + |\alpha| < l_{\mathcal{D},i}$ then $|\alpha| < m_i$ and (2) $\sum_{j+|\alpha|=l_{\mathcal{D},i},|\alpha|=m_i} c_{j,\alpha}(0,0)\hat{\xi}^\alpha \ne 0$ where $\hat{\xi} = (1,0,\cdots,0)$.

Lemma 1. *If the equation $D(u) = 0$ has a linear part with order m, then there exists a sufficiently large $\nu_0 > 0$ such that for $\nu \ge \nu_0$ $NP(D;\nu) = NP(\mathcal{D};\nu)$ holds.*

Definition 1. *Put $S = S_\theta(T)$ and $S_0 = S_{\theta_0}(T_0)$ Let $\gamma > 0$. $Asy^0_{\{\gamma\}}(S\times D_R)$ is the set of all functions $f(t,x) \in \mathcal{O}(S\times D_R)$ such that for any $S_0 \subset\subset S$*

$$|f(t,x)| \le C_0 \exp(-c_0|t|^{-\gamma})$$

where c_0 depends on S_0.

Then we have same result as Ōuchi's that in the case that $\sum_{|q|\ge 0} t^{\sigma_q} c_q(t,x) Z^q$ is analytic in Z.

Theorem 1. *Let $c_0(t,x) \in Asy^0_{\{\gamma^*_{\mathcal{D},1}\}}(S_\theta(T)\times D_R)$ $(0 < \theta < \pi/2\gamma^*_{\mathcal{D},1})$. Suppose that Condition 2 and 3 hold for $\mathcal{L}_i$ $(i = 1,\dots,p_{\mathcal{D}}-1)$. Then there is a solution $u(t,x) \in Asy^{\{0\}}_{\gamma^*_{\mathcal{D},1}}(S_{\theta'}(T)\times D_\rho)$ $(0 < \theta' < \pi/2\gamma^*_{\mathcal{D},p_{\mathcal{D}}-1})$ of $D(u) = 0$ for $0 < \rho < R$.*

In this paper it is main purpose that we investigate the relation between formal power series solutions and genuine solutions of some nonlinear partial differential equations. For this purpose we apply Theorem 1.

Now we consider the following equation:

$$Lu(t,x) = F\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u(t,x)\Big\}_{j+|\alpha|\le m}\Big) \tag{1.2}$$

where $L := \sum_{j+|\alpha|\le l} a_{j,\alpha}(x)(t\partial/\partial t)^j(\partial/\partial x)^\alpha$ and $a_{j,\alpha}(x) \in \mathcal{O}(D_R)$.

We define the operator $L(x,k,\partial/\partial x)$:

$$L\Big(x,k,\frac{\partial}{\partial x}\Big) := \sum_{j+|\alpha|\le l} a_{j,\alpha}(x)k^j\Big(\frac{\partial}{\partial x}\Big)^\alpha, \tag{1.3}$$

and we simply denote $L(x,k,\partial/\partial x)$ by L_k.

Assumption 1. *The operator L_k is a differential operator with order m_L.*

We put

$$P.S.L(x,k,\xi) := \sum_{j+|\alpha|=l} a_{j,\alpha}(x)k^j\xi^\alpha. \tag{1.4}$$

Assumption 2. *$P.S.L(x,k,\hat{\xi})$ is a polynomial in k with degree $l-m_L$ and does not vanish for $k=1,2,\dots$ where $\hat{\xi}=(1,0,\dots,0)\in\mathbb{N}^n$.*

Assumption 3. *The function $F(t,x,Z)$ is holomorphic in a neigborhood of the origin in (t,x,Z) and satisfies the following two conditions:*

$$i)\quad F(0,x,0)\equiv 0$$
$$ii)\quad \frac{\partial F}{\partial Z_{j,\alpha}}(0,x,0)\equiv 0 \quad \text{for} \quad j+|\alpha|\le m.$$

By Assumption 3 $F(t,x,Z)$ has the following expansion:

$$F(t,x,Z) = g(t,x) + \sum_{|q|\ge 1} t^{\sigma_q} g_q(t,x)Z^q =: g(t,x) + G_1(t,x)(Z) \tag{1.5}$$

where $g_q(0,x)\not\equiv 0$ and $\sigma_q>0$ for $|q|=1$. We may assume that the coefficients $g(t,x)$ and $g_q(t,x)$ belong to $\mathcal{O}(S_\theta(T)\times D_{R_0})$.

For thus type equations Gérard-Tahara [1] and Ōuchi [2] studied an existence of a formal power series solution with Gevrey type estimate and further Ōuchi [2] studied a formal solution of a more general form $\sum u_k(x)t^{q_k}$ with $q_k\in\mathbb{R}$.

