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On Convergence of Sum of White noises to Periodic Stochastic Process

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Abstract

This paper gives a proof of the convergence of a weighted linear sum of white noises to a periodic stochastic process. This problem is rooted in the classical problem on macroeconomic dynamics, especially business cycle theory that the source of energy which maintains the economic cycles are erratic shocks. In order to solve this problem, we represent the covariance of the sum of white noises as a characteristic function by making use of a spectral measure and show that the spectral measure weakly converges to a measure by which the periodic stochastic process can be represented.

1 Introduction

Much interest has long been taken in the problem of the economic oscillations by many mathematical economists. It is concerned with a question of whether the time series of economic growth could be well elucidated by a certain relevant economic model. The major difficulty underlying this problem is that if one adopts an aggregate model, in other words, a one sector growth model, the dynamics deduced from the model, in most cases, do not describe sustained fluctuating paths. In a seminal paper, Frisch(1933) constructed a dynamic macroeconomic model and showed that the paths obtained from his model damped to 0. In order to reconcile the theory to the actual data which we observe in reality, he needed to overcome this pathology and so suggested an alternative way which left a great influence to the successors. He introduced a stream of erratic shocks that constantly upsets the evolution. By doing so he conjectured that the system have an energy necessary to maintain the swings. Along this line of thought, since the optimal paths of a one sector optimal growth model in the sense of Ramsey-Cass-Koopmans model monotonously converges to a steady state (Ramsey(1928), Cass(1966), Koopmans(1965)), the so called Real Business Cycle Theory in which the exogenous random shocks are inserted into the one sector model has been brought about(e.g. Kydland and Prescott (1982), Long and Plosser (1983)). In these celebrated works, the experimental numerical results are successfully established. However, the exact and general mathematical law about the movement of the *cumulation* of the random shocks have not been discovered yet.

The objective of this paper is to give an analytical explanation for the assertion that the cumulation of stochastic shocks would possibly be a cyclic stochastic process.

In order to solve this problem, we are greatly indebted to the monumental paper of Slutzky(1937) for the method of assigning the weights to the series of erratic shocks. Sargent(1987) tried to prove that the Slutzky-way-weighted linear sum of the white noises converges to a periodic stochastic process as the number of the sum goes to infinity by representing the covariance of the sum of white noises as a characteristic function by making use of a spectral measure. However he did not show that the spectral measure weakly converges to a measure by which the periodic stochastic process can be represented, which means that the linear sum could never converges to a periodic stochastic process. In this paper we show that a weighted linear sum of white noises on the basis of Slutzky converges to a periodic stochastic process in probability and also give the sufficient conditions for almost everywhere convergence.

This paper is organized as follows. The next section offers the model. The third section presents our results. The fourth section gives a concluding remarks.

2 The Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let E denote the associated expectation operator, namely, $E[x] = \int x(w)dP$ for any random variable $x : \Omega \to \mathbb{R}$. Let $\varepsilon : \mathbb{Z} \times \Omega \to \mathbb{R}$ be a stochastic process which satisfies the following conditions; $\varepsilon_t \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and

$$E[\varepsilon_t] = 0, \qquad E[\varepsilon_t^2] = \sigma^2, \qquad E[\varepsilon_t \varepsilon_s] = 0 \quad t \neq s$$

where $\varepsilon(t, \omega) = \varepsilon_t(\omega)$. This stochastic process is called the white noise. We define a operator L^n so that;

$$L^n \varepsilon_t = \varepsilon_{t-n}, \qquad n \in \mathbb{N}.$$

We named it a lag operator. We consider the following stochastic process.

$$y_t^n(\omega) = (1-L)^{\alpha n}(1+L)^n \varepsilon_t(\omega), \qquad \alpha \in \mathbb{N},\ n \in \mathbb{N}$$

where $L^1 = L$. Write $(1-L)^{\alpha n}(1+L)^n$ as $b_0 + b_1 L + b_2 L^2 + b_3 L^3 \cdots + b_{(1+\alpha)n} L^{(1+\alpha)n}$. Then,

$$y_t^n(\omega) = b_0 \varepsilon_t(\omega) + b_1 \varepsilon_{t-1}(\omega) + b_2 \varepsilon_{t-2}(\omega) \cdots + b_{(1+\alpha)n} \varepsilon_{t-(1+\alpha)n}(\omega).$$

Note that $E[(y_t^n)^2] = (b_0^2 + b_1^2 + \cdots b_{(1+\alpha)n}^2)\sigma^2$ does not depend on ω and t. Let

$$A_n = \frac{1}{\sqrt{E[(y_t^n)^2]}}$$

and define

$$Y_t^n(\omega) = A_n(1-L)^{\alpha n}(1+L)^n \varepsilon_t(\omega).$$

We see that $Y_t \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and the convariance of this is

$$E[Y_t^n Y_s^n] = A_n^2 \Big(\sum_{j=0}^{(1+\alpha)n-u} b_j b_{j+u} \Big) \sigma^2, \quad u = t-s \geq 0.$$

The above covariance only depends on the difference of the comparing periods. In general, the $\mathcal{L}^2$ stochastic process of which the covariance satisfies such a property is called a weakly stationary stochastic process. So $\{Y_t^n\}$ is a weakly stationary stochastic process. We write the covariance function as $\rho_n(u)$. We can easily see that $\rho_n(u) = \rho_n(-u)$ and $\rho_n(u) = 0$ for $u > (1+\alpha)n$.

