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Uniform perfectness of fiberwise Julia sets of fibered rational maps

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Abstract

We consider fiber-preserving complex dynamics on fiber bundles whose fibers are Riemann spheres and whose base spaces are compact metric spaces. In this context, we show that the fiberwise Julia sets are C_1-uniformly perfect and that the Hausdorff dimensions are greater than a positive constant C_2, where the constants C_1 and C_2 do not depend on any points in the base space. From this result, we show that, for any semigroup G generated by a compact family of rational maps on $\overline{\mathbb{C}}$ of degree two or greater, there exists a positive constant C such that the Julia set of any subsemigroup H of G is C-uniformly perfect. In particular, we show that for any such a semigroup H, if there exists a super attracting fixed point z_0 of some element of H in the Julia set of H, then z_0 belongs to the interior of the Julia set of H.

1 Introduction and the main results

1.1 Results on fibered rational maps

In this section, we state the results on fibered rational maps. The proofs are given in Section 2.4. First, we provide some notation and definitions regarding the dynamics of fibered rational maps.

Definition 1.1. ([J2]) A triplet (π, Y, X) is called a '$\overline{\mathbb{C}}$-bundle' if

1. Y and X are compact metric spaces,
2. $\pi : Y \to X$ is a continuous and surjective map,
3. and there exists an open covering $\{U_i\}$ of X such that, for each i, there exists a homeomorphism $\Phi_i : U_i \times \overline{\mathbb{C}} \to \pi^{-1}(U_i)$ such that $\Phi_i(\{x\} \times$

$\overline{\mathbb{C}}) = \pi^{-1}(x)$ and $\Phi_j^{-1} \circ \Phi_i : \{x\} \times \overline{\mathbb{C}} \to \{x\} \times \overline{\mathbb{C}}$ is a Möbius map for each $x \in U_i \cap U_j$, under the identification $\{x\} \times \overline{\mathbb{C}} \cong \overline{\mathbb{C}}$.

Remark 1. By the condition 3, each fiber $Y_x := \pi^{-1}(x)$ has a complex structure. Furthermore, given $x_0 \in X$, one may find a continuous family $i_x : \overline{\mathbb{C}} \to Y_x$ of homeomorphisms, for x close to x_0. Such a family $\{i_x\}$ is called a 'local parameterization'. Since X is compact, we may assume throughout this paper that there exists a compact subset M_0 of the set of Möbius transformations of $\overline{\mathbb{C}}$ such that $i_x \circ j_x^{-1} \in M_0$ for any two local parameterizations $\{i_x\}$ and $\{j_x\}$.

Moreover, throughout this paper, we assume the following condition:

- there exists a smooth $(1,1)$-form $\omega_x > 0$ inducing the distance on Y_x from Y, and $x \mapsto \omega_x$ is continuous. That is, if $\{i_x\}$ is a local parameterization, then the pull back $i_x^* \omega_x$ is a positive smooth form on $\overline{\mathbb{C}}$ that depends continuously on x.

Definition 1.2. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ and $g : X \to X$ be continuous maps. Let f be called a fibered rational map over g (or a rational map fibered over g), if

1. $\pi \circ f = g \circ \pi$
2. $f|_{Y_x} : Y_x \to Y_{g(x)}$ is a rational map, for any $x \in X$. That is, $(i_{g_x})^{-1} \circ f \circ i_x$ is a rational map from $\overline{\mathbb{C}}$ to itself, for any local parameterization i_x at $x \in X$ and $i_{g(x)}$ at $g(x)$.

Notation: If $f : Y \to Y$ is a fibered rational map over $g : X \to X$, then we set $f_x^n = f^n|_{Y_x}$, for any $x \in X$ and $n \in \mathbb{N}$. Furthermore, we set $d_n(x) = \deg(f_x^n)$ and $d(x) = d_1(x)$, for any $x \in X$ and $n \in \mathbb{N}$.