We define the index γ_1 as follows;

$$\gamma_1 = \min_{q\in J}\left\{\frac{\sigma_q+|q|-1}{l_q-l};\ l_q>l\right\} \tag{1.6}$$

where $J=\{q:\ g_q(t,x)\not\equiv 0\}$.

Let us define some function class.

Definition 2. *Let $s\ge 0$. We define $G^{\{s\}}$ as the set of all formal series of the form $u(t,x)=\sum_{k\ge1}u_k(x)t^k$ with $u_k(x)\in\mathcal{O}(D_R)$ such that the series $\sum_{k\ge1}u_k(x)t^k/k!^s$ is convergent in a neigborhood of the origin.*

Definition 3. *Put $S=S_\theta(T)$ and $S_0=S_{\theta_0}(T_0)$. Let $\gamma>0$. $Asy_{\{\gamma\}}(S\times D_R)$ is the set of all function $f(t,x)\in\mathcal{O}(S\times D_R)$ such that for any $S_0\subset\subset S$*

$$\left|f(t,x)-\sum_{k=0}^{N-1}f_k(x)t^k\right| \le A_0B_0^N|t|^N\Gamma(N/\gamma+1)\quad t\in S_0$$

where $f_k(x)\in\mathcal{O}(D_R)$ $(k\in\mathbb{N})$ holds for constants A_0 and B_0 where A_0 and B_0 depend on S_0.

Then we have;

Theorem 2. *Suppose that the equation (1.2) satisfies Assumption 1, 2 and 3. Then the equation (1.2) has a formal solution $\widetilde{u}(t,x)=\sum_{k\ge1}u_k(x)t^k$ with*

$$\left(\frac{\partial}{\partial x_1}\right)^h u_k(0,x')\equiv 0 \quad \text{for}\quad h=0,1,\dots,m_L-1. \tag{1.7}$$

Further (1) if $l\ge m$ the formal solution is convergent in a neigborhood of the origin, (2) if $l<m$ the formal solution belongs to $G^{\{1/\gamma_1\}}$.

Remark 1. *Let (E') be an equation obtained by substituting $u(t,x) = (\partial/\partial x_1)^{m_L} w(t,x)$ into the equation (1.2). We apply Gérard-Tahara's result ([1] Chapter 10) for this equation (E') with respect to $w(t,x)$. Then we can obtain Theorem 2.*

In the case $l < m$, under some conditions we will investigate that for some sector S_1 and $r > 0$ there exists a genuine solution $u_{S_1}(t,x) \in Asy_{\{\gamma_1\}}(S_1 \times D_r)$ where γ_1 defined in (1.6).

Let us start a construction of a genuine solution. It follows from Theorem 2 that for the formal solution $\widetilde{u}(t,x)$ there exists a function $u_0(t,x) \in Asy_{\{\gamma_1\}}(S_\theta(T) \times D_R)$ with $0 < \theta < \pi/2\gamma_1$ such that for any $S_0 \subset\subset S_\theta(T)$

$$|u_0(t,x) - \sum_{k=1}^{N-1} u_k(x)t^k| \le A_0 {B_0}^N |t|^N \Gamma(N/\gamma_1 + 1) \quad \text{for} \quad t \in S_0$$

where A_0 and B_0 depend on S_0.

We construct a genuine solution as the following form:

$$u(t,x) = u_0(t,x) + v(t,x).$$

Put

$$D(u(t,x)) = Lu(t,x) - F\Big(t, x, \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u(t,x)\Big\}_{j+|\alpha|\le m}\Big).$$

We consider the following equation with respect to $v(t,x)$as follows:

$$D^{u_0}(v) := D(u_0 + v) = 0. \tag{1.8}$$

Theorem 3. *Let $S = S_\theta(T)$ with $0 < \theta < \pi/2\gamma^*_{\mathcal{D}^{u_0}, p_{\mathcal{D}^{u_0}}-1}$. Under the assumptions of Theorem 2, suppose that Condition 2 and 3 for all $i = 1, \ldots, p_{\mathcal{D}^{u_0}} - 1$ hold for the equation (1.8). Then for $S_1 \subset\subset S$ there exists a solution $u_{S_1}(t,x) \in Asy_{\{\gamma_1\}}(S_1 \times D_r)$ of the equation (1.2).*

Proof. Put $\widetilde{g}(t,x) := Lu_0(t,x) - G_1(t,x)(u_0(t,x))$. Let $g_0(t,x) = g(t,x) - \widetilde{g}(t,x)$, then we have

$$|g_0(t,x)| \le Ce^{-c|t|^{-\gamma_1}} \quad \text{for} \quad t \in S_0. \tag{1.9}$$

For the proof, see Ōuchi ([3],Proposition 5.2)

For the equation $D^{u_0}(v) = 0$ by showing $\gamma^*_{\mathcal{D}^{u_0},1} = \gamma_1$ we can obtain Theorem 3 from Theorem 1.