3 Results

First of all, let us state the main theorem in this paper. Let $\{Y_t^n\}$ be the weakly stationary stocahstic process constructed in the previous section.

Theorem 1. There exists a stochastic process $X_t^n(\omega)$ which is periodic in t for each n and ω such that

$$\|Y_t^n(\omega) - X_t^n(\omega)\|_{\mathcal{L}^2(\Omega)} \longrightarrow 0 \qquad (n \to \infty)$$

for all $t \in \mathbb{Z}$.

This theorem could be interpreted as follows. Let us now consider the case that for every $t (\in \mathbb{Z})$ a shock comes down from somewhere the magnitude of which is ε_t. Then Y_t^n may be considered as a cumulation of these random shocks from the time $t-(1+\alpha)n$ to t. Theorem 1 claims that when the n which expresses the number of the cumulation goes to infinity, Y_t^n converges to the periodic stochastic process in probability, which represents the situation that a cumulation of random shocks eventually leads to periodicity.

To prove this result, we need some following lemmas. Let ρ_n be a covariance function defined at the previous section. We define a spectral measure as follows.

Definition. A Radon measure μ_n [1] on $[-\pi,\pi]$ is a spectral measure of ρ_n *if* it satisfies

$$\rho_n(t)=\int_{-\pi}^{\pi} e^{it\theta}\mu_n(d\theta) \quad t\in\mathbb{Z}.$$

Lemma 1. We can caluculate the spectral probability density function of the covariance ρ_n as follows,

$$p_n(\theta)=\frac{1}{2\pi}\sum_{u=0,\pm1,\pm2,\cdots\pm(1+\alpha)n}\rho_n(u)e^{-iu\theta}$$

(proof) We have to show that $p_n(\theta)d\theta$ is a spectral probability measure of the covariance $\rho_n(t)$, namely, it satisfies $p_n(\theta)\geq 0$, $\int_{-\pi}^{\pi}p_n(\theta)d\theta=1$ and

$$\rho_n(t)=\int_{-\pi}^{\pi} e^{it\theta}p_n(\theta)d\theta,\quad t\in\mathbb{Z}. \tag{1}$$

The right hand side of (1)

$$=\int_{-\pi}^{\pi} e^{it\theta}\Big(\frac{1}{2\pi}\sum_{u=0,\pm1,\pm2,\cdots\pm(1+\alpha)n}\rho_n(u)e^{-iu\theta}\Big)d\theta$$

$$=\int_{-\pi}^{\pi} e^{it\theta}\Big[\frac{1}{2\pi}\Big(\rho_n(0)+\sum_{u=1}^{(1+\alpha)n}\rho_n(u)[e^{-iu\theta}+e^{iu\theta}]\Big)\Big]d\theta$$

$$=\int_{-\pi}^{\pi} e^{it\theta}\Big[\frac{1}{2\pi}\Big(\rho_n(0)+\sum_{u=1}^{(1+\alpha)n}2\rho_n(u)\cos u\theta\Big)\Big]d\theta$$

$$=\frac{1}{2\pi}\rho_n(0)\int_{-\pi}^{\pi}e^{it\theta}d\theta+\frac{1}{\pi}\sum_{u=1}^{(1+\alpha)n}\rho_n(u)\int_{-\pi}^{\pi}e^{it\theta}\cos u\theta d\theta$$

[1] Let X be a topological space and $\mathcal{B}(X)$ be a Borel - σ - field. Consider a finite measure μ defined on that. Then μ is a Radon measure *if* for any $A\in\mathcal{B}(X)$ and $\varepsilon>0$ there exists a compact set $K\subset A$ such that $\mu(A\setminus K)<\varepsilon$.

$$
\begin{aligned}
&= \frac{1}{2\pi}\rho_n(0)\Big(\int_{-\pi}^{\pi} \cos t\theta d\theta + i\int_{-\pi}^{\pi} \sin t\theta d\theta\Big) \\
&\quad + \frac{1}{\pi}\sum_{u=1}^{(1+\alpha)n} \rho_n(u)\Big(\int_{-\pi}^{\pi} \cos t\theta \cos u\theta d\theta + i\int_{-\pi}^{\pi} \sin t\theta \cos u\theta d\theta\Big)
\end{aligned}
$$

Note that

$$
\int_{-\pi}^{\pi} \cos t\theta d\theta = \begin{cases} 2\pi & t = 0 \\ 0 & t \neq 0, \end{cases}
$$

$$
\int_{-\pi}^{\pi} \cos t\theta \cos u\theta d\theta = \begin{cases} \pi & u = t \\ 0 & u \neq t, \end{cases}
$$

and for all $t, u \in \mathbb{Z}$

$$
\int_{-\pi}^{\pi} \sin t\theta d\theta = 0,
$$

$$
\int_{-\pi}^{\pi} \sin t\theta \cos u\theta d\theta = 0.
$$

Then,

$$
\text{The right hand side of (1)} = \begin{cases} \rho_n(t) & t = 0, \pm 1, \pm 2, \cdots \pm (1+\alpha)n \\ 0 & \text{otherwise} \end{cases}
$$

Therefore we obtain (1). Note that

$$
\int_{-\pi}^{\pi} p_n(\theta)d\theta = \rho_n(0) = E[(Y_t^n)^2] = \frac{(b_0^2 + b_1^2 + \cdots b_{(1+\alpha)n}^2)\sigma^2}{(b_0^2 + b_1^2 + \cdots b_{(1+\alpha)n}^2)\sigma^2} = 1
$$

for all $n \in \mathbb{N}$. Let us now caluculate $p_n(\theta)$ concretely.