Definition 1.3. Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Then, for any $x \in X$, we denote by $F_x(f)$(simply F_x) the set of points $y \in Y_x$ that has a neighborhood U of y in Y_x such that $\{f_x^n\}_{n \in \mathbb{N}}$ is a normal family in U; that is, $y \in F_x$ if and only if the family $Q_x^n = i_{x_n}^{-1} \circ f_x^n \circ i_x$ of rational maps on $\overline{\mathbb{C}}$ ($x_n := g^n(x)$) is normal near $i_x^{-1}(y)$. Note that, by Remark 1, this does not depend on the choices of local parameterizations at x and x_n. Equivalently, F_x is the open subset of Y_x, where the family $\{f_x^n\}$ of mappings from Y_x into Y is locally equicontinuous. We set $J_x(f)$(simply J_x)$= Y_x \setminus F_x$.

Furthermore, we set $\tilde{J}(f) = \overline{\bigcup_{x \in X} J_x}$, where the closure is taken in the space Y, $\tilde{F}(f) = Y \setminus \tilde{J}(f)$, and $\hat{J}_x(f)$(simply $\hat{J}_x$)$= \tilde{J}(f) \cap Y_x$, for each $x \in X$.

Remark 2. There exists a fibered rational map $f : Y \to Y$ such that $\bigcup_{x \in X} J_x$ is NOT compact.

Now we define uniform perfectness.

Definition 1.4. Let C be a positive number. Let K be a closed subset of $\overline{\mathbb{C}}$. We say that K is C**-uniformly perfect** if, $\sharp K \geq 2$ and for any doubly connected domain A in $\overline{\mathbb{C}}$ such that A separates K; i.e., such that $A \subset \overline{\mathbb{C}} \setminus K$ and both of the two connected components of $\overline{\mathbb{C}} \setminus A$ have non-empty intersections with K, mod A (the modulus of A. For the definition, see [LV]) is less than C.

Theorem 1.5. (**Main theorem A**) *Let* (π, Y, X) *be a* $\overline{\mathbb{C}}$*-bundle. Let* $f : Y \to Y$ *be a fibered rational map over* $g : X \to X$ *such that* $d(x) \geq 2$, *for any* $x \in X$. *Then, it follows that:*

1. *There exists a positive constant* C *such that, for any* $x \in X$, J_x *and* $\hat{J}_x$ *are* C*-uniformly perfect. Furthermore, there exists a positive constant* C_1 *such that diam* $J_x > C_1$, *for each* $x \in X$, *with respect to the distance in* Y_x *induced by* ω_x. *Moreover, there exists a positive constant* C_2 *such that, for each* $x \in X$, *the Hausdorff dimension* $\dim_H(J_x)$ *of* J_x, *with respect to the distance on* Y_x *induced by* ω_x, *satisfies the condition that* $\dim_H(J_x) \geq C_2$. *(Note that* C_1 *and* C_2 *do not depend on* x. *)*
2. *Suppose further that* $f(\tilde{F}(f)) \subset \tilde{F}(f)$*(for example, assume that* $g : X \to X$ *is an open map). If a point* $z \in Y$ *satisfies* $f^n_{\pi(z)}(z) = z$ *and* $(f^n_{\pi(z)})'(z) = 0$ *for some* $n \in \mathbb{N}$ *and* $z \in \hat{J}_{\pi(z)}$, *then* z *belongs to the interior of* $\hat{J}_{\pi(z)}$ *with respect to the topology of* $Y_{\pi(z)}$.

Example 1.6. Let $z_0 \in \overline{\mathbb{C}}$ be a point. Let h_1 and h_2 be two rational maps on $\overline{\mathbb{C}}$ of degree two or greater. Let $f : \Sigma_2 \times \overline{\mathbb{C}} \to \Sigma_2 \times \overline{\mathbb{C}}$ be the fibered rational map associated with the generator system $\{h_1, h_2\}$. Suppose that z_0 is a superattracting fixed point of h_1 and is a repelling fixed point of h_2. Then, it can easily be seen that $z_0 \in \hat{J}_x$, where $x = (1, 1, \dots) \in \Sigma_2$. Since the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is an open map, by Theorem 1.5 it follows that z_0 belongs to the interior of $\hat{J}_x$.