The equation $D^{u_0}(v) = 0$ has the following expansion:

$$\begin{aligned} Lv(t,x) = g_0(t,x) \quad &+ \quad \sum_{|q|\ge 1} t^{\sigma_q} g_q(t,x) \prod_{j+|\alpha|\le m} \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha (u_0 + v)\Big\}^{q_{j,\alpha}} \\ &- \quad \sum_{|q|\ge 1} t^{\sigma_q} g_q(t,x) \prod_{j+|\alpha|\le m} \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u_0\Big\}^{q_{j,\alpha}}. \end{aligned} \tag{1.10}$$

By $\widetilde{u}(t,x) = \sum_{k\ge 1} u_k(x)t^k$ we can put $(t\partial/\partial t)^j (\partial/\partial x)^\alpha u_0(t,x) =: tc_{j,\alpha}(t,x)$ for $c_{j,\alpha}(t,x)$ $\in \mathcal{O}(S \times D_R)$. Then $\mathcal{D}^{u_0} v(t,x)$ becomes

$$\begin{aligned} Lv(t,x) \quad - \quad &\sum_{|q|\ge 1} \sum_{j+|\alpha|\le m} q_{j,\alpha} t^{\sigma_q + |q| - 1} g_q(t,x) \{c_{j,\alpha}(t,x)\}^{q_{j,\alpha}-1} \\ \times \quad &\Big(\prod_{\substack{j'+|\alpha'|\le m \\ (j'\alpha')\ne(j,\alpha)}} \{c_{j',\alpha'}(t,x)\}^{q_{j',\alpha'}}\Big) \Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha v(t,x). \end{aligned} \tag{1.11}$$

From (1.11) we understand that $\mathcal{D}^{u_0} v(t,x)$ is constructed by all differentials of the equation (??). Therefore we have $\gamma^*_{\mathcal{D}^{u_0},1} = \gamma_1$ by the definition of γ_1. Hence we obtain Theorem 3. Q.E.D.

2 Preparation of Theorem 1

In this section it is our purpose to prepare a proof of Theorem 1. Let $0 < R < R_0$. Let us define an operator P as follows:

$$P := \left(t\frac{\partial}{\partial t}\right)^{L^*-m^*}\left(\frac{\partial}{\partial x_1}\right)^{m^*} - \sum_{\substack{|\alpha|=m^* \\ \alpha_1<m^*}} a_\alpha(t,x)\left(t\frac{\partial}{\partial t}\right)^{L^*-m^*}\left(\frac{\partial}{\partial x}\right)^{\alpha} - \sum_{\substack{L\le L^* \\ |\alpha|<m^*}} t^{-\gamma(L^*-L)} b_{L,\alpha}(t,x)\left(t\frac{\partial}{\partial t}\right)^{L-|\alpha|}\left(\frac{\partial}{\partial x}\right)^{\alpha}$$

where $a_\alpha(t,x)$ and $b_{L,\alpha}(t,x)$ belong to $\mathcal{O}(S_\theta(T)\times D_R)$.

Let us consider the following equation:

$$Pu = \sum_{|q|\ge 1} t^{\sigma_q-\sigma_{L^*}} c_q(t,x) \prod_{j+|\alpha|\le m}\left\{\left(t\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial x}\right)^{\alpha} u\right\}^{q_{j,\alpha}} + g(t,x) \tag{2.1}$$

where $c_q(t,x) \in \mathcal{O}(S_\theta(T)\times D_R)$ for $|q|\ge 1$ and $g(t,x) \in Asy^0_{\{\gamma\}}(S_\theta(T)\times D_R)$, and the orders $\sigma_q - \sigma_{L^*}$ satisfy as follows;

$$\sigma_q - \sigma_{L^*} = \begin{cases} -\gamma(L^*-l_q)+J_q^1 & (J_q^1>0) \quad \text{for} \quad l_q\le L^* \\ \gamma^*(l_q-L^*)+J_q^2 & (J_q^2\ge 0) \quad \text{for} \quad l_q > L^* \end{cases} \tag{2.2}$$

where $0\le\gamma<\gamma^*\le\infty$. If $\{q;\ l_q>L^*\}=\emptyset$ then we define $\gamma^*=\infty$.

Let us introduce the following functional class $X_{p,q,c,\gamma}$ where $p\in\mathbb{N}$ and $q,c,\gamma\ge 0$.