$$
\begin{aligned}
p_n(\theta) &= \frac{1}{2\pi} \sum_{u=0,\pm 1,\pm 2,\cdots \pm (1+\alpha)n} \rho_n(u)e^{-iu\theta} \\
&= \frac{1}{\pi}[\rho_n(0) + \rho_n(1)[e^{-i\theta} + e^{i\theta}] + \rho_n(2)[e^{-2i\theta} + e^{2i\theta}] \\
&\qquad + \cdots + \rho_n((1+\alpha)n)[e^{-(1+\alpha)ni\theta} + e^{(1+\alpha)ni\theta}]]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi} A_n^2 \sigma^2 [\sum_{j=0}^{(1+\alpha)n} b_j^2 + \sum_{j=1}^{(1+\alpha)n} b_j b_{j-1}[e^{-i\theta} + e^{i\theta}] + \sum_{j=2}^{(1+\alpha)n} b_j b_{j-2}[e^{-2i\theta} + e^{2i\theta}] \\
&\qquad + \cdots + \sum_{j=(1+\alpha)n}^{(1+\alpha)n} b_j b_{j-(1+\alpha)n}[e^{-(1+\alpha)ni\theta} + e^{(1+\alpha)ni\theta}]] \\
&= \frac{1}{2\pi} A_n^2 \sigma^2 (b_0 + b_1 e^{-i\theta} + b_2 e^{-2i\theta} + \cdots b_{(1+\alpha)n} e^{-(1+\alpha)ni\theta}) \\
&\qquad \times (b_0 + b_1 e^{i\theta} + b_2 e^{2i\theta} + \cdots b_{(1+\alpha)n} e^{(1+\alpha)ni\theta}) \\
&= \frac{1}{2\pi} A_n^2 \sigma^2 (1 - e^{-i\theta})^{\alpha n} (1 + e^{-i\theta})^n (1 - e^{i\theta})^{\alpha n} (1 + e^{i\theta})^n \\
&= \frac{1}{2\pi} A_n^2 \sigma^2 [(1 - e^{-i\theta})(1 - e^{i\theta})]^{\alpha n} [(1 + e^{-i\theta})(1 + e^{i\theta})]^n \\
&= \frac{1}{2\pi} A_n^2 \sigma^2 [2(1 - \cos\theta)]^{\alpha n} [2(1 + \cos\theta)]^n \\
&= \frac{1}{2\pi} A_n^2 \sigma^2 2^{(1+\alpha)n} (1 - \cos\theta)^{\alpha n} (1 + \cos\theta)^n. \qquad (*)
\end{aligned}
$$

So we can say $p_n(\theta) \geq 0$. Then we get the desired results. $\square$

In the following, A_n^2 is more to be specific. Considering the fact $\int_{-\pi}^{\pi} p_n(\theta) d\theta = 1$ and $(*)$ we get

$$
\int_{-\pi}^{\pi} (1 - \cos\theta)^{\alpha n} (1 + \cos\theta)^n d\theta = 2\pi \frac{1}{A_n^2 \sigma^2 2^{(1+\alpha)n}} \qquad (**)
$$

Let $\theta = 2\phi$.

$$
\begin{aligned}
&\int_{-\pi}^{\pi} (1 - \cos\theta)^{\alpha n} (1 + \cos\theta)^n d\theta \\
&= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2\phi)^{\alpha n} (1 + \cos 2\phi)^n d\phi \\
&= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2 \sin^2 \phi)^{\alpha n} (2 \cos^2 \phi)^n d\phi \\
&= 4 \cdot 2^{(1+\alpha)n} \int_0^{\frac{\pi}{2}} \sin^{2\alpha n} \phi \cos^{2n} \phi d\phi.
\end{aligned}
$$

For the Beta function

$$
B(x, y) = \int_0^1 u^{x-1} (1 - u)^{y-1} du, \quad x > 0, y > 0
$$

we put $u = \sin^2 \phi$ and so get

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} (\sin\phi)^{2x-1}(\cos\phi)^{2y-1} d\phi, \quad x > 0,\, y > 0.$$

The Beta function can be expressed by the Gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-s} s^{x-1} ds, \quad x > 0$$

as follows;

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x > 0, y > 0.$$

(See e.g. Abramowitz and Stegun (1970), p258, 6.2.1). Therefore the left hand side of $(**)$ is

$$\begin{aligned}\int_{-\pi}^{\pi} (1-\cos\theta)^{\alpha n}(1+\cos\theta)^n d\theta &= 2\cdot 2^{(1+\alpha)n}\cdot \underbrace{2\int_0^{\frac{\pi}{2}} \sin^{2\alpha n}\phi \cos^{2n}\phi d\phi}_{B(\alpha n+1/2,n+1/2)} \\ &= 2\cdot 2^{(1+\alpha)n}\frac{\Gamma(\alpha n+1/2)\Gamma(n+1/2)}{\Gamma((1+\alpha)n+1)}.\end{aligned}$$

Since this equals to the right hand side of $(**)$, we get

$$\frac{1}{A_n^2} = \frac{2^{2(1+\alpha)n}}{\pi}\frac{\Gamma(\alpha n+1/2)\Gamma(n+1/2)}{\Gamma((1+\alpha)n+1)}\sigma^2.$$