Remark 3. Uniform perfectness implies many useful properties ([BP],[Po],[Su]). This terminology was introduced in [Po]. For a survey on uniform perfectness, see [Su]. We now consider the following:

1. In [BP], it was shown that a closed subset K of $\mathbb{C}$ is C-uniformly perfect if and only if there exists a constant δ such that, for any component U of $\overline{\mathbb{C}} \setminus K$,

$$\lambda_U(z) > \delta / \mathrm{dist}(z, \partial U), \tag{1}$$

where $\lambda_U(z)$ denotes the density of the hyperbolic metric of U at z and $\mathrm{dist}(z, \partial U)$ denotes the Euclidian distance of the point z from the set ∂U. (If K is bounded and U is the unbounded component of $\overline{\mathbb{C}} \setminus K$, then (1) will hold only for all $z \in U$ sufficiently close to K.)

In the above discussion, δ depends only on C, and C depends only on δ. Detailed inequalities regarding the relationships among δ, C and other invariants are presented in [Su]. .

2. If a closed subset K of $\overline{\mathbb{C}}$ is C-uniformly perfect, then the Hausdorff dimension $\dim_H(K)$ of K with respect to the spherical metric satisfies $\dim_H(K) \geq C' > 0$, where C' is a positive constant that depends only on C (Theorem 7.2 in [Su]).

1.2 Results on rational semigroups

In this section, we present several results on rational semigroups. The proofs are given in Section 2.5. Before stating results, we will first establish some notation and definitions regarding the dynamics of rational semigroups.

For a Riemann surface S, let End(S) denote the set of all holomorphic endomorphisms of S. In other words, it is a semigroup whose semigroup operation constitutes a composition of maps. A **rational semigroup** is a subsemigroup of End($\overline{\mathbb{C}}$) without any constant elements. We say that a rational semigroup G is a **polynomial semigroup** if each element of G is a polynomial. Research on the dynamics of rational semigroups was initiated by A. Hinkkanen and G. J. Martin ([HM1]), who were interested in the role that the dynamics of polynomial semigroups plays in research on various one-complex-dimensional moduli spaces for discrete groups, and by F. Ren's group([GR]).

Definition 1.7. Let G be a rational semigroup. We set

$$F(G) = \{z \in \overline{\mathbb{C}} \mid G \text{ is normal in a neighborhood of } z\}, \quad J(G) = \overline{\mathbb{C}} \setminus F(G).$$

$F(G)$ is called the **Fatou set** for G, and $J(G)$ is called the **Julia set** for G. The backward orbit $G^{-1}(z)$ of z and the **set of exceptional points** $E(G)$ are defined by: $G^{-1}(z) = \cup_{g \in G} g^{-1}(z)$ and $E(G) = \{z \in \overline{\mathbb{C}} \mid \sharp G^{-1}(z) \leq 2\}$. For any subset A of $\overline{\mathbb{C}}$, set $G^{-1}(A) = \cup_{g \in G} g^{-1}(A)$. We denote by $\langle h_1, h_2, \dots \rangle$ the rational semigroup generated by the family $\{h_i\}$. For a rational map g, we denote by $J(g)$ the Julia set of the dynamics of g.

We now present a result on uniform perfectness of Julia sets of rational semigroups.

Theorem 1.8. (**Main theorem B**) *Let Λ be a compact set in the space $\{h : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \mid h : \text{holomorphic}, \deg(h) \geq 2\}$ with topology induced by uniform convergence on $\overline{\mathbb{C}}$. Let G be a rational semigroup generated by the set Λ. Then there exists a positive constant C such that each subsemigroup H of G satisfies the condition that $J(H)$ is C-uniformly perfect. Furthermore, for any subsemigroup H of G, if a point $z_0 \in J(H)$ satisfies the condition that there exists an element $h \in H$ such that $h(z_0) = z_0$ and $h'(z_0) = 0$, then it follows that $z_0 \in$int $J(H)$.*

Example 1.9. Let $G = \langle h_1, h_2 \rangle$ where $h_1(z) = 2z^2 + z^2$ and $h_2(z) = \frac{z^3}{z-a}$, $a \in \mathbb{C}$ with $a \neq 0$. Then, $0 \in J(G)$ and 0 is a superattracting fixed point of h_2. Hence, $0 \in$int $J(G)$, by Theorem 1.8. Since ∞ is a common attracting fixed point of h_1 and h_2, we have $\infty \in F(G)$. Furthermore, let H be a subsemigroup of G such that $0 \in J(H)$ and $h_2 \in H$. Then, by Theorem 1.8 again, we have $0 \in$int $J(H)$. Moreover, we have $\infty \in F(H)$.