Let $\rho>0$. For $\varphi(x)=\sum_{\beta\in\mathbb{N}^n} a_\beta x^\beta$ we define the norm $||\varphi||_\rho$ by

$$||\varphi||_\rho = \sum_{\beta\in\mathbb{N}^n} |a_\beta|\frac{\beta!}{|\beta|!}\tau^{\beta_1}\rho^{|\beta|}. \tag{2.3}$$

For a fixed $a>0$ we put

$$\Theta^{(k)} = \frac{ak!}{(k+1)^{m+2}} \quad \text{and} \quad \Theta^{(k)}_{R-\rho} = \frac{1}{(R-\rho)^k}\Theta^{(k)}.$$

Definition 4. *$X_{p,q,c,\gamma}(S_\theta(T)\times D_\rho)$ is the set of all function $\varphi(t,x)\in\mathcal{O}(S_\theta(T)\times D_\rho)$ with the following bounds; There exists a positive constant Φ such that for all $s\in\mathbb{N}$*

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^{s}\varphi(t,\cdot)\right|\right|_\rho \le \Phi\zeta^s|t|^q\exp(-c|t|^{-\gamma})\Theta^{(s+p)} \quad \textit{for} \quad t\in S_\theta(T). \tag{2.4}$$

The norm of $\varphi(t,x)$ is defined by the infimum of Φ in (2.4) and is denoted by $||\varphi||_{p,q,c,\gamma}$.

This definition is a little bit different from that in Ōuchi [3].

We fix a positive constant δ such that $0<\delta<\min\{J_q^1;\ q \text{ with } l_q\le L^*\}$ and $\gamma^*/\delta\in\mathbb{N}$. We define P_k by $P_k=[\delta k/\gamma*]+(L^*-m^*)k$. If $\{q;\ l_q>L^*\}=\emptyset$ then $P_k=(L^*-m^*)k$ by $\gamma^*=\infty$ where $[a]$ denote the integral part of a.

Theorem 4. *Let $S=S_\theta(T)$. Suppose that for all $s\in\mathbb{N}$ there exists a positive constant G such that*

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^{s} g(t,\cdot)\right|\right|_R \le G\zeta^s\exp(-c_0|t|^{-\gamma})\Theta^{(s+m^*)} \quad \textit{for} \quad t\in S. \tag{2.5}$$

Then the equation (2.1) has a formal solution $u(t,x)=\sum_{k\geq 0}u_k(t,x)$ *that satisfies for* $k\geq 0$

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s u_k(t,\cdot)\right|\right|_\rho \leq U_k\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)k)!}\Theta_{R-\rho}^{(s+P_k)} \quad \text{for} \quad t\in S \tag{2.6}$$

for a sufficiently small $T>0$, $0<\rho<R$ *and* $0<c<c_0$ *where the series* $\sum_{k\geq 0}U_k t^k$ *converges in a neigborhood of the origin* $t=0$.

Let us introduce some lemmas for the functional class $X_{p,q,c,\gamma}$.

Lemma 2. *Assume*

$$||u||_\rho \leq \Theta_{R-\rho}^{(k)} \quad \text{for} \quad 0<\rho<R. \tag{2.7}$$

(1) Let $k>0$. *Then we have*

$$\left|\left|\frac{\partial}{\partial x_1}u\right|\right|_\rho \leq \frac{M_0 e}{\tau}\Theta_{R-\rho}^{(k+1)} \quad \text{for} \quad 0<\rho<R \tag{2.8}$$

and have

$$\left|\left|\frac{\partial}{\partial x_i}u\right|\right|_\rho \leq M_0 e\Theta_{R-\rho}^{(k+1)} \quad \text{for} \quad 0<\rho<R \tag{2.9}$$

for $i=2,\ldots,n$ *where* $M_0=2^{m+2}$.
(2) Let $k>1$. *Then we have*

$$\left|\left|\left(\frac{\partial}{\partial x_1}\right)^{-1}u\right|\right|_\rho \leq 2\tau\Theta_{R-\rho}^{(k-1)} \quad \text{for} \quad 0<\rho<R. \tag{2.10}$$

Proposition 1. *Let* $0\leq L'\leq L\leq m$ *and* $P,\ P'>0$. *For functions* $u(t,x)$ *and* $v(t,x)$ *we assume that there exist positive constants* U *and* V *such that for* $t\in S$

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s u\right|\right|_\rho \leq U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta^{(s+P+L)}$$

and

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s v\right|\right|_\rho \leq V\zeta^s|t|^{q'}\exp(-c|t|^{-\gamma})\frac{1}{P'!}\Theta^{(s+P'+L')}.$$

