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On p-adic families of Hilbert cusp forms of finite slope

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0. Introduction

Let p be an odd prime number. We fix an algebraic closure $\bar{\mathbb{Q}}$ of the field $\mathbb{Q}$ of rational numbers in the field $\mathbb{C}$ of complex numbers and an embedding $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, where $\bar{\mathbb{Q}}_p$ is an algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. We denote by i_∞ the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Then we take the p-adic completion $\mathbb{C}_p$ of $\bar{\mathbb{Q}}_p$ and fix an isomorphism $\mathbb{C}_p \cong \mathbb{C}$ of fields which is compatible with the embeddings i_p and i_∞. We denote by ord_p the normalized p-adic valuation in $\mathbb{C}_p$ so that $\mathrm{ord}_p(p) = 1$ and by $|\cdot|$ the absolute value given by ord_p. In this section, we would like to see the author's motivation, which is a story over $\mathbb{Q}$, for working on p-adic families of Hilbert cusp forms of finite slope.

Let N be a positive integer prime to p and $k \geq 2$ an integer. We take a normalized cuspidal Hecke eigenform f of level Np and weight k whose Fourier expansion is given by $f(q) = \sum_{n\geq 1} a_n(f)q^n$ with $a_1(f) = 1$. Then we know that the Fourier coefficint a_n is the $T(n)$-eigenvalue of f for each $n \geq 1$, where $T(n)$ is the Hecke operator at n. In particular, all $a_n(f)$'s belong to $\bar{\mathbb{Q}}$. We then put $\alpha := \mathrm{ord}_p(i_p(a_p(f)))$ and call it the $T(p)$-*slope* of f, which is a non-negative rational number in this case. Then it is known that if f satisfies some technical assumptions, then there exists a family $\{f_{k'}\}_{k'\in\mathcal{K}}$ of normalized cuspidal Hecke eigenforms $f_{k'}$ of weight k' and level Np having fixed $T(p)$-slope α parametrizd by an arithmetic progression $\mathcal{K}$ of radius p^m starting from k with some non-negative integer m. This fact has been proved in the case where $\alpha = 0$, i.e., *ordinary* case, by Hida [8] and [9], and his result has been generalized to the case where α is any non-negative rational number by Coleman [5] and [6].

The author [16, Main Theorem] used such families of finite $T(p)$-slopes to prove Gouvêa's conjecure in the unobstructed case, which asserts that all deformations of the mod p Galois representation associated with f to complete Noetherian local rings are associated with Katz's generalized p-adic modular forms of tame level N (for the details of this conjecture, see [16]). The author would like to generalize this result to the case over totally real fields.

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Now let us recall Coleman's arguments in [6] to obtain p-adic families $\{f_{k'}\}_{k'\in\mathcal{K}}$ of eigenforms having fixed $T(p)$-slope α as above. He constucted in [6, Section B4] the Banach module $S^{\dagger}(N)$ consisting of *families of overconvergent cusp forms* which is specialized to the Banach space $S_k^{\dagger}(N)$ of *overconvergent cusp forms* of weight k. One of the key points is that the Hecke operator $T(p)$ acts on these spaces *completely continuously.* The space $S_k^{\mathrm{cl}}(Np)$ of classical cusp forms of weight k and level Np is included in $S^{\dagger}(N)$. For any non-negative rational number α, we denote by $S_k^{\dagger}(N)^{\alpha}$ (resp. $S_k^{\mathrm{cl}}(Np)^{\alpha}$) the subspace of $S_k^{\dagger}(N)$ (resp. $S_k^{\mathrm{cl}}(Np)$) generated by all generalized $T(p)$-eigenspaces for all $T(p)$-eigenvalues whose p-adic valuation are α. Coleman [5, Theorem 8.1] proved that if $k > \alpha + 1$, then

$$S_k^{\dagger}(N)^{\alpha} = S_k^{\mathrm{cl}}(Np)^{\alpha},$$

i.e., the classciality of overconvergent cusp forms of small $T(p)$-slope, and that if $k \equiv k' \pmod{p^{m(\alpha)}}$ with some non-negative integer $m(\alpha)$ depending on α, then we have

$$\dim_{\mathbb{C}_p} S_k^{\dagger}(N)^{\alpha} = \dim_{\mathbb{C}_p} S_{k'}^{\dagger}(N)^{\alpha},$$

i.e, the local constancy of $\dim_{\mathbb{C}_p} S_k^{\dagger}(N)^{\alpha}$ with respect to weights k (cf. [6, Theorem B3.4]). Then as an application of these facts, under some technical conditions, he constructed p-adic families $\{f_{k'}\}_{k'\in\mathcal{K}}$ as above by means of the duality theorems between then classical Hecke algebras and the spaces of classical cusp forms and the theory of newforms and oldforms (see [6, Corollary B5.7.1]).

The aim of this article is to generalize Coleman's argments above to the case over totally real fields. Namely, we shall define in Section 1.1 the spaces $S_{(n,v)}^{\mathrm{cl}}(G;\Gamma_1(N);\mathbb{C}_p)$ of classical Hilbert cusp forms which are interpolated by the Banach module $S(G;\Gamma_1(N))$ of "p-adic Hilbert cusp forms" defined in Section 1.2. Then in Section 2.1 we shall define the Hecke operator $T(\pi)$ which acts on them completely continuously, and prove in Section 2.2 the classicality of p-adic Hilbert cusp forms of small $T(\pi)$-slope and in Section 2.3 the local constancy of dimensions of submodules having fixed $T(\pi)$-slope α. The method which we shall use is based on works of Buzzard [3] on "eigenvariety machine," and of Chenevier [4] dealing with automorphic forms on any twisted form of GL_n over $\mathbb{Q}$ which is compact at infinity modulo center.

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1. Classical and p-adic automorphic forms

In this section, we define spaces of classical automorphic forms and p-adic ones on the algebraic groups defined by the unit groups of totally definite quaternion algebras over totally real fields. In this article, we assume that p is an odd prime number for simplicity, although the case of $p = 2$ can be also done as well.