In particular, let H_0 be a subsemigroup of G that is generated by $G \setminus \langle h_1 \rangle$. Then we have all of the following:

1. $0 \in$ int $J(H_0)$.
2. $\infty \in F(H_0)$.
3. For any finitely generated subsemigroup H_1 of H_0, we have $0 \in F(H_1)$.

For, since $0 \in$int $J(G)$, $J(G) = \overline{\cup_{g \in G} J(g)}$ (Corollary 3.1 in [HM1]), and $J(h_1)$ is nowhere dense, we obtain that there exists a suquence (g_n) in H_0 such that $d(0, J(g_n)) \to 0$ as $n \to \infty$. Hence, $0 \in J(H_0)$. Since 0 is a superattracting fixed point of h_2 and $h_2 \in H_0$, by Theorem 1.8 we obtain $0 \in$ int $J(H_0)$. Since $H_0 \subset G$ and $\infty \in F(G)$, we obtain $\infty \in F(H_0)$. For any finitely generated subsemigroup H_1 of H_0, since 0 is a common attracting fixed point of any element of H_0, it follows that $0 \in F(H_1)$.

Remark 4. In [HM2], by A. Hinkkanen and G. Martin, it was shown that a finitely generated rational semigroup G such that each $g \in G$ is of degree two or greater satisfies the condition that $J(G)$ is uniformly perfect. In [St], by R. Stankewitz, it was shown that, if Λ (this is allowed to have an element of degree one) is a family of rational maps on $\overline{\mathbb{C}}$ such that the Lipschitz constant of each element of Λ with respect to the spherical metric on $\overline{\mathbb{C}}$ is uniformly bounded, then the Julia set of semigroup G generated by Λ is uniformly perfect. However, there has been no research done on the uniform perfectness of the Julia sets of subsemigroups of such a semigroup. In [HM2] and [St], the proofs were based on the density of repelling fixed points in the Julia sets (which was shown by an application of Ahlfors's five-island theorem), whereas, in this paper, the proof of Theorem 1.8 is based on the combination of the density of repelling fixed points in a Julia set with Proposition 2.2 (potential theory).

2 Tools and Proofs

We now present the proofs of the main results, meanwhile providing further notation and tools.

2.1 Fundamental properties of fibered rational maps

By means of definitions, the following lemma can be easily shown.

Lemma 2.1. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$. Then,*

1. *For each $x \in X$, $f_x^{-1}(F_{g(x)}) = F_x$, $f_x^{-1}(J_{g(x)}) = J_x$. Furthermore, $f(\tilde{J}(f)) \subset \tilde{J}(f)$.*
2. *If $g : X \to X$ is an open map, then $f^{-1}(\tilde{J}(f)) = \tilde{J}(f)$ and $f(\tilde{F}(f)) \subset \tilde{F}(f)$.*
3. *If $g : X \to X$ is a surjective and open map, then $f^{-1}(\tilde{J}(f)) = \tilde{J}(f) = f(\tilde{J}(f))$ and $f^{-1}(\tilde{F}(f)) = \tilde{F}(f) = f(\tilde{F}(f))$.*