Then we have for $t\in S$

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s (uv)\right|\right|_\rho \leq UV\zeta^s|t|^{q+q'}\exp(-c|t|^{-\gamma})\frac{1}{(P+P')!}\Theta^{(s+P+P'+L)}.$$

Proposition 2. *Let* $P\geq 0$. *For a function* $u(t,x)$ *we assume that there exists a positive constant* U *such that for* $t\in S$

$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s u\right|\right|_\rho \leq U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta_{R-\rho}^{(s+P)}.$$

Then we have the following estimates;
there exists a positive constant C *such that for* $t\in S$

(1)
$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s\left\{\left(t\frac{\partial}{\partial t}\right)^{-1}u\right\}\right|\right|_\rho \leq \frac{|t|^{\gamma+1}}{c\gamma}U\zeta^s\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta_{R-\rho}^{(s+P)} \quad \text{if} \quad q=0,$$
$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s\left\{\left(t\frac{\partial}{\partial t}\right)^{-1}u\right\}\right|\right|_\rho \leq \frac{1}{q}U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta_{R-\rho}^{(s+P)} \quad \text{if} \quad q>0,$$

(2)
$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s\left\{t^{-\gamma}\left(t\frac{\partial}{\partial t}\right)^{-1}u\right\}\right|\right|_\rho \leq \frac{C}{c\gamma}U\zeta^s|t|^q\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta_{R-\rho}^{(s+P)},$$

(3)
$$\left|\left|\left(t\frac{\partial}{\partial t}\right)^s(t^{-\gamma}u)\right|\right|_\rho \leq \frac{C}{c\gamma}U\zeta^{s+1}|t|^q\exp(-c|t|^{-\gamma})\frac{1}{P!}\Theta_{R-\rho}^{(s+P+1)}.$$

Let us prove Theorem 4. We consider a solution of the following equation:

$$P'w(t,x) = W(t,x) \tag{2.11}$$

where

$$P' := P\Big(t\frac{\partial}{\partial t}\Big)^{-L^*+m^*}\Big(\frac{\partial}{\partial x}\Big)^{-m^*} \tag{2.12}$$

For the equation (2.11) we assume for $0<\rho<R$ and $t\in S$

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s W\Big|\Big|_\rho \le W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P+m^*)}.$$

Let $A_\alpha = ||a_\alpha||_{0,0,0,\gamma}$ and $B_{L,\alpha} = ||b_{L,\alpha}||_{0,0,0,\gamma}$.

Proposition 3. *Let $P\ge 0$. The equation (2.11) has a singular solution $w(t,x)$ that satisfies*

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s w\Big|\Big|_\rho \le \frac{1}{1-C(\zeta,\tau)}W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P+m^*)}$$

for $t\in S$ and $0<\rho<R$ where

$$C(\zeta,\tau) := \sum_{\substack{|\alpha|=m^*\\ \alpha_1<m^*}} A_\alpha(M_0e)^{|\alpha'|}(2\tau)^{m^*-\alpha_1} + \sum_{\substack{L\le L^*\\ |\alpha|<m^*}} B_{L,\alpha}\Big(\frac{C}{c\gamma}\Big)^{L^*-L}\zeta^{m^*-|\alpha|}(M_0e)^{|\alpha'|}(2\tau)^{m^*-\alpha_1}.$$

Proof. We construct a solution $w(t,x)=\sum_{\beta=0}^\infty w_\beta(t,x)$ as follows:

$$\begin{aligned} w_0(t,x) &= W(t,x)\\ w_\beta(t,x) &= \sum_{\substack{|\alpha|=m^*\\ \alpha_1<m^*}} a_\alpha(t,x)\Big(\frac{\partial}{\partial x}\Big)^{\alpha-e_1m^*} w_{\beta-1}(t,x)\\ &\quad+\sum_{\substack{L\le L^*\\ |\alpha|<m^*}} t^{-\gamma(L^*-L)}b_{L,\alpha}(t,x)\Big(t\frac{\partial}{\partial t}\Big)^{L-L^*+m^*-|\alpha|}\Big(\frac{\partial}{\partial x}\Big)^{\alpha-e_1m^*} w_{\beta-1}(t,x) \end{aligned}$$

for $\beta\ge 1$.

By Lemma 2, Proposition 1 and Proposition 2-(3) we can show that the following estimate holds for $\beta\ge 0$:

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s w_\beta\Big|\Big|_\rho \le \{C(\zeta,\tau)\}^\beta W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P+m^*)} \tag{2.13}$$

for $t\in S$ and $0<\rho<R$.