Then from $(*)$ we obtain

$$p_n(\theta) = \frac{1}{2\cdot 2^{(1+\alpha)n}}\frac{\Gamma((1+\alpha)n+1)}{\Gamma(\alpha n+1/2)\Gamma(n+1/2)}(1-\cos\theta)^{\alpha n}(1+\cos\theta)^n \tag{2}$$

Differentiating it as for θ, we get

$$\begin{aligned}p_n'(\theta) = &\frac{1}{2\cdot 2^{(1+\alpha)n}}\frac{\Gamma((1+\alpha)n+1)}{\Gamma(\alpha n+1/2)\Gamma(n+1/2)} \\ &\times [n(1-\cos\theta)^{\alpha n-1}(1+\cos\theta)^{n-1}\sin\theta(\alpha(1+\cos\theta)-(1-\cos\theta))].\end{aligned}$$

Let θ^* satisfies $\cos\theta^* = (1-\alpha)/(1+\alpha)$. p_n takes the minimum value 0 at $\theta = -\pi, \pi, 0$ and maximum value

$$\frac{1}{2\cdot 2^{(1+\alpha)n}}\frac{\Gamma((1+\alpha)n+1)}{\Gamma(\alpha n+1/2)\Gamma(n+1/2)}\left[\frac{2\cdot(2\alpha)^{\alpha}}{(1+\alpha)^{(1+\alpha)}}\right]^n$$

at $\theta = -\theta^*, \theta^*$. It deserves a special notice that p_n is an even function.

Lemma 2. As $n \to \infty$, we get $p_n(\theta^*) = p_n(-\theta^*) \longrightarrow \infty$ and $p_n(\theta) \longrightarrow 0$ $(\theta \neq \theta^*, -\theta^*)$.

(proof) First we note that

$$\Gamma(n+1) = n!$$

(Abramowitz and Stegun (1970), p255, 6.1.6)

$$\Gamma(n+1/2) = \frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}$$

(Abramowitz and Stegun (1970), p255, 6.1.12).

Then we obtain

$$\begin{aligned}\frac{\Gamma((1+\alpha)n+1)}{\Gamma(\alpha n+1/2)\Gamma(n+1/2)} &= \frac{(1+\alpha)n!}{\frac{(2\alpha n-1)(2\alpha n-3)\cdots 5\cdot 3\cdot 1}{2^{\alpha n}}\sqrt{\pi}\cdot\frac{(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1}{2^n}\sqrt{\pi}} \\ &= \frac{(1+\alpha)n!}{\frac{(2\alpha n)!}{2^{2\alpha n}(\alpha n)!}\sqrt{\pi}\cdot\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}} \\ &= \frac{2^{2(1+\alpha)n}}{\pi}\frac{((1+\alpha)n)!(\alpha n)!n!}{(2\alpha n)!(2n)!}.\end{aligned}$$

We apply the Stirling's formula

$$x! = \sqrt{2\pi}x^{x+\frac{1}{2}}\exp(-x+\frac{\varepsilon}{12x}), \quad x>0, \quad 0<\varepsilon<1,$$

to the factorial term of the right hand side of the above equation. (Abramowitz and Stegun (1970), p257, 6.1.38.) Then,

$$\begin{aligned}\frac{((1+\alpha)n)!(\alpha n)!n!}{(2\alpha n)!(2n)!} &= \sqrt{2\pi}\{(1+\alpha)n\}^{(1+\alpha)n+\frac{1}{2}}\exp(-(1+\alpha)n+\mathcal{O}(n^{-1})) \\ &\times\sqrt{2\pi}(\alpha n)^{\alpha n+\frac{1}{2}}\exp(-\alpha n+\mathcal{O}(n^{-1})) \\ &\times\sqrt{2\pi}n^{n+\frac{1}{2}}\exp(-n+\mathcal{O}(n^{-1})) \\ &\div\sqrt{2\pi}(2\alpha n)^{2\alpha n+\frac{1}{2}}\exp(-2\alpha n+\mathcal{O}(n^{-1})) \\ &\div\sqrt{2\pi}(2n)^{2n+\frac{1}{2}}\exp(-2n+\mathcal{O}(n^{-1})) \\ &= \frac{1}{2^{2(1+\alpha)n}}\sqrt{\frac{\pi(1+\alpha)n}{2}}\left\{\frac{(1+\alpha)^{(1+\alpha)}}{\alpha^\alpha}\right\}^n\exp(\mathcal{O}(n^{-1}))\end{aligned}$$

where $\mathcal{O}(n^{-1})$ means that when $n \to \infty$, $\mathcal{O}(n^{-1}) \to 0$ and $\mathcal{O}(n^{-1}) \times n$ is bounded. Then we get

$$\frac{\Gamma((1+\alpha)n+1)}{\Gamma(\alpha n+1/2)\Gamma(n+1/2)} = \sqrt{\frac{(1+\alpha)n}{2\pi}}\left\{\frac{(1+\alpha)^{(1+\alpha)}}{\alpha^{\alpha}}\right\}^{n}\exp(\mathcal{O}(n^{-1}))$$

From (2) the spectral probability density function is

$$p_n(\theta) = \sqrt{\frac{(1+\alpha)n}{2^3\pi}}\left\{\frac{(1+\alpha)^{(1+\alpha)}}{2(2\alpha)^{\alpha}}(1-\cos\theta)^{\alpha}(1+\cos\theta)\right\}^{n}\exp(\mathcal{O}(n^{-1})). \quad (3)$$