1.1. Classical automorphic forms

Let F be a totally real field of degree g and O its ring of integers. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be all prime ideals of F above p. Then the set I of all embeddings $\sigma : F \hookrightarrow \bar{\mathbb{Q}}$ has the partition $I = \bigsqcup_{i=1}^r I_i$, where I_i is the subset of I consisting of embeddings σ such that the completion of $i_p(F^\sigma)$ in $\mathbb{C}_p$ coincides with the $\mathfrak{p}_i^\sigma$-adic completion $F^\sigma_{\mathfrak{p}_i^\sigma}$ of F^σ.

In this article, we shall formulate "modular forms" as "automorphic forms" on adelic groups on quaternion algebras defined over F. Let B be a totally definite quaternion algebra over F. We fix a maximal order R of B and a finite Galois extension K_0 over $\mathbb{Q}$ containing F for which there is an isomorphism

$$B \otimes_{\mathbb{Q}} K_0 \cong M_2(K_0)^I$$

such that we have $R \otimes_{\mathbb{Z}} O_0 \cong M_2(O_0)^I$, where $M_2(A)$ with some ring A stands for the ring of 2×2 matrices with coefficients in A and $\mathbb{Z}$ and O_0 are the rings of integers in $\mathbb{Q}$ and K_0, respectively. Then we may assume that for a prime ideal $\mathfrak{l}$ at which B is unramified, this isomorphism induces an isomorphism

$$B \otimes_F F_{\mathfrak{l}} \cong M_2(F_{\mathfrak{l}})$$

such that we have $R \otimes_O O_{\mathfrak{l}} \cong M_2(O_{\mathfrak{l}})$, where $O_{\mathfrak{l}}$ is the $\mathfrak{l}$-adic completion of O. We fix this isomorphism in this article. Let G be the algebraic group defined over $\mathbb{Q}$ given by

$$G(A) := (B \otimes_{\mathbb{Q}} A)^\times$$

for $\mathbb{Q}$-algebras A. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\mathbb{A}_{\mathfrak{f}}$ its finite part. We denote by K the p-adic completion of $i_p(K_0)$ in $\mathbb{C}_p$ whose ring of integers is denoted by $\mathcal{O}$. For $\gamma \in G(\mathbb{A}_{\mathfrak{f}})$, under the natural identification

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^r F_{\mathfrak{p}_i},$$

we then take the *σ-projection* $\gamma_\sigma \in \mathrm{GL}_2(K)$ of the p-part $\gamma_p = (\gamma_i)_{i=1}^r \in G(\mathbb{Q}_p) = \prod_{i=1}^r (B \otimes_F F_{\mathfrak{p}_i})^\times$ of γ as the image in $\mathrm{GL}_2(K)$ of γ_i under the projection σ with the subscript i determined by the condition that $\sigma \in I_i$ for each $\sigma \in I$.

Let N be an integral ideal of F at which B is unramified. We put $\hat{R} := R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} := \prod_{l:\text{prime}} \mathbb{Z}_l$ with the rings $\mathbb{Z}_l$ of l-adic integers. We then define an open compact subgroup

$$\Gamma_1(N) := \{x \in \hat{R}^\times \text{with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a-1, c, d-1 \in NO_N\}$$

of $\hat{R}^\times$, where x_N is the N-part of x and $O_N := \prod_{\mathfrak{l}|N:\text{prime}} O_{\mathfrak{l}}$. By the approximation theorem, there exist $t_1, \ldots, t_h \in G(\mathbb{A})$ for some positive integer h such that $(t_i)_N = 1$ and $(t_i)_\infty = 1$ for each $i = 1, \ldots, h$ and

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{h} G(\mathbb{Q}) t_i \Gamma_1(N) G(\mathbb{R})_+, \tag{1}$$

where $G(\mathbb{R})_+$ is the connected component of $G(\mathbb{R})$ with the indentity. We fix the decomposition (1) in this article and put $\Gamma_i := (t_i^{-1} G(\mathbb{Q}) t_i) \cap \Gamma_1(N) G(\mathbb{R})_+$ for each $i = 1, \ldots, h$, which is a discrete subgroup of $G(\mathbb{R})_+$ (cf. [10, Section 2]). Since we assume that B is totally definite, we see that the quotient subgroup $\Gamma_i / \Gamma_i \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ of $G(\mathbb{R})_+ / G(\mathbb{R})_+ \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times$ is finite for each $i = 1, \ldots, h$.

Let $\mathbb{Z}[I]$ be the free $\mathbb{Z}$-module generated by I. We define an equivalence relation $\sim$ in $\mathbb{Z}[I]$ as follows: for $a, b \in \mathbb{Z}[I]$, $a \sim b$ if and only if $a - b \in \mathbb{Z} t_0$, where $t_0 := \sum_{\sigma \in I} \sigma$. We then put

$$W^{\text{cl}} := \{(n, v) \in \mathbb{Z}[I] \times \mathbb{Z}[I] \mid n + 2v \sim 0, n > 0\},$$

where we mean by $n > 0$ that n is *positive*, i.e., all coefficients n_σ of n are positive integers. We call W^{cl} the set of *classical weights*. For $(n, v) \in W^{\text{cl}}$ and any $\mathcal{O}$-algebra A, we denote by $L(n, v; A)$ the left $\mathrm{GL}_2(\mathcal{O})^I$-module consisting of polynomials P of $2g$-parameters $(X_\sigma, Y_\sigma)_{\sigma \in I}$ with coefficients in A which are homogeneous of degree n_σ for each variable (X_σ, Y_σ), on which $\gamma = (\gamma_\sigma)_{\sigma \in I} \in \mathrm{GL}_2(\mathcal{O})^I$ acts by

$$\gamma \cdot P := \det(\gamma)^v P(((X_\sigma, Y_\sigma)^t \gamma_\sigma^\iota)_{\sigma \in I}). \tag{2}$$

Here we define $\det(\gamma)^v := \prod_{\sigma \in I} \det(\gamma_\sigma)^{v_\sigma}$ and for a 2×2 matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put $x^\iota := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Definition 1.1. For $(n, v) \in W^{\text{cl}}$ and an $\mathcal{O}$-algebra A, we put

$$S^{\text{cl}}_{(n,v)}(G; \Gamma_1(N); A) := \{f : G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathrm{f}}) \to L(n, v; A) : \text{ function } \mid f(xu) = u^{-1} \cdot f(x) \text{ for } u \in \Gamma_1(N), x \in G(\mathbb{A}_{\mathrm{f}})\},$$

which we call the space of *classical automorphic forms of level* $\Gamma_1(N)$ *and weight* (n, v) *on* G *(defined over* A*)*.