By the definition of $C(\zeta,\tau)$ we have $C(\zeta,\tau)<1$ for a sufficiently small $\tau>0$. Hence for $w(t,x)=\sum_{\beta\ge 0} w_\beta(t,x)$ we have

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s w\Big|\Big|_\rho \le \frac{1}{1-C(\zeta,\tau)}W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P+m^*)}.$$

Q.E.D.

By Proposition 3 we have the following proposition;

Proposition 4. *Let $P \geq 1$. For an equation*

$$Pu(t,x) = W(t,x)$$

assume that there exists a positive constant W such that for $t \in S$

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s W\Big|\Big|_\rho \leq W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta^{(s+P+m^*)}_{R-\rho}.$$

Then we have for $t \in S$

$$\begin{aligned}\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s u\Big|\Big|_\rho &\leq \Big(\frac{L^*-m^*}{\delta}\Big)^{L^*-m^*}(2\tau)^{m^*}\frac{1}{1-C(\zeta,\tau)}W\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma}) \\ &\times \frac{1}{((L^*-m^*)k)!}\Theta^{(s+P)}_{R-\rho}.\end{aligned}$$

Proof. For the solution $w(t,x)$ of the equation (2.11), we have

$$u(t,x) = \Big(t\frac{\partial}{\partial t}\Big)^{-(L^*-m^*)}\Big(\frac{\partial}{\partial x_1}\Big)^{-m^*} w(t,x). \tag{2.14}$$

By Lemma 2-(2), Proposition 2-(1) and Proposition 3, we obtain the desired result. Q.E.D.

Proof of Theorem 4.
We put $C_q := ||c_q||_{0,0,0,\gamma}$. We construct a formal solution $u(t,x) = \sum_{k\geq 0} u_k(t,x)$ of the equation (2.1) by same method as Ōuchi [3]:

$$\begin{aligned}
&Pu_0(t,x) = g(t,x) \\
&Pu_k(t,x) = \sum_{\substack{1\leq|q|\leq k \\ l_q\leq L^*}} t^{\sigma_q-\sigma_{L^*}}c_q(t,x)\sum_{|k(q)|+1=k}\prod_{j+|\alpha|\leq m}\prod_{i=1}^{q_{j,\alpha}}\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{k_i}(t,x) \\
&\qquad + \sum_{\substack{1\leq|q|\leq k \\ l_q> L^*}} t^{\sigma_q-\sigma_{L^*}}c_q(t,x)\sum_{|k(q)|+\frac{\gamma^*}{\delta}(l_q-L^*)=k}\prod_{j+|\alpha|\leq m}\prod_{i=1}^{q_{j,\alpha}}\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{k_i}(t,x) \\
&\qquad =: W_{1,k}(u_{k'};\ k'<k) + W_{2,k}(u_{k'};\ k'<k) =: W_k(u_{k'};\ k'<k).
\end{aligned} \tag{2.15}$$

Let us give $u_k(t,x)$ $(k \geq 0)$ an estimate. By the assumption we have

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s g\Big|\Big|_R \leq G\zeta^s\exp(-c_0|t|^{-\gamma})\Theta^{(s+m^*)}.$$

By Lemma 2-(2), Proposition 2-(1) and 3 we have for $t \in S = S_\theta(T)$

$$\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s u_0\Big|\Big|_\rho \leq G\zeta^s\exp(-c|t|^{-\gamma})\Theta^{(s)}_{R-\rho}$$

for a sufficiently small $T > 0$. So we put $U_0 = G$, then we have the desired result for $k = 0$.

At next we assume that we have the desired result for $k' = 0, 1, \ldots, k-1$. By Lemma 2-(1) for $k_i < k$ we have

$$\begin{aligned}&\Big|\Big|\Big(t\frac{\partial}{\partial t}\Big)^s\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{k_i}\Big\}\Big|\Big|_\rho \\ &\leq \frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\zeta^s|t|^{\delta k_i}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)k_i)!}\Theta^{(s+P_{k_i}+j+|\alpha|)}_{R-\rho}.\end{aligned}$$

Put $P' = \sum_{j+|\alpha|\le m}\sum_{i=1}^{q_{j,\alpha}} P_{k_i}$. We have $\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}} P_{k_i}!/((L^*-m^*)k_i)! \le P'!/((L^*-m^*)|k(q)|!$, then by Proposition 1 we have

$$\begin{aligned}
&\Big\|\Big(t\frac{\partial}{\partial t}\Big)^s\Big\{c_q\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u_{k_i}\Big\}\Big\|_\rho\\
&\le \frac{C_q}{(R-\rho)^{l_q(|q|-1)}}\Big\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\Big\}\zeta^s|t|^{\delta|k(q)|}\exp(-c|t|^{-\gamma})\\
&\times\frac{1}{((L^*-m^*)|k(q)|)!}\Theta_{R-\rho}^{(s+P'+l_q)}.
\end{aligned}\tag{2.16}$$