Since $(1-\cos\theta)^{\alpha}(1+\cos\theta)$ takes the maximum value

$$\frac{2\cdot(2\alpha)^{\alpha}}{(1+\alpha)^{(1+\alpha)}}$$

at $\theta = \theta^*, -\theta^*$, we obtain

$$\frac{(1+\alpha)^{(1+\alpha)}}{2(2\alpha)^{\alpha}}(1-\cos\theta)^{\alpha}(1+\cos\theta)\begin{cases} =1 & \theta = \theta^*, -\theta^* \\ <1 & \theta \neq \theta^*, -\theta^* \end{cases} \quad (4)$$

In general, for a, k which satisfy $0 < a < 1$, $k > 0$ we have

$$\lim_{x\to\infty} x^k \cdot a^x = 0$$

Considering together $\exp(\mathcal{O}(n^{-1})) \to 1$, (3) and (4), we get

$$\lim_{n\to\infty} p_n(\theta) = \begin{cases} \infty & \theta = \theta^*, -\theta^* \\ 0 & \theta \neq \theta^*, -\theta^* \end{cases}$$

So we obtain the desired result. □

Lemma 3. Let δ_{θ^*}、$\delta_{-\theta^*}$ be Dirac measures concentrated at θ^* and $-\theta^*$ respectively. The spectral probability measure $p_n(\theta)d\theta$ weakly converges to

$$\frac{1}{2}\delta_{-\theta^*} + \frac{1}{2}\delta_{\theta^*}$$

as $n \to \infty$. [2]

[2]Let X be a topological space and $\mathcal{B}(X)$ be a Borel-σ-field. Suppose that $\{\mu_n\}_{n\in\mathbb{N}}$, μ are measures on that. μ_n weakly converges to μ *if* for any bounded real valued function $f : X \to \mathbb{R}$,

$$\lim_{n\to\infty}\int_X f d\mu_n = \int_X f d\mu.$$

(proof) It suffices to show that for any closed set $F \subset [-\pi, \pi]$

$$\overline{\lim}_{n\to\infty} \int_F p_n(\theta)d\theta \leq \begin{cases} \frac{1}{2} & \theta^* \in F \text{or } -\theta^* \in F \quad \text{exclusively} \\ 0 & \theta^* \notin F \text{and } -\theta^* \notin F \\ 1 & \theta^*, -\theta^* \in F \end{cases} \tag{5}$$

1) $\theta^* \notin F$, $-\theta^* \in F$.
Since $F \cap [0, \pi]$ is compact and p_n is continuous, there exists a maximum solution $\hat{\theta} \in F \cap [0, \pi]$ of p_n. From the shape of p_n, $\hat{\theta}$ doesnot depend on n.(Take the minimum distance from θ^*). Since $\hat{\theta} \neq \theta^*, -\theta^*$, taking account of lemma 2,

$$p_n(\hat{\theta}) \longrightarrow 0 \qquad (n \to \infty).$$

Then from bounded convergence theorem,

$$\int_{F\cap[0,\pi]} p_n(\theta)d\theta \longrightarrow 0 \qquad (n \to \infty).$$

We see

$$\int_{F\cap[-\pi,0]} p_n(\theta)d\theta \leq \int_{-\pi}^{0} p_n(\theta)d\theta = \frac{1}{2}$$

then

$$\int_F p_n(\theta)d\theta = \int_{F\cap[-\pi,0]} p_n(\theta)d\theta + \int_{F\cap[0,\pi]} p_n(\theta)d\theta$$
$$\leq \frac{1}{2} + \int_{F\cap[0,\pi]} p_n(\theta)d\theta.$$

Therefore we get

$$\overline{\lim}_{n\to\infty} \int_F p_n(\theta)d\theta \leq \frac{1}{2}.$$

In the case that $\theta^* \in F$ and $-\theta^* \notin F$, the similar discussion apply.

2)$\theta^* \notin F$ and $-\theta^* \notin F$. The same as 1).

3)$\theta^*, -\theta^* \in F$.
Trivially

$$\overline{\lim}_{n\to\infty} \int_F p_n(\theta)d\theta \leq 1$$

holds. So we get (5). □

Lemma 4. For all $t \in \mathbb{Z}$

$$\rho_n(t) \longrightarrow \cos\theta^* t \qquad (n \to \infty).$$

(proof) Since p_n is even and sin is odd, we get

$$\rho_n(t) = \int_{-\pi}^{\pi} e^{i\theta t} p_n(\theta) d\theta = \int_{-\pi}^{\pi} \cos\theta t p_n(\theta) d\theta,$$

which weakly converges to

$$\int_{-\pi}^{\pi} \cos\theta t d(\frac{1}{2}\delta_{-\theta^*} + \frac{1}{2}\delta_{\theta^*}) = \cos\theta^* t$$

by Lemma 3. □

Now we prove Theorem 1.