Proof. This proof is the same as that for Lemma 2.4 in [S1]. □

2.2 Fundamental properties of rational semigroups

For a rational semigroup G, for each $f \in G$, it holds that $f(F(G)) \subset F(G)$ and $f^{-1}(J(G)) \subset J(G)$. Note that this equality does not hold in general. If $\sharp J(G) \geq 3$, then $J(G)$ is a perfect set, $\sharp E(G) \leq 2$, $J(G)$ is the smallest closed backward invariant set containing at least three points, and $J(G)$ is the closure of the union of all repelling fixed points of the elements of G, which implies that $J(G) = \overline{\bigcup_{g \in G} J(g)}$. If a point z is not in $E(G)$, then, for every $x \in J(G)$, $x \in \overline{G^{-1}(z)}$. In particular, if $z \in J(G) \setminus E(G)$ then $\overline{G^{-1}(z)} = J(G)$. For more precise statements, see Lemma 2.3 in [S3], for which the proof is based on [HM1] and [GR]. Furthermore, if G is generated by a precompact subset Λ of $\text{End}(\overline{\mathbb{C}})$, then $J(G) = \overline{\bigcup_{f \in \Lambda} f^{-1}(J(G))} = \bigcup_{h \in \overline{\Lambda}} h^{-1}(J(G))$. In particular, if Λ is compact, then $J(G) = \bigcup_{f \in \Lambda} f^{-1}(J(G))$([S1]). We call this property of the Julia set **backward self-similarity**.

Remark 5. In the context of backward self-similarity, the existing research on the Julia sets of rational semigroups may be considered as a kind of generalization of the research on self-similar sets constructed by some similarity transformations from $\mathbb{C}$ to itself, which can be regarded as the Julia sets of some rational semigroups. It can be easily seen that the Sierpiński gasket is the Julia set of a rational semigroup $G = \langle h_1, h_2, h_3 \rangle$, where $h_i(z) = 2(z - p_i) + p_i, i = 1, 2, 3$, with $p_1 p_2 p_3$ a regular triangle.

2.3 Potential theory and measure theory

For the proof of results on uniform perfectness, Johnness, etc., let us borrow some notation from [J2] and [S1], concerning potential theoretic aspects. By the arguments in [J2] and [S1], for a fibered rational map $f : Y \to Y$ over $g : X \to X$ with $d(x) \geq 2$, for each $x \in X$, one can show a result corresponding to Proposition 2.5 in [S1], using the arguments in §3 in [J2] and from pp. 580-581 in [S1] . In this paper, the following statements, and especially the lower semicontinuity of $x \mapsto J_x(f)$, are necessary. (Proposition 2.2.3).

Proposition 2.2. *Let (π, Y, X) be a $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a rational map fibered over $g : X \to X$. Assume that $d(x) \geq 2$, for each $x \in X$. Then, for each $x \in X$, there exists a Borel probability measure μ_x on Y satisfying all of the following.*

1. *$x \mapsto \mu_x$ is continuous with respect to the weak topology of probability measures in Y.*

2. *$supp(\mu_x) = J_x$, for each $x \in X$.*

3. *$x \mapsto J_x$ is lower semicontinuous with respect to the Hausdorff metric in the space of the non-empty compact subsets of Y. That is, if $x, x^n \in X, x^n \to x$ as $n \to \infty$ and $y \in J_x$, then there exists a sequence (y_n) of points in Y with $y_n \in J_{x^n}$, for each $n \in \mathbb{N}$, such that $y_n \to y$ as $n \to \infty$.*

Furthermore , $J_x(f)$ is a non-empty perfect set, for each $x \in X$.

Proof. Since $d(x) \geq 2$, for each $x \in X$, and $x \mapsto d(x)$ is continuous, we can demonstrate these statements in the same way as in §3 in [J2], using the argument from pp. 580-581 in [S1]. The statement 3 follows easily from 1 and 2.

□

2.4 Proof of main theorem A

In this section, we present a proof of main theorem A in Section 1.1.

Proof of Theorem 1.5. First, we prove statement 1. Since, for each $x \in X$, J_x is a non-empty perfect set (Proposition 2.2), it has uncountably many points. Combined with the lower semicontinuity of the map $x \mapsto J_x$ (statement 3 in Proposition 2.2) and the compactness of X, this suggests that, for any $x \in X$, one can take four points $z_{x,1}, z_{x,2}, z_{x,3}$ and $z_{x,4}$ in J_x so that $d(z_{x,i}, z_{x,j}) > C_1$, whenever $i \neq j$ and $x \in X$, for some constant C_1 independent of (i, j) and $x \in X$.