Under $1\le|q|\le k$ and $|k(q)|+1=k$, it follows from $\sigma_q-\sigma_{L^*}=-\gamma(L^*-l_q)+J_q^1$, $P'+l_q\le P_k+l_q-L^*+m^*$ and $P_k+m^*-(P'+L^*)\le|q|$ that by Proposition 2-(3) we have

$$\begin{aligned}
\Big\|\Big(t\frac{\partial}{\partial t}\Big)^s W_{1,k}\Big\|_\rho \le& \sum_{\substack{1\le|q|\le k\\ l_q\le L^*}}\Big(\frac{C}{c\gamma}\Big)^{L^*-l_q}\frac{C_q}{(R-\rho)^{l_q(|q|-1)}}\sum_{|k(q)|+1=k}\Big\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\Big\}\\
&\times\zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P_k+m^*)}.
\end{aligned}$$

At next let us estimate $W_{2,k}(t,x)$. As $W_{1,k}(t,x)$ under the conditions $1\le|q|\le k$ and $|k(q)|+\frac{\gamma^*}{\delta}(l_q-L^*)=k$ the inequality (2.16) holds for $t\in S$. By $P_k+m^*-(P'+l_q)\ge(L^*-m^*)(k-|k(q)|-1)\ge0$ and $\sigma_q-\sigma_{L^*}=\gamma^*(l_q-L^*)+J_q^2$ and $P_k+m^*-(P'+l_q)\le P_k+L^*-(P'+L)\le|q|$ we have for $t\in S$

$$\begin{aligned}
\Big\|\Big(t\frac{\partial}{\partial t}\Big)^s W_{2,k}\Big\|_\rho \le& \sum_{\substack{1\le|q|\le k\\ L_q>L^*}}\frac{M_0^{|q|}C_q}{(R-\rho)^{L_q(|q|-1)}}\sum_{|k(q)|+\gamma^*(l_q-L^*)/\delta=k}\Big\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\Big\}\\
\times&\ \zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P_k+m^*)}.
\end{aligned}$$

Therefore we obtain the following estimate for $W_k=W_{1,k}+W_{2,k}$:

$$\begin{aligned}
\Big\|\Big(t\frac{\partial}{\partial t}\Big)^s W_k\Big\|_\rho \le& \sum_{\substack{1\le|q|\le k\\ l_q\le L^*}}\Big(\frac{C}{c\gamma}\Big)^{L^*-l_q}\frac{M_0^{|q|}C_q}{(R-\rho)^{l_q(|q|-1)}}\sum_{|k(q)|+1=k}\Big\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\Big\}\\
\times&\ \zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P_k+m^*)}\\
+&\sum_{\substack{1\le|q|\le k\\ l_q>L^*}}\frac{M_0^{|q|}C_q}{(R-\rho)^{l_q(|q|-1)}}\sum_{|k(q)|+\gamma^*(l_q-L^*)/\delta=k}\Big\{\prod_{j+|\alpha|\le m}\prod_{i=1}^{q_{j,\alpha}}\frac{\zeta^j(M_0e)^{|\alpha|}}{\tau^{\alpha_1}}U_{k_i}\Big\}\\
\times&\ \zeta^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P_k+m^*)}.
\end{aligned}$$

Put

$$M:=\Big(\frac{L^*-m^*}{\delta}\Big)^{L^*-m^*}(2\tau)^{m^*}\frac{1}{1-C(\zeta,\tau)}.$$

By Proposition 4, we have for $t\in S$

$$\Big\|\Big(t\frac{\partial}{\partial t}\Big)^s u_k\Big\|_\rho\le U_k{\xi_0}^s|t|^{\delta k}\exp(-c|t|^{-\gamma})\frac{1}{((L^*-m^*)(k-1))!}\Theta_{R-\rho}^{(s+P_k+m^*)}$$

where

$$U_k = M \sum_{\substack{1\le|q|\le k\\ l_q\le L^*}} \Big(\frac{C}{c\gamma}\Big)^{L^*-l_q} \frac{M_0^{|q|}C_q}{(R-\rho)^{l_q(|q|-1)}} \sum_{|k(q)|+1=k} \Big\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k_i} \Big\}$$

$$+ \; M \sum_{\substack{1\le|q|\le k\\ l_q> L^*}} \frac{M_0^{|q|}C_q}{(R-\rho)^{l_q(|q|-1)}} \sum_{|k(q)|+\gamma^*(l_q-L^*)/\delta=k} \Big\{ \prod_{j+|\alpha|\le m} \prod_{i=1}^{q_{j,\alpha}} \frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} U_{k_i} \Big\}.$$