(the proof of Theorem 1) Let

$$X_t^n(\omega) = Y_0^n(\omega)\cos\theta^* t + \frac{1}{\sin\theta^*}[Y_1^n(\omega) - \cos\theta^* Y_0^n(\omega)]\sin\theta^* t.$$

Note that the above is a periodic in t for each n, ω, the periodicity of which is $2\pi/\theta^*$.

$$\begin{aligned}
&||Y_t^n(\omega) - X_t^n(\omega)||^2_{\mathcal{L}^2(\Omega)} \\
&= ||Y_t^n(\omega) - (Y_0^n(\omega)\cos\theta^* t + \frac{1}{\sin\theta^*}[Y_1^n(\omega) - \cos\theta^* Y_0^n(\omega)]\sin\theta^* t)||^2_{\mathcal{L}^2(\Omega)} \\
&= E[|Y_t^n(\omega) - (Y_0^n(\omega)\cos\theta^* t + \frac{1}{\sin\theta^*}[Y_1^n(\omega) - \cos\theta^* Y_0^n(\omega)]\sin\theta^* t)|^2] \\
&= \int_\Omega |Y_t^n(\omega)|^2 dP - 2\cos\theta^* t \int_\Omega Y_t^n(\omega) Y_0^n(\omega) dP - 2\frac{1}{\sin\theta^*}\sin\theta^* t \int_\Omega Y_t^n(\omega) Y_1^n(\omega) dP \\
&\qquad + 2\frac{\cos\theta^*}{\sin\theta^*}\sin\theta^* t \int_\Omega Y_t^n(\omega) Y_0^n(\omega) dP + \cos^2\theta^* t \int_\Omega |Y_0^n(\omega)|^2 dP \\
&\qquad\quad + \frac{1}{\sin^2\theta^*}\sin^2\theta^* t \int_\Omega |Y_1^n(\omega) - \cos\theta^* Y_0^n(\omega)|^2 dP \\
&\qquad\qquad + 2\cos\theta^* t \sin\theta^* t \frac{1}{\sin\theta^*} \int_\Omega Y_0^n(\omega)(Y_1^n(\omega) - \cos\theta^* Y_0^n(\omega)) dP \\
&= \rho_n(0) - 2\cos\theta^* t \rho_n(t) - 2\frac{1}{\sin\theta^*}\sin\theta^* t \rho_n(t-1) \\
&\qquad + 2\frac{\cos\theta^*}{\sin\theta^*}\sin\theta^* t \rho_n(t) + \cos^2\theta^* t \rho_n(0) \\
&\qquad\quad + \frac{1}{\sin^2\theta^*}\sin^2\theta^* t(\rho_n(0) - 2\cos\theta^* \rho_n(1) + \cos^2\theta^* \rho_n(0))
\end{aligned}$$

$$+2\cos\theta^* t \sin\theta^* t \tfrac{1}{\sin\theta^*}(\rho_n(1) - \cos\theta^* \rho_n(0))$$

by Lemma 4 as $n \to \infty$ this converges to

$$\begin{aligned}
&1 - 2\cos^2\theta^* t - 2\frac{1}{\sin\theta^*}\sin\theta^* t\cos\theta^*(t-1) + 2\frac{\cos\theta^*}{\sin\theta^*}\sin\theta^* t\cos\theta^* t + \cos^2\theta^* t \\
&\qquad + \frac{1}{\sin^2\theta^*}\sin^2\theta^* t(1 - 2\cos^2\theta^* + \cos^2\theta^*) + 2\cos\theta^* t\sin\theta^* t\frac{1}{\sin\theta^*}(\cos\theta^* - \cos\theta^*) \\
&= 1 - \cos^2\theta^* t - \sin^2\theta^* t = 0
\end{aligned}$$

□

By the chebyshev's inequality, the following holds.

Corollary. There exists a stochastic process $X_t^n(\omega)$ which is periodic in t for each n and ω such that

$$|Y_t^n(\omega) - X_t^n(\omega)| \longrightarrow 0 \qquad \text{in probability} \qquad (n \to \infty)$$

for all $t \in \mathbb{Z}$

Remark The case $\alpha \in \mathbb{Q}$: In this paper, we consider only the case $\alpha \in \mathbb{N}$. But we can also consider the case $\alpha \in \mathbb{Q}$ if $\alpha n \in \mathbb{N}$. Let $\alpha = q/p$, $p, q \in \mathbb{N}$. Taking the subsequence of n as $p, 2p, 3p, \cdots, kp, \cdots$, $k \in \mathbb{N}$, the same discussion can apply.

Until this, the weighted sum is on the basis of Slutzky(1937). However if we adequately change the weights, almost everywhere convergence holds.

Theorem 2. Consider the following stochastic process

$$Y_t^n(\omega) = A_n(1 - L^2)^{n^2}\varepsilon_t(\omega)$$

where

$$A_n = \frac{1}{\sqrt{E[(y_t^n)^2]}}$$

and $y_t^n(\omega) = (1 - L^2)^{n^2}\varepsilon_t(\omega)$. Then there exists a stochastic process $X_t^n(\omega)$ which is periodic in t for each n and ω such that

$$|Y_t^n(\omega) - X_t^n(\omega)| \longrightarrow 0 \qquad \text{almost everywhere} \qquad (n \to \infty)$$

for all $t \in \mathbb{Z}$

(proof) If we put $\alpha = 1$ and substitute n for n^2, we can apply the same discussion. So we have the spectral probability density function and the covariance respectively as follows;

$$p_n(\theta) = \frac{1}{2 \cdot 2^{2n^2}} \frac{\Gamma(2n^2+1)}{\Gamma(n^2+1/2)\Gamma(n^2+1/2)} \sin^{2n^2}\theta, \tag{6}$$

$$\rho_n(t) = \int_{-\pi}^{\pi} \cos\theta t p_n(\theta) d\theta \tag{7}$$

Next let

$$X_t^n(\omega) = Y_0^n(\omega)\cos\frac{\pi}{2}t + Y_1^n(\omega)\sin\frac{\pi}{2}t$$

Note that for n、 ω it is a periodic function in t.