Suppose that there exists a sequence of annuli $\{D_j\}$ with $D_j \subset Y_{x_j}$, $x_j \in X$ such that D_j separates J_{x_j}, for each j and mod $D_j \to \infty$ as $j \to \infty$. Let D'_j and D''_j be the two components of $Y_{x_j} \setminus D_j$. We may assume that

$$\text{diam } D''_j \to 0 \text{ as } j \to \infty. \tag{2}$$

For, by the existence of $\{z_{x,i}\}_{x,i}$, it may be assumed that $\inf_{j \in \mathbb{N}} \text{diam } D'_j > 0$. Then, since mod $D_j \to \infty$ as $j \to \infty$, by Lemma 6.1 on p. 34 in [LV] it follows that diam $D''_j \to 0$ as $j \to \infty$.

It may also be assumed that $(\text{int} D''_j) \cap J_{x_j} \neq \emptyset$, for each $j \in \mathbb{N}$. Hence, there exists a smallest positive integer n_j such that diam $f^{n_j+1}(D''_j) \geq C_1$.

Then, there exists a constant l_0 such that $l_0 C_1 <$ diam $f^{n_j}(D''_j)$, for each j. Since diam $f^{n_j}(D''_j) < C_1$, there exist three distinct points in $\{z_{x'_j,i}\}_{i=1,\dots,4}$ none of which belongs to $f^{n_j}(D''_j)$, where $x'_j = g^{n_j}(x_j)$. Since $D_j \subset F_{x_j}$, it follows that none of these three points belongs to $f^{n_j}(D_j)$, or to $f^{n_j}(D_j \cup D''_j)$. Let $\varphi_j : \{|z| < 1\} \to D_j \cup D''_j$ be a Riemann map such that $\varphi_j(0) = y_j \in D''_j$ (Note that we may assume that int $D'_j \neq \emptyset$ for each j). Then, from the above, it follows that, if we set $\alpha_j = i_{x'_j}^{-1} f^{n_j} \varphi_j : \{|z| < 1\} \to \overline{\mathbb{C}}$ then $\{\alpha_j\}_{j\in\mathbb{N}}$ is normal in $\{|z| < 1\}$. But this causes a contradiction, because diam $\varphi_j^{-1}(D''_j) \to 0$ as $j \to \infty$, which follows from mod $\varphi_j^{-1}(D_j) =$ mod $D_j \to \infty$ as $j \to \infty$, and $l_0 C_1 <$ diam $f^{n_j}(D''_j)$, for each j.

Next, suppose that there exists a sequence of annuli $\{D_j\}$ with $D_j \subset Y_{x_j}$, $x_j \in X$ such that D_j separates $\hat{J}_{x_j}$, for each j, and mod $D_j \to \infty$ as $j \to \infty$. Let D'_j and D''_j be the two components of $Y_{x_j} \setminus D_j$. As in the previous paragraph, it may be assumed that diam $D''_j \to 0$ as $j \to \infty$.

Fix any $j \in \mathbb{N}$. Let $y \in D''_j \cap \hat{J}_{x_j}$ be a point. There exists a sequence $((x_{j,n}, y_{j,n}))_n$ in $X \times Y$ with $y_{j,n} \in J_{x_{j,n}}$, for each $n \in \mathbb{N}$, such that $y_{j,n} \to y$ as $n \to \infty$. Then it follows that there exists a number $n(j) \in \mathbb{N}$ such that $J_{x_{j,n(j)}} \subset D'_j \cup D''_j$. Since $J_{x_{j,n(j)}} \cap D''_j \neq \emptyset$ (take $n(j)$ sufficiently large), by the previous paragraph, it must be that $J_{x_{j,n(j)}} \subset D''_j$, for large j. However, this contradicts the existence of $\{z_{x,i}\}_{x,i}$, because it is also the case that diam $D''_j \to 0$ as $j \to \infty$. Hence, we have proved the first and the second statements in 1. The third statement in 1 follows from the uniform perfectness of J_x , the continuity of ω_x, and Theorem 7.2 in [Su].

Next, we prove statement 2. Suppose the point z belongs to the boundary of $\hat{J}_x$ with respect to the topology of Y_x, where $x = \pi(z)$. Under a coordinate exchange, the map f_x^n around z is conjugate to $\alpha(z) = z^l$ for some $l \in \mathbb{N}$. Since $f_x^n(Y_x \setminus \hat{J}_x) \subset Y_x \setminus \hat{J}_x$, there exists an annulus A around z in Y_x that separates $\hat{J}_x$ and is isomorphic to a round annulus $A' = \{r < |z| < R\}$ in the above coordinate. Then, mod $(f_x^{ns}(A)) =$ mod $(\alpha^s(A')) \to \infty$ as $s \to \infty$. In addition, $f_x^{ns}(A)$ separates $\hat{J}_x$, for each $s \in \mathbb{N}$. This contradicts the fact that $\hat{J}_x$ is uniformly perfect. □

Corollary 2.3. *(Corollary of Theorem 1.5) Let $(\pi, Y = X \times \overline{\mathbb{C}},\ X)$ be a trivial $\overline{\mathbb{C}}$-bundle. Let $f : Y \to Y$ be a fibered rational map over $g : X \to X$ such that f_x is a polynomial with $d(x) \geq 2$, for each $x \in X$. Let $R > 0$ be a number such that for each $x \in X$, $\pi_{\overline{\mathbb{C}}} J_x(f) \subset \{z \mid d(z, \infty) > R\}$, where $\pi_{\overline{\mathbb{C}}} : Y \to \overline{\mathbb{C}}$ denotes the projection and d denotes the spherical metric. (Note that such an R exists, since $d(x) \geq 2$ for each $x \in X$.) Let $\rho_x(z)|dz|$ be the hyperbolic metric on $\pi_{\overline{\mathbb{C}}} A_x(f)$. Let $\delta_x(z) = \inf_{w \in \partial(\pi_{\overline{\mathbb{C}}} A_x(f))} |z - w|$ for each $z \in \pi_{\overline{\mathbb{C}}} A_x(f) \cap \mathbb{C}$. Then, there exists a positive constant C depending only on f and R such that for each $x \in X$ and each $z \in \pi_{\overline{\mathbb{C}}} A_x(f) \cap \{z \mid d(z, \infty) > R\}$, we have $C \leq \rho_x(z)\delta_x(z) \leq 1$.*

Proof. $\rho_x(z)\delta_x(z) \leq 1$ follows easily from the Schwarz lemma.

Next, as in the proof of Theorem 1.5, for any $x \in X$, we can take two points $z_{x,1}$ and $z_{x,2}$ in $J_x(f)$ so that $d(z_{x,1}, z_{x,2}) > c_0$, whenever $x \in X$, for some positive constant c_0 independent of $x \in X$. Let $\psi_x(z)$ be a Möbius transformation such that $\psi_x(z_{x,1}) = \infty$ and ψ_x preserves the spherical metric. Let $B_x = \psi_x(\pi_{\overline{\mathbb{C}}} A_x(f))$ $(\subset \mathbb{C})$. By Theorem 1.5 and Theorem 2.16 in [Su], there exists a positive constant c_1 such that for each $x \in X$ and each $z \in B_x \cap \mathbb{C}$, $\rho_{x,1}(z) \inf_{w \in \partial B_x} |z - w| \geq c_1$, where $\rho_{x,1}(z)|dz|$ denotes the hyperbolic metric on B_x.

Let $z \in A_x(f) \cap \{y \mid d(y, z_{x,1}) > c_0/2, d(y, \infty) > R\}$. Then, we have $\rho_x(z) = \rho_{x,1}(\psi_x(z))|\psi_x'(z)|$. Since $\psi_x(z) \in \{y \mid d(y, \psi_x(\infty)) > R, d(y, \infty) > c_0/2\}$ and $\psi_x^{-1}(\{y \mid d(y, \psi_x(\infty)) > R\}) \subset \{y \mid d(y, \infty) > R\}$, by the Cauchy formula, we have $|\psi_x'(z)| = |(\psi_x^{-1})'(\psi_x(z))|^{-1} \geq c_2$, where c_2 is a positive constant independent of z and x.

Next, let $w_0 \in \partial(\pi_{\overline{\mathbb{C}}} A_x(f)) = \pi_{\overline{\mathbb{C}}} J_x(f)$ be a point such that $\delta_x(z) = |z - w_0|$.

Suppose case (1): $w_0 \in \{y \mid d(y, z_{x,1}) \leq c_0/4\}$. Then, $|z - w_0| \geq c_3$, where c_3 is a positive constant depending only on c_2. Further, $\inf_{w \in \partial B_x} |\psi_x(z) - w| \leq |\psi_x(z) - \psi_x(z_{x,2})| \leq c_4$, where c_4 is a positive constant depending only on c_0. Hence, $\delta_x(z) \geq \frac{c_3}{c_4} \inf_{w \in \partial B_x} |\psi_x(z) - w|$.

Suppose case (2): $w_0 \in \{y \mid d(y, z_{x,1}) > c_0/4\}$. Let γ be the Euclidean segment connecting z and w_0. Then, $d(z_{x,1}, \gamma) \geq c_5$, where c_5 is a positive constant depending only on c_0, which implies $d(\infty, \psi_x(\gamma)) \geq c_5$. Hence, by the Cauchy formula, we have $\inf_{w \in \partial B_x} |\psi_x(z) - w| \leq |\psi_x(z) - \psi_x(w_0)| \leq \sup_{w \in \gamma} |\psi_x'(w)| \cdot |z - w_0| \leq c_6 \delta_x(z)$, where c_6 is a positive constant independent of z and x_0.

From these arguments, we find that there exists a positive constant c_7 depending only on f and R, such that for each $z \in (\pi_{\overline{\mathbb{C}}} A_x(f)) \cap \{y \mid d(y, z_{x,1}) > c_0/2, d(y, \infty) > R\}$, we have $\rho_x(z)\delta_x(z) \geq c_7$. Similarly, we find that there exists a positive constant c_8 depending only on f and R, such that for each $z \in (\pi_{\overline{\mathbb{C}}} A_x(f)) \cap \{y \mid d(y, z_{x,2}) > c_0/2, d(y, \infty) > R\}$, we have $\rho_x(z)\delta_x(z) \geq c_8$. Since $d(z_{x,1}, z_{x,2}) > c_0$, we obtain the statement of the Corollary.

□

2.5 Proof of main theorem B

In this section, we prove main theorem B in Section 1.2.

Proof of Theorem 1.8. Let $f : \Lambda^{\mathbb{N}} \times \overline{\mathbb{C}} \to \Lambda^{\mathbb{N}} \times \overline{\mathbb{C}}$ be the fibered rational map over the shift map $g : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ $(g(h_1, h_2, h_3, \cdots) = (h_2, h_3, \cdots))$ defined as $f((h_1, h_2, \cdots),\ y) = ((h_2, h_3, \cdots),\ h_1(y))$. Then, by Theorem 1.5, there exist positive constants C_0 and C_1 such that each $g \in G$ satisfies the conditions that $J(g)$ is C_0-uniformly perfect and that diam $J(g) > C_1$.

Let H be any subsemigroup of G. Let A be an annulus that separates

$J(H)$. Let V_1 and V_2 be two connected components of $\overline{\mathbb{C}} \setminus A$. Since $J(G) = \overline{\bigcup_{g \in G} J(g)}$ (Corollary 3.1 in [HM1]) , there exist elements g_1 and g_2 in H such that $J(g_1) \cap V_1 \neq \emptyset$ and $J(g_2) \cap V_2 \neq \emptyset$.

If $J(g_1) \cap V_2 \neq \emptyset$ or $J(g_2) \cap V_1 \neq \emptyset$, then mod $A \leq C_0$. If $J(g_1) \cap V_2 = \emptyset$ and $J(g_2) \cap V_1 = \emptyset$, then, by Lemma 3.1 on p. 34 in [LV], there exists a constant $C = C(C_1)$ that depends only on C_1 such that mod $A \leq C$. The second statement follows from the uniform perfectness of $J(H)$ and Theorem 4.1 in [HM2]. □

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