At last we show that $\sum_{k\ge 0} U_k t^k$ is a convergent power series in a neigborhood of the origin. Here let us consider the following equation:

$$Y = G + Mt \sum_{\substack{|q|\ge 1\\ l_q\le L^*}} \Big(\frac{C}{c\gamma}\Big)^{L^*-l_q} \frac{M_0^{|q|}C_q}{(R-\rho)^{m(|q|-1)}} \prod_{j+|\alpha|\le m} \Big(\frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} Y\Big)^{q_{j,\alpha}}$$

$$+ \; M \sum_{\substack{|q|\ge 1\\ l_q> L^*}} t^{(l_q-L^*)\gamma^*/\delta} \frac{M_0^{|q|}C_q}{(R-\rho)^{m(|q|-1)}} \prod_{j+|\alpha|\le m} \Big(\frac{\zeta^j (M_0 e)^{|\alpha|}}{\tau^{\alpha_1}} Y\Big)^{q_{j,\alpha}}.$$

Remark 2. *The right hand side in the above equation is a convergent power series in Y.*

By substituting $Y(t) = \sum_{k\ge 0} Y_k t^k$ into the above equation, we have $U_k \le Y_k$ for $k \ge 0$. So it is sufficient to prove that the above equation has a holomorphic solution. By implicit function theorem at $(t, Y) = (0, G)$ we have a holomorphic solution with $Y(0) = G$. Hence $\sum_{k\ge 0} U_k t^k$ converges in a neigborhood of the origin $t = 0$. Q.E.D.

3 Proof of Theorem 1

In this section we give a proof of Theorem 1. By Theorem 4 we obtain the following estimate; there exist positive constants A and B such that for $t \in S_\theta(T)$

$$(3.1)\qquad \begin{aligned} &||u_k(t,\cdot)||_\rho \le AB^k|t|^{\delta k}\exp(-c|t|^{-\gamma}) \quad \text{if} \quad \{q;\ l_q > L^*\} = \emptyset \\ &||u_k(t,\cdot)||_\rho \le AB^k|t|^{\delta k}\exp(-c|t|^{-\gamma})\Gamma\Big(\frac{\delta k}{\gamma^*}+1\Big) \quad \text{if} \quad \{q;\ l_q > L^*\} \ne \emptyset. \end{aligned}$$

By (3.1), if $\{q;\ l_q > L^*\} = \emptyset$ then the formal solution $\sum_{k\ge 0} u_k(t,x)$ in Theorem 4 becomes a genuine solution of the equation (2.1).

From now we discuss the case $\{q;\ l_q > L^*\} \ne \emptyset$. It is our purpose to show the following two propositions;

Let $S = S_\theta(T)$ and $S_0 = S_{\theta_0}(T_0)$ with $|\theta_0| \le \pi/(2\gamma^*)$.Put

$$g_{S_0}(t,x) := Pu_{S_0}(t,x) - \sum_{|q|\ge 1} t^{\sigma_q-\sigma_{L^*}} c_q(t,x) \prod_{j+|\alpha|\le m} \Big\{\Big(t\frac{\partial}{\partial t}\Big)^j \Big(\frac{\partial}{\partial x}\Big)^\alpha u_{S_0}(t,x)\Big\}^{q_{j,\alpha}} - g(t,x).$$

Proposition 5. *Let $u_k(t,x)$ be constructed in Theorem 4. Then there exists a function $u_{S_0}(t,x) \in Asy^0_{\{\gamma*\}}(S_0 \times D_\rho)$ such that*

$$||u_{S_0} - \sum_{k=0}^{N} u_k||_\rho \le AB^{N+1}\Gamma\Big(\frac{(N+1)\delta}{\gamma*}+1\Big)|t|^{(N+1)\delta}\exp(-c|t|^{-\gamma}) \quad \textit{for} \quad t \in S_0$$

with $|\theta_0| < \pi/(2\gamma*)$.

Proposition 6. *We have* $g_{S_0}(t,x) \in Asy^0_{\{\gamma *\}}(S_0 \times D_\rho)$ *for* $0 < \rho_0 < \rho$.

Proof of Theorem 1

We can prove Theorem 1 by applying Theorem 4, Proposition 5 and 6. We omit a detailed proof. Q.E.D.

References

[1] Gérard, R. and Tahara, H., *Singular Nonlinear Partial Differential Equations*, Vieweg, 1996.

[2] Ōuchi, S., *Formal solutions with Gevrey type estimates of nonlinear partial differential equations*, J. Math. Sci. Univ. Tokyo, 1(1994), 205–237.

[3] ———, *Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations*, J. Math. Sci. Univ Tokyo, 2(1995), 375–417.