$$\begin{aligned} &E[|Y_t^n(\omega) - (Y_0^n(\omega)\cos\frac{\pi}{2}t + Y_1^n(\omega)\sin\frac{\pi}{2}t)|^2] \\ &= \rho_n(0) - 2\cos\frac{\pi}{2}t\rho_n(t) - 2\sin\frac{\pi}{2}t\rho_n(t-1) \\ &\qquad + (\cos^2\frac{\pi}{2}t + \sin^2\frac{\pi}{2}t)\rho_n(0) + 2\cos\frac{\pi}{2}t\sin\frac{\pi}{2}t\rho_n(1) \end{aligned}$$

from (7)

$$= \int_{-\pi}^{\pi} [2 - 2(\cos\frac{\pi}{2}t\cos\theta t + \sin\frac{\pi}{2}t\cos\theta(t-1)) + 2\cos\frac{\pi}{2}t\sin\frac{\pi}{2}t\cos\theta] p_n(\theta) d\theta$$

for each $t \in \mathbb{Z}$, $\cos\frac{\pi}{2}t$=0 or $\sin\frac{\pi}{2}t$=0 so $2\cos\frac{\pi}{2}t\sin\frac{\pi}{2}t\cos\theta = 0$. Therefore

$$= \int_{-\pi}^{\pi} [2 - 2(\cos\frac{\pi}{2}t\cos\theta t + \sin\frac{\pi}{2}t\cos\theta(t-1))] p_n(\theta) d\theta$$

Denote

$$F_t(\theta) = 2 - 2(\cos\frac{\pi}{2}t\cos\theta t + \sin\frac{\pi}{2}t\cos\theta(t-1)).$$

We see for $t = 2k, 2k+1$, $k \in \mathbb{Z}$

$$F_t(\theta) = 2 + (-1)^{k+1} 2\cos 2k\theta \tag{8}$$

i) $t=0,1(k=0)$ We see $F_t(\theta)=0$.

ii) $t\neq 0,1$ $(k\neq 0)$
Think of

$$e^{i(2k)\theta}=(\cos\theta+i\sin\theta)^{2k}.$$

Because the real part and imaginary part of both sides of the above are equal, we can deduce as follows,

$$\cos 2k\theta=\cos^{2k}-\binom{2k}{2}\cos^{2k-2}\theta(1-\cos^2\theta)+\binom{2k}{4}\cos^{2k-4}\theta(1-\cos^2\theta)^2 \\ -\binom{2k}{6}\cos^{2k-6}\theta(1-\cos^2\theta)^3\cdots(-1)^k(1-\cos^2\theta)^k.$$

Take the coefficients $a_1^k, a_2^k, \cdots a_k^k \in \mathbb{R}$ appropriately

$$=a_1^k\cos^{2k}\theta+a_2^k\cos^{2k-2}\theta+a_3^k\cos^{2k-4}\theta+\cdots+a_k^k\cos^2\theta+(-1)^k. \tag{9}$$

By (8), (9)

$$F_t(\theta)=2[1+(-1)^{k+1}(a_1^k\cos^{2k}\theta+a_2^k\cos^{2k-2}\theta+a_3^k\cos^{2k-4}\theta+\cdots+a_k^k\cos^2\theta)+(-1)^{2k+1}]$$

taking account of $(-1)^{2k+1}=-1$, we take the coefficients $c_1^k, c_2^k, \cdots c_k^k \in \mathbb{R}$ appropriately

$$=2(c_1^k\cos^{2k}\theta+c_2^k\cos^{2k-2}\theta+c_3^k\cos^{2k-4}\theta+\cdots+c_k^k\cos^2\theta). \tag{10}$$

We also have

$$\begin{aligned}&E[|Y_t^n(\omega)-X_t^n(\omega)|^2]\\ &=\int_{-\pi}^{\pi}F_t(\theta)p_n(\theta)d\theta\\ &=2\int_0^{\pi}F_t(\theta)p_n(\theta)d\theta\\ &=4\int_0^{\pi}(c_1^k\cos^{2k}\theta+c_2^k\cos^{2k-2}\theta+c_3^k\cos^{2k-4}\theta+\cdots+c_k^k\cos^2\theta)p_n(\theta)d\theta.\end{aligned} \tag{11}$$

Let us now caluculate the each term of (11). The same discussion as that between Lemma 1 and Lemma 2 can also apply to this case, then we get

$$\int_0^{\pi}\cos^{2m}\theta\sin^{2n^2}\theta d\theta=2\int_0^{\frac{\pi}{2}}\cos^{2m}\theta\sin^{2n^2}\theta d\theta=\frac{\Gamma(n^2+1/2)\Gamma(m+1/2)}{\Gamma(n^2+m+1)}. \tag{12}$$

Then from (6),

$$\int_0^\pi \cos^{2m}\theta p_n(\theta)d\theta$$
$$= \frac{1}{2\cdot 2^{2n^2}}\frac{\Gamma(2n^2+1)}{\Gamma(n^2+1/2)\Gamma(n^2+1/2)}\int_0^\pi \cos^{2m}\theta\sin^{2n^2}\theta d\theta$$