Remark 1.1. In the case where we regard $A = \mathbb{C}$ as an $\mathcal{O}$-algebra via the fixed isomorphism $\mathbb{C}_p \xrightarrow{\sim} \mathbb{C}$ and B is unramified at all finite places of F (hence g must be even by Hasse principle (cf. [15, XIII, Sections 3 and 6])), it is known that $S^{\mathrm{cl}}_{(n,v)}(G;\Gamma_1(N);\mathbb{C})$ are isomorphic to the spaces of classical holomorphic Hilbert cusp forms of weight $(n_\sigma+2)_{\sigma\in I}$ and level N by a result of Jacquet-Langlands and Shimizu (cf. [10, Theorem 2.1]).

1.2. p-Adic automorphic forms

We fix a classical weight $(n,v)\in W^{\mathrm{cl}}$. Let N be an integral ideal of F which is not prime to p and unramified in B. We now take *arbitrarily* $s(\le r)$ prime ideals above p which divide N. We may denote them by $\mathfrak{p}_1,\dots,\mathfrak{p}_s$. We then put $I' := \sqcup_{i=1}^{s} I_i \subset I$ and denote the cardinality of I' by $g'(\le g)$. We fix a prime element π_i of the $\mathfrak{p}_i$-adic completion $F_{\mathfrak{p}_i}$ of F at $\mathfrak{p}_i$ for each $i=1,\dots,s$. We then denote by $\begin{pmatrix}1&0\\0&\pi\end{pmatrix}$ the element of $G(\mathbb{A}_{\mathrm{f}})$ whose $\mathfrak{p}_i$-part is the diagonal matrix $\begin{pmatrix}1&0\\0&\pi_i\end{pmatrix}$ for each $i=1,\dots,s$ and other parts are trivial. In the following, for an element $\gamma\in\Gamma_1(N)$, we write its σ-projection as

$$\gamma_\sigma = \begin{pmatrix}1+\pi_i^\sigma a_\sigma & b_\sigma\\ \pi_i^\sigma c_\sigma & 1+\pi_i^\sigma d_\sigma\end{pmatrix}$$

with some $a_\sigma, b_\sigma, c_\sigma, d_\sigma \in \mathcal{O}$ for each $\sigma\in I$ with i such that $\sigma\in I_i$. Then we have

$$(3)\qquad (X_\sigma,Y_\sigma)^t\gamma_\sigma{}^\iota = ((1+\pi_i^\sigma d_\sigma)X_\sigma - b_\sigma Y_\sigma, Y_\sigma + \pi_i^\sigma(a_\sigma Y_\sigma - c_\sigma X_\sigma))$$

for all $\sigma\in I'$ with i such that $\sigma\in I_i$, and

$$(4)\qquad (X_\sigma,Y_\sigma)^t\begin{pmatrix}1&0\\0&\pi\end{pmatrix}_\sigma^\iota = \begin{cases}(\pi_i^\sigma X_\sigma, Y_\sigma) & (\sigma\in I_i\subset I'),\\ (X_\sigma,Y_\sigma) & (\sigma\in I\setminus I').\end{cases}$$

For any elements $\gamma=\gamma_1\begin{pmatrix}1&0\\0&\pi\end{pmatrix}\gamma_2$ with $\gamma_1,\gamma_2\in\Gamma_1(N)$ of the double coset $\Gamma_1(N)\begin{pmatrix}1&0\\0&\pi\end{pmatrix}\Gamma_1(N)$, using actions (3) and (4), we define a K-endomorphism $[\gamma]_{(n,v)}$ on $L(n,v;K)$ with normalization of the $\det^v$-part by

$$(5)\ [\gamma]_{(n,v)}\cdot P := \prod_{\tau\in I\setminus I'}\det(\gamma_\tau)^{v_\tau}\prod_{\sigma\in I'}\det(\gamma_{1\sigma}\gamma_{2\sigma})^{v_\sigma} \times P(((X_\tau,Y_\tau)^t\gamma_\tau^\iota)_{\tau\in I}).$$

Let $K\langle x_\sigma | \sigma \in I' \rangle$ be the strictly convergent power series ring of g'-variables $(x_\sigma)_{\sigma \in I'}$ with coefficients in K, which is the subring of the formal power series ring $K[\![x_\sigma | \sigma \in I']\!]$ consisting of power series $P(x) = \sum_{(i_\sigma)_{\sigma \in I'} \in \mathbb{Z}_{\geq 0}^{I'}} a_{(i_\sigma)_{\sigma \in I'}} \prod_{\sigma \in I'} x_\sigma^{i_\sigma}$ such that $|a_{(i_\sigma)_{\sigma \in I'}}| \to 0$ as $\sum_{\sigma \in I'} i_\sigma \to \infty$. This is an orthonormalizable K-Banach algebra with sup norm $|\cdot|$ with respect to coefficients in K (for the notion in the p-adic Banach theory, see [6, Chapter A]). We can take the set $\{\prod_{\sigma \in I'} x_\sigma^{i_\sigma} | i_\sigma \geq 0, \sigma \in I'\}$ as an orthonormal basis of $K\langle x_\sigma | \sigma \in I' \rangle$. We define actions on the variables $(x_\sigma)_{\sigma \in I'}$ of the σ-projections of $\gamma \in \Gamma_1(N)$ and $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ for $\sigma \in I'$ as follows:

$$(6)\, \gamma_\sigma \cdot x_\sigma := \frac{-b_\sigma + (1 + \pi_i^\sigma d_\sigma) x_\sigma}{1 + \pi_i^\sigma (a_\sigma - c_\sigma x_\sigma)} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}_\sigma \cdot x_\sigma := \pi_i^\sigma x_\sigma$$

with i such that $\sigma \in I_i$. Note that the denominator $1 + \pi_i^\sigma (a_\sigma - c_\sigma x_\sigma)$ in the action (6) is a unit in $\mathcal{O}\langle x_\sigma \rangle$. Then by [6, Lemma A1.6], we see that elements in the double coset $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$ give completely continuous K-endomorphisms on $K\langle x_\sigma | \sigma \in I' \rangle$ whose operator norms are at most 1. Here the operator norm $|L|$ of a continuous endomorphism L on a Banach module M is defined by

$$|L| := \sup_{0 \neq m \in M} \frac{|L(m)|}{|m|}.$$

Now we define a Banach module S over the strictly convergent power series ring $K\langle \xi_\sigma | \sigma \in I' \rangle$ of g'-variables $(\xi_\sigma)_{\sigma \in I'}$ as follows: S is the set of polynomials P of $2(g-g')$-parameters $(X_\tau, Y_\tau)_{\tau \in I \setminus I'}$ with coefficients in $K\langle \xi_\sigma, x_\sigma | \sigma \in I' \rangle$ which are homogeneous of degree n_τ for each variable (X_τ, Y_τ). We can take the set

$$\{(\prod_{\tau \in I \setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau}) \prod_{\sigma \in I'} x_\sigma^{m_\sigma} | a_\tau + b_\tau = n_\tau \text{ with } a_\tau, b_\tau \geq 0,\ m_\sigma \geq 0\}$$

as an orthonormal basis of S over $K\langle \xi_\sigma | \sigma \in I' \rangle$. Let $e(\mathfrak{p}_i)$ be the ramification index of the prime ideal $\mathfrak{p}_i$ in $F/\mathbb{Q}$. In order to define an action of $\Gamma_1(N)$ on S, we assume the condition that

$$\text{(ram)} \qquad e(\mathfrak{p}_i) < p - 1 \quad \text{for each} \quad i = 1, \ldots, s$$

is satisfied in the following. We see that $j_\sigma(\gamma_\sigma)$ for elements γ of $\Gamma_1(N)$ and $\Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$, and $\det(\gamma_\sigma)$ for $\gamma \in \Gamma_1(N)$ are of the form $1 + \pi_i^\sigma a$ with some $a \in \mathcal{O}$ for each $\sigma \in I'$ with i such that $\sigma \in I_i$. Then we can define their powers with any element s in $\mathbb{C}_p$ (resp. $\mathbb{C}_p\langle\xi_\sigma\rangle$) such that $|s| \leq 1$ by a convergent power series as

$$(7) \qquad (1+\pi_i^\sigma a)^s := 1 + \sum_{k\geq 1} \frac{s(s-1)\cdots(s-k+1)}{k!}(\pi_i^\sigma)^k a^k$$

in $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\mathcal{O}_{\mathbb{C}_p}\langle\xi_\sigma\rangle$) because of the assumption (ram) (cf. [4, Lemme 3.6.1]). Here we denote by $\mathcal{O}_{\mathbb{C}_p}$ the ring of *p-adic integers* in $\mathbb{C}_p$, i.e., the subring of $\mathbb{C}_p$ consisting of elements s such that $|s| \leq 1$. We then define an action $[\gamma]$ of $\gamma \in \Gamma_1(N)$ on S as

$$(8) \quad [\gamma]\cdot P := \prod_{\tau\in I\setminus I'} \det(\gamma_\tau)^{v_\tau}\Big(\prod_{\sigma\in I'} j_\sigma(\gamma_\sigma)^{\xi_\sigma}\det(\gamma_\sigma)^{\frac{\mu(n,v)-\xi_\sigma}{2}}\Big)$$
$$\times P(((X_\tau,Y_\tau)^t\gamma_\tau^\iota)_{\tau\in I\setminus I'};(\xi_\sigma,\gamma_\sigma\cdot x_\sigma)_{\sigma\in I'}).$$

As for $\gamma = \gamma_1\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\gamma_2 \in \Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N)$ with $\gamma_1,\gamma_2\in\Gamma_1(N)$, we define a $K\langle\xi_\sigma|\sigma\in I'\rangle$-endomorphism on S as

$$(9) \quad [\gamma]\cdot P := \prod_{\tau\in I\setminus I'} \det(\gamma_\tau)^{v_\tau}\Big(\prod_{\sigma\in I'} j_\sigma(\gamma_\sigma)^{\xi_\sigma}\det(\gamma_{1\sigma}\gamma_{2\sigma})^{\frac{\mu(n,v)-\xi_\sigma}{2}}\Big)$$
$$\times P(((X_\tau,Y_\tau)^t\gamma_\tau^\iota)_{\tau\in I\setminus I'};(\xi_\sigma,\gamma_\sigma\cdot x_\sigma)_{\sigma\in I'}),$$

which is completely continuous with operator norm ≤ 1.

Definition 1.2. We denote by $\mathcal{W}_{(n,v)}$ the g'-dimensional closed affinoid ball over K of radius 1 around $(n_\sigma)_{\sigma\in I'}$. Then the set $\mathcal{W}_{(n,v)}(\mathbb{C}_p)$ of its $\mathbb{C}_p$-valued points coincides with $\mathcal{O}_{\mathbb{C}_p}^{I'}$ and $K\langle\xi_\sigma|\sigma\in I'\rangle$ is the affinoid algebra associated to $\mathcal{W}_{(n,v)}$. (For the details of affinoid algebras and affinoid varieties, see [1, Part B and Chapter 7] and [6, Section A5].) We call it the space of the *I'-parts of p-adic weights associated to (n,v)*. We then associate $(t_\sigma := \frac{\mu(n,v)-s_\sigma}{2})_{\sigma\in I'}$ to any point $(s_\sigma)_{\sigma\in I'}\in\mathcal{W}_{(n,v)}(\mathbb{C}_p)$, and put the *p-adic weight* (s,t) as

$$s := \sum_{\sigma\in I'} s_\sigma\sigma + \sum_{\tau\in I\setminus I'} n_\tau\tau \quad \text{and}$$
$$t := \frac{\mu(n,v)t_0 - s}{2} = \sum_{\sigma\in I'} t_\sigma\sigma + \sum_{\tau\in I\setminus I'} v_\tau\tau.$$

Further, we denote by $W^{\mathrm{cl}}_{(n,v)}$ the subset of $\mathcal{W}_{(n,v)}(\mathbb{C}_p)$ consisting of elements $(n'_\sigma)_{\sigma\in I'}$ whose components are positive integers of the same parity as $\mu(n,v)$ for all $\sigma\in I'$. We call it the set of the *I'-parts of classical weights associated to (n,v)*. For $(n'_\sigma)_{\sigma\in I'}\in W^{\mathrm{cl}}_{(n,v)}$, we put $(v'_\sigma := \frac{\mu(n,v)-n'_\sigma}{2})_{\sigma\in I'}$ and define (n',v') as well as (s,t). By the definition of $W^{\mathrm{cl}}_{(n,v)}$, we see that v'_σ are also integers for all $\sigma \in I'$ and that $n' + 2v' = \mu(n,v)t_0$.

For $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$, we denote by $K_{(s,t)}$ the p-adic completion in $\mathbb{C}_p$ of the fraction field of $K\langle\xi_\sigma|\sigma \in I'\rangle/(\xi_\sigma - s_\sigma|\sigma \in I')$. We denote by $S_{(s,t)}$ the specialized orthonormalizable $K_{(s,t)}$-Banach space $S\otimes_{K\langle\xi_\sigma|\sigma\in I'\rangle} K_{(s,t)}$. Then we denote by $[\gamma]_{(s,t)}$ the specialized $K_{(s,t)}$-endomorphism $[\gamma]\otimes K_{(s,t)}$ on $S_{(s,t)}$ for elements γ of $\Gamma_1(N)$ and $\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N)$.

Definition 1.3. (1) Assume the condition (ram). We define the space of *p-adic automorphic forms of level* $\Gamma_1(N)$ *on* G *(with coefficients in* K*)* as

$$S(G;\Gamma_1(N)) := \{f : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}}) \to S : \text{function}| \\ f(xu) = [u^{-1}]\cdot f(x), u \in \Gamma_1(N), x \in G(\mathbb{A}_{\mathrm{f}})\}.$$

We then have a K-isomorphism

$$(10)\qquad S(G;\Gamma_1(N)) \xrightarrow{\sim} \oplus_{i=1}^h S^{\Gamma_i},\ f \mapsto (f(t_1),\dots,f(t_h)),$$

where $t_1,\dots,t_h \in G(\mathbb{A})$ are the fixed representatives of the decomposition (1). Here each S^{Γ_i} is the submodule of the orthonormalizable $K\langle\xi_\sigma|\sigma\in I'\rangle$-module S consisting of elements fixed under the action of $\Gamma_i = (t_i^{-1}G(\mathbb{Q})t_i)\cap\Gamma_1(N)G(\mathbb{R})_+$. Since Γ_i acts on S via the finite quotient group $\Gamma_i/\Gamma_i\cap(F\otimes_{\mathbb{Q}}\mathbb{R})^\times$ because of the assumption $n+2v\sim 0$, we then see that S^{Γ_i} satisfies the property (Pr) of [3, Section 2] for each $i = 1,\dots,h$. We now define a norm in $S(G;\Gamma_1(N))$ via this isomorphism as

$$|f| := \sup_{1\le i\le h}|f(t_i)|.$$

Therefore, $S(G;\Gamma_1(N))$ can be regarded as a $K\langle\xi_\sigma|\sigma\in I'\rangle$-Banach module with the norm $|\cdot|$ which satisfies the property (Pr) of [3, Section 2].

(2) Let $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$. Assume the condition (ram) in the case where $(s_\sigma)_{\sigma\in I'} \notin W^{\mathrm{cl}}_{(n,v)}$. We define the space of *p-adic automorphic forms of weight* (s,t) *and level* $\Gamma_1(N)$ *on* G *(defined over* $K_{(s,t)}$*)* as

$$S_{(s,t)}(G;\Gamma_1(N)) := \{f : G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}}) \to S_{(s,t)} : \text{function}| \\ f(xu) = [u^{-1}]_{(s,t)}\cdot f(x), u \in \Gamma_1(N), x \in G(\mathbb{A}_{\mathrm{f}})\}.$$

Then we have an isomorphism

$$(11)\quad S_{(s,t)}(G;\Gamma_1(N)) \xrightarrow{\sim} \oplus_{i=1}^h S^{\Gamma_i}_{(s,t)},\ f \mapsto (f(t_1),\dots,f(t_h))$$

of $K_{(s,t)}$-Banach spaces satisfying the property (Pr) of [3, Section 2], where we define a norm in $S_{(s,t)}(G;\Gamma_1(N))$ as

$$|f| := \sup_{1\le i\le h} |f(t_i)|.$$

Putting $x_\sigma = \frac{X_\sigma}{Y_\sigma}$ for each $\sigma \in I'$, we then see easily the following

Lemma 1.1. *For any $(n'_\sigma)_{\sigma\in I'} \in W^{\mathrm{cl}}_{(n,v)}$, we have a natural K-inclusion*

$$L(n',v';K) \hookrightarrow S_{(n',v')},$$
$$P((X_\tau,Y_\tau)_{\tau\in I}) \mapsto P((X_\tau,Y_\tau)_{\tau\in I\setminus I'};(x_\sigma,1)_{\sigma\in I'})$$

which is compatible with $[\gamma]_{(n',v')}$ for all γ in $\Gamma_1(N)$ and the double coset $\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N)$ on these spaces. Thus we have an inclusion

$$S^{\mathrm{cl}}_{(n',v')}(G;\Gamma_1(N);K) \hookrightarrow S_{(n',v')}(G;\Gamma_1(N))$$

of K-Banach spaces satisfying the property (Pr) of [3, Section 2].

2. p-Adic automorphic forms of small $T(\pi)$-slope

Let the notation be as in Section 1.2. In this section, we shall introduce the Hecke operator $T(\pi)$ on the spaces of p-adic automorphic forms. Then we shall investigate some properties of p-adic automorphic forms having small $T(\pi)$-slope.

2.1. The Hecke operator $T(\pi)$

In this subsection, we assume the condition (ram), i.e., $e(\mathfrak{p}_i) < p-1$ for all $i = 1,\dots,s$, unless we deal with the I'-parts of classical weights in $W^{\mathrm{cl}}_{(n,v)}$. In order to define the Hecke operator $T(\pi)$, we decompose the double coset $\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N)$ in a disjoint union of right cosets as

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N) = \bigsqcup_{i=1}^{l}\zeta_i\Gamma_1(N).$$

For $f \in S(G;\Gamma_1(N))$ (resp. $S_{(s,t)}(G;\Gamma_1(N))$ for $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$), we put

$$(f|T(\pi))(x) := \sum_{i=1}^{l}[\zeta_i]\cdot f(x\zeta_i) \ \left(\text{resp. } \sum_{i=1}^{l}[\zeta_i]_{(s,t)}\cdot f(x\zeta_i)\right) \tag{12}$$

for $x \in G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})$. Note that this definition is independent of choices of representatives $\{\zeta_i\}$ and $f|T(\pi)$ is also an element of $S(G;\Gamma_1(N))$ (resp. $S_{(s,t)}(G;\Gamma_1(N))$) (cf. [10, Section 2]).

Proposition 2.1. *Assume the condition* (ram) *unless* $(s_\sigma)_{\sigma\in I'} \in W^{\mathrm{cl}}_{(n,v)}$. *The Hecke operator* $T(\pi)$ *is completely continuous on* $S(G;\Gamma_1(N))$ *and* $S_{(s,t)}(G;\Gamma_1(N))$ *for any* $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$ *with operator norm* ≤ 1.

Proof. We shall prove the proposition for $S(G;\Gamma_1(N))$, because we can prove in the case of $S_{(s,t)}(G;\Gamma_1(N))$ as well. To see the complete continuity of $T(\pi)$, we calculate the action of $T(\pi)$ on $\oplus_{j=1}^h S^{\Gamma_j}$ via the isomorphism (10) by means of the decomposition

$$\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N) = \bigsqcup_{i=1}^{l} \zeta_i\Gamma_1(N).$$

For $f \in S(G;\Gamma_1(N))$, the image of $f|T(\pi)$ under the isomorphism (10) is

$$\begin{aligned}((f|T(\pi))(t_1),\dots,(f|T(\pi))(t_h))\\ = \sum_{i=1}^{l}([\zeta_i]\cdot f(t_1\zeta_i),\dots,[\zeta_i]\cdot f(t_h\zeta_i)).\end{aligned}$$

We fix $1 \le i \le l$. For each $j = 1,\dots,h$, there exist $1 \le \sigma_i(j) \le h$ and $u_i(j) \in \Gamma_1(N)$ such that

$$t_j\zeta_i = t_{\sigma_i(j)}u_i(j)$$

in $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$. Then we see that

$$f(t_j\zeta_i) = f(t_{\sigma_i(j)}u_i(j)) = [u_i(j)^{-1}]\cdot f(t_{\sigma_i(j)})$$

by the definition of automorphic forms of level $\Gamma_1(N)$. Therefore we see that

$$\begin{aligned}((f|T(\pi))(t_1),\dots,(f|T(\pi))(t_h))\\ = \sum_{i=1}^{l}([\zeta_i u_i(1)^{-1}]\cdot f(t_{\sigma_i(1)}),\dots,[\zeta_i u_i(h)^{-1}]\cdot f(t_{\sigma_i(h)})).\end{aligned}$$

Thus the proposition is proven, because the endomorphisms $[\cdot]$ given by the double coset $\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma_1(N)$ on S are completely continuous with operator norm ≤ 1. □

We denote by $K\langle\xi_\sigma|\sigma\in I'\rangle\{\{X\}\}$ the subring of the formal power series ring $K\langle\xi_\sigma|\sigma\in I'\rangle[\![X]\!]$ consisting of power series $\sum_{i\ge 0} c_iX^i$ such that

$$|c_i|M^i \to 0 \quad \text{as} \quad i\to\infty$$

for all $M \in \mathbb{R}$. By Proposition 2.1 and the arguments in [3, Section 2] dealing with Banach modules satisfying the property (Pr), we have the following

Proposition 2.2. *Assume the condition* (ram). *We have the characteristic power series*

$$P((\xi_\sigma)_{\sigma\in I'}, X) := \det(1 - XT(\pi)|_{S(G;\Gamma_1(N))})$$
$$= 1 + \sum_{i\geq 1} c_i X^i \in K\langle \xi_\sigma | \sigma \in I' \rangle \{\{X\}\}$$

of $T(\pi)$ *on* $S(G;\Gamma_1(N))$ *with* $|c_i| \leq 1$. *Furthermore, for any* $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$, *we see that*

$$P((s_\sigma)_{\sigma\in I'}, X) = 1 + \sum_{i\geq 1} c_i((s_\sigma)_{\sigma\in I'}) X^i \in K_{(s,t)}\{\{X\}\}$$

is the characteristic power series of $T(\pi)$ *on* $S_{(s,t)}(G;\Gamma_1(N))$.

Let α be a non-negative rational number. For $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$, let $S_{(s,t)}(G;\Gamma_1(N))^{\alpha}_{\mathbb{C}_p}$ be the $\mathbb{C}_p$-subspace of $S_{(s,t)}(G;\Gamma_1(N)) \otimes_{K_{(s,t)}} \mathbb{C}_p$ generated by all generalized $T(\pi)$-eigenspaces for all eigenvalues λ such that $\mathrm{ord}_p(\lambda) = \alpha$. In the following subsections, we shall investigate p-adic automorphic forms which have small $T(\pi)$-slope.

2.2. Classicality of p-adic automorphic forms

In Lemma 1.1 without the condition (ram), we have seen that the spaces of classical automorphic forms are included in the ones of p-adic automorphic forms. Now we shall see that p-adic automorphic forms of small $T(\pi)$-slope are classical. Namely,

Theorem 2.3. *Let* $\alpha \in \mathbb{Q}_{\geq 0}$ *and* $(n'_\sigma)_{\sigma\in I'} \in W^{\mathrm{cl}}_{(n,v)}$. *If the condition*

$$\alpha < \nu_{n'} := \min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}(\min_{\sigma\in I_i}\{n'_\sigma\} + 1)\}$$

is satisfied, then we have (without the condition (ram)*)*

$$S_{(n',v')}(G;\Gamma_1(N))^{\alpha}_{\mathbb{C}_p} = S^{\mathrm{cl}}_{(n',v')}(G;\Gamma_1(N);\mathbb{C}_p)^{\alpha}.$$

Proof. By the isomorphism (11) in Section 1, we see that the $\mathbb{C}_p$-Banach quotient space $(S_{(n',v')}(G;\Gamma_1(N)) \otimes_K \mathbb{C}_p)/S^{\mathrm{cl}}_{(n',v')}(G;\Gamma_1(N);\mathbb{C}_p)$ is isomorphic to a direct summand of the direct sum of h-copies of the orthonormalizable $\mathbb{C}_p$-Banach quotient space $S_{(n',v')} \otimes_K \mathbb{C}_p/L(n',v';\mathbb{C}_p)$

whose orthonormal basis is

$$\{(\prod_{\tau\in I\setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau}) \prod_{\sigma\in I'} x_\sigma^{m_\sigma} | a_\tau + b_\tau = n_\tau \text{ with } a_\tau, b_\tau \geq 0,\ m_\sigma \geq 0$$
$$\text{and } m_\sigma > n'_\sigma \text{ for some } \sigma\}.$$

By the actions (3), (4) and (6) on the variables X_τ, Y_τ and x_σ in Section 1.2, we then see easily that

$$|T(\pi)| \leq p^{-\nu_{n'}}$$

on $(S_{(n',v')} \otimes_K \mathbb{C}_p / L(n', v'; \mathbb{C}_p))^h$. Hence we see that if $\alpha < \nu_{n'}$, then the image of any generalized $T(\pi)$-eigenvector of slope α is 0 in the quotient space $(S_{(n',v')}(G; \Gamma_1(N)) \otimes_K \mathbb{C}_p)/S^{\mathrm{cl}}_{(n',v')}(G; \Gamma_1(N); \mathbb{C}_p)$. So we have

$$S_{(n',v')}(G; \Gamma_1(N))^\alpha_{\mathbb{C}_p} = S^{\mathrm{cl}}_{(n',v')}(G; \Gamma_1(N); \mathbb{C}_p)^\alpha.$$

□

Remark 2.1. It is known that the spaces of definite quaternionic automorphic forms over $\mathbb{Q}$ defined by means of homogeneous polynomials of degree n are isomorphic to the spaces of elliptic cusp forms of weight $k = n + 2$ by Jacquet-Langlands' theorem (cf. [2, Theorem 2]). Coleman [5, Theorem 6.1 and Theorem 8.1] showed that p-adic overconvergent modular forms of weight k and U_p-slope α are classical if $\alpha < k - 1 (= n + 1)$. Since $s = 1$ and $e(p) = 1$ in the case of $F = \mathbb{Q}$, Theorem 2.3 is a generalization of the result of Coleman to the case over totally real fields.

2.3. The local constancy of $\dim_{\mathbb{C}_p} S_{(s,t)}(G; \Gamma_1(N))^\alpha_{\mathbb{C}_p}$

We assume the condition (ram), i.e., $e(\mathfrak{p}_i) < p-1$ for all $i = 1, \ldots, s$. Let $\alpha \in \mathbb{Q}_{\geq 0}$. In this subsection, we shall give an explicit description of $m(\alpha)$ such that if $(s_\sigma)_{\sigma\in I'}, (s'_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$ satisfy that $|s_\sigma - s'_\sigma| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$, then we have

$$\dim_{\mathbb{C}_p} S_{(s,t)}(G; \Gamma_1(N))^\alpha_{\mathbb{C}_p} = \dim_{\mathbb{C}_p} S_{(s',t')}(G; \Gamma_1(N))^\alpha_{\mathbb{C}_p}$$

by applying Chenevier's argument in [4, Section 5] to our case.

By Definition 1.3 (2), we regard $S_{(s,t)}(G; \Gamma_1(N))$ as a direct summand of the orthonormalizable $K_{(s,t)}$-Banach module $S^h_{(s,t)}$ for which we can also have the characteristic power series

$$P'((s_\sigma)_{\sigma\in I'}, X) =: 1 + \sum_{i\geq 1} c'_i((s_\sigma)_{\sigma\in I'}) X^i \in K_{(s,t)}\{\{X\}\}$$

with $|c'_i((s_\sigma)_{\sigma\in I'})|\le 1$. To obtain $m(\alpha)$ as above, we shall investigate the Newton polygon $N'_{(s,t)}$ of $P'((s_\sigma)_{\sigma\in I'},X)$. We can take the set

$$\{e_{M,a}:=(0,\dots,M,\dots,0)\}_{M\in\mathfrak{M},\ 1\le a\le h}$$

as an orthonormal basis of $S^h_{(s,t)}$, where we put the set of monomials

$$\mathfrak{M}:=\{(\prod_{\tau\in I\setminus I'}X_\tau^{a_\tau}Y_\tau^{b_\tau})\prod_{\sigma\in I'}x_\sigma^{m_\sigma}|a_\tau+b_\tau=n_\tau \text{ with } a_\tau,b_\tau\ge 0,\ m_\sigma\ge 0\}$$

and M sits in the a-th component in $e_{M,a}$. We shall calculate the p-adic valuations of coefficients $c'_i((s_\sigma)_{\sigma\in I'})$ of $P'((s_\sigma)_{\sigma\in I'},X)$ by means of this basis. For $\gamma=\gamma_1\begin{pmatrix}1&0\\0&\pi\end{pmatrix}\gamma_2\in\Gamma_1(N)\begin{pmatrix}1&0\\0&\pi\end{pmatrix}\Gamma_1(N)$ with $\gamma_1,\gamma_2\in\Gamma_1(N)$ and a monomial $M=(\prod_{\tau\in I\setminus I'}X_\tau^{a_\tau}Y_\tau^{b_\tau})\prod_{\sigma\in I'}x_\sigma^{m_\sigma}\in\mathfrak{M}$, we have

$$(13)\quad [\gamma]_{(s,t)}\cdot M=\prod_{\tau\in I\setminus I'}\det(\gamma_\tau)^{v_\tau}(\prod_{\sigma\in I'}j_\sigma(\gamma_\sigma)^{s_\sigma}\det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_\sigma})$$
$$\times((\prod_{\tau\in I\setminus I'}X_\tau^{a_\tau}Y_\tau^{b_\tau})^t\gamma_\tau^\iota)\prod_{\sigma\in I'}(\gamma_\sigma\cdot x_\sigma)^{m_\sigma}.$$

By the definition of $j_\sigma(\gamma_\sigma)^{s_\sigma}$ and the action (6) on the variable x_σ in Section 1.2 for each $\sigma\in I'$, we see that the p-adic valuations of all coefficients of monomials of the form $(\prod_{\tau\in I\setminus I'}X_\tau^{a'_\tau}Y_\tau^{b'_\tau})\prod_{\sigma\in I'}x_\sigma^{k_\sigma}$ in the expansion of (13) in $S_{(s,t)}$ are at least $\lambda\sum_{\sigma\in I'}k_\sigma$, where we put the positive rational number $\lambda:=\min_{1\le i\le s}\{\frac{1}{e(\mathfrak{p}_i)}\}-\frac{1}{p-1}$. Now we order the basis $\{e_{M,a}\}_{M,a}$ as follows: For $k\ge 0$, we define the subset

$$\mathcal{A}_k:=\{e_{M,a}|1\le a\le h,\ M \text{ is of the form } (\prod_{\tau\in I\setminus I'}X_\tau^{a_\tau}Y_\tau^{b_\tau})\prod_{\sigma\in I'}x_\sigma^{k_\sigma} \text{ with } \sum_{\sigma\in I'}k_\sigma=k\}$$

of $\{e_{M,a}\}_{M,a}$. Then we see that the cardinality $\sharp\mathcal{A}_k=h_n\binom{k+g'-1}{g'-1}$ for $k\ge 0$, where $h_n:=h\prod_{\tau\in I\setminus I'}(n_\tau+1)$, and that for $k\ge 1$,

$$(14)\qquad \sum_{q=0}^{k}q\cdot\sharp\mathcal{A}_q=h_n g'\binom{k+g'}{g'+1}.$$

We then exhibit elements of $\mathcal{A}_0$ as $e_1^{(0)},\dots,e_{h_n}^{(0)}$ arbitrarily. Next we exhibit elements of $\mathcal{A}_1$ as $e_{h_n+1}^{(1)},\dots,e_{h_n(g'+1)}^{(1)}$ arbitrarily. We then repeat this operation for all $k\ge 2$ as

$$e^{(k)}_{h_n\binom{k+g'-1}{g'}+1},\dots,e^{(k)}_{h_n\binom{k+g'}{g'}}.$$

We are going to obtain the representation matrix of infinite degree of $T(\pi)$ with respect to the basis $\{e_j^{(l)}\}_{j,l}$ ordered as above. For each $e_j^{(l)}$, we write

$$e_j^{(l)}|T(\pi) = \sum_{i_0=1}^{h_n} \alpha_{i_0}^{(0)}(j,l)e_{i_0}^{(0)} + \sum_{k\geq 1} \sum_{i_k=h_n\binom{k+g'-1}{g'-1}+1}^{h_n\binom{k+g'}{g'}} \alpha_{i_k}^{(k)}(j,l)e_{i_k}^{(k)}$$

with $\alpha_{i_k}^{(k)}(j,l) \in \mathcal{O}_{(s,t)}$ for all $k \geq 0$, where $\mathcal{O}_{(s,t)}$ is the ring of integers in $K_{(s,t)}$. As mentioned above, we then see that

$$\text{(15)} \qquad \operatorname{ord}_p(\alpha_{i_k}^{(k)}(j,l)) \geq k\lambda$$

for all $k \geq 0$, $j \geq 1$ and $l \geq 0$. The representation matrix of $T(\pi)$ with respect to the ordered basis $\{e_1^{(0)}, \dots, e_{h_n}^{(0)}, \dots\}$ is of the form

$$\begin{pmatrix} \alpha_1^{(0)}(1,0) & \cdots & \alpha_1^{(0)}(h_n,0) & \cdots \\ \vdots & & \vdots & \\ \alpha_{h_n}^{(0)}(1,0) & \cdots & \alpha_{h_n}^{(0)}(h_n,0) & \cdots \\ \vdots & & \vdots & \\ \vdots & & \vdots & \end{pmatrix}.$$

It is known that the coefficient $c_i'((s_\sigma)_{\sigma\in I'})$ of $P'((s_\sigma)_{\sigma\in I'}, X)$ is given by $(-1)^i\times$ (the convergent sum of i-th minors of the above matrix) for each $i \geq 1$ (cf. [13, Proposition 7 (a)]). So we see easily that

$$\operatorname{ord}_p(c_i'((s_\sigma)_{\sigma\in I'})) > i^{1+\frac{1}{g'}} \frac{2\lambda g'}{(g'+1)(g'+2)^2}\left(\frac{g'!}{h_n}\right)^{\frac{1}{g'}}$$

by (14) and (15) in the case where

$$h_n\binom{k+g'-1}{g'} + 1 \leq i \leq h_n\binom{k+g'}{g'}$$

with some $k \geq 2$. On the other hand, in the case where $1 \leq i \leq h_n(g'+1)$, we see that $\operatorname{ord}_p(c_i'((s_\sigma)_{\sigma\in I'})) \geq 0$ by Proposition 2.2. Therefore we have

(16)

$$\operatorname{ord}_p(c_i'((s_\sigma)_{\sigma\in I'})) \geq \frac{2\lambda g'}{(g'+1)(g'+2)^2}\left(\frac{g'!}{h_n}\right)^{\frac{1}{g'}} i\left(i^{\frac{1}{g'}} - (h_n(g'+1))^{\frac{1}{g'}}\right)$$

for all $i \geq 1$. We put the function

$$\mu(x) := \frac{2\lambda g'}{(g'+1)(g'+2)^2}\left(\frac{g'!}{h_n}\right)^{\frac{1}{g'}} x\left(x^{\frac{1}{g'}} - (h_n(g'+1))^{\frac{1}{g'}}\right)$$

on $\mathbb{R}_{\geq 0}$, which is a monotone increasing function. Since the Newton polygon $N_{(s,t)}$ of the characteristic power series $P((s_\sigma)_{\sigma\in I'}, X)$ of $T(\pi)$ acting on $S_{(s,t)}(G;\Gamma_1(N))$ is bounded by $N'_{(s,t)}$ from the bottom, we then obtain the following

Proposition 2.4. *Assume the condition* (ram). *Then we have*

$$N_{(s,t)}(x) \geq \mu(x)$$

for all $(s_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$ *and* $x \in \mathbb{R}_{\geq 0}$.

Secondly, the characteristic power series $P((\xi_\sigma)_