From (12) and the similar argument of Lemma 2,

$$= \frac{1}{2\pi}\frac{(2n^2)!}{(2n^2-1)(2n^2-3)\cdots 3\cdot 1\cdot(2n^2-1)(2n^2-3)\cdots 3\cdot 1}$$
$$\times\frac{\pi}{2^{m+n^2}}\frac{(2n^2-1)(2n^2-3)\cdots 3\cdot 1\cdot(2m-1)(2m-3)\cdots 3\cdot 1}{(m+n^2)!}$$
$$= \frac{1}{2\pi}\frac{2^{n^2}(n^2)!}{(2n^2-1)(2n^2-3)\cdots 3\cdot 1}\frac{\pi}{2^{m+n^2}}\frac{(2n^2-1)(2n^2-3)\cdots 3\cdot 1(2m-1)(2m-3)\cdots 3\cdot 1}{(m+n^2)!}$$
$$= \frac{1}{2}\frac{1}{2^m(n^2+1)(n^2+2)\cdots(n^2+m)}(2m-1)(2m-3)\cdots 3\cdot 1.$$

Therefore for each $m\in\mathbb{N}$

$$\sum_{n=1}^{\infty}\int_0^\pi \cos^{2m}\theta p_n(\theta)d\theta < \infty \tag{13}$$

By (11)

$$\sum_{n=1}^{\infty} E[|Y_t^n(\omega)-X_t^n(\omega)|^2] < \infty. \tag{14}$$

Then,

$$\lim_{k\to\infty}\sum_{n=k}^{\infty} E[|Y_t^n(\omega)-X_t^n(\omega)|^2] = 0 \tag{15}$$

By the chebyshev's inequality, for any $\varepsilon>0$, we have

$$P\{\omega\in\Omega \mid \sup_{n\geq k}|Y_t^n(\omega)-X_t^n(\omega)|\geq\varepsilon\} \leq \sum_{n=k}^{\infty} P\{\omega\in\Omega \mid |Y_t^n(\omega)-X_t^n(\omega)|\geq\varepsilon\}$$
$$\leq \frac{1}{\varepsilon^2}\sum_{n=k}^{\infty}E[|Y_t^n(\omega)-X_t^n(\omega)|^2] \tag{16}$$

Since the almost everywhere convergence is equivalent to that for each $\varepsilon>0$ the following holds

$$\lim_{k\to\infty}P\{\omega\in\Omega \mid \sup_{n\geq k}|Y_t^n(\omega)-X_t^n(\omega)|\geq\varepsilon\} = 0,$$

by (15), (16), the desired result is obtained. □

4 Concluding Remarks

In this section we mainly compare our results with those of Slutzky(1937) and Sargent(1987). First, since Slutzky(1937) proved the convergence without using the spectral measure, only the convergence of $\rho_n(1)$ and $\rho_n(2)$ are directly verified and so can not be clarified for all $t \in \mathbb{Z}$. Then his discussion left the possibility that for fixed n as t goes to infinity

$$\|Y_t^n(\omega) - X_t^n(\omega)\|_{\mathcal{L}^2} \longrightarrow \infty.$$

In this paper, to the contrary, for fixed n we can easily verify that for each n there exists $M > 0$ such that $\sup_{t\in\mathbb{Z}} \|Y_t^n(\omega) - X_t^n(\omega)\|_{\mathcal{L}^2(\Omega)} \leq M$.

Secondly, though Slutzky only refered to the convergence in probability, we show in this paper the almost everywhere convergence.

Lastly let us briefly mention the idea of the proof of this main results. In general, that a stochastic process is a periodic function with the periodicity T is equivalent to that the spectral measure of the convariance of a stochastic process (we call it ν) has the weights only on the discrete points $2\pi k/T$, $k \in \mathbb{Z}$. [3] If we denote the spectral distribution function as

$$F(x) = \nu((-\infty, x]) \quad x \in \mathbb{R},$$

this is a step function so that it jumps at $2\pi k/T$, $k \in \mathbb{Z}$ discontinuously and on the outside of these points takes constant value. If there exists a density function, denote $p : [-\pi, \pi] \to \mathbb{R}$, we see

$$F(x) = \int_{-\pi}^{x} p(t)dt$$

Very roughly speaking, if we differentiate F, $F' = p$ holds so that p jumps up to infinity at the points $2\pi k/T$, $k \in \mathbb{Z}$. Sargent(1987, p273-p275) expressed the covariance using the spectral measure and only proved that spectral density goes to infinity at θ^* and $-\theta^*$. However his setting also allowed that the weights on a neighborhood of θ^* donot vanish and even go to infinity so that the spectral measure leaves the weights on the outside of θ^* and $-\theta^*$. Then the proof of the existence of the periodic stochastic process failed.

5 Appendix

In this section we mention the following proposition which characterizes a periodic stochastic process when it is a weakly stationary process.

[3]See Appendix.

Proposition[4]. Let $X : \mathbb{Z} \times \Omega \to \mathbb{R}$ be a weakly stationary stochastic process. Then the following conditions are equivalent.

(1)The covariance ρ of $X(t,\omega)$ is a periodic function with the periodicity $T \in \mathbb{N}$, namely, for every $u \in \mathbb{Z}$

$$\rho(u+T) = \rho(u)$$

holds.

(2)$X(t,\omega)$ is a periodic function as for t almost everywhere with the periodicity $T \in \mathbb{N}$, namely, for every $t \in \mathbb{Z}$

$$X(t+T,\omega) = X(t,\omega) \qquad \text{a.e.} \quad \omega$$

holds.

(3)Let ν be a spectral measure of the covariance of $X(t,\omega)$. For every $E \in \mathcal{B}([-\pi,\pi])$ which satisfies

$$E \cap \{2k\pi/T | k \in \mathbb{Z}\} = \emptyset,$$

$\nu(E) = 0$ holds where $\mathcal{B}([-\pi,\pi])$ is a Borel-σ-field on $[-\pi,\pi]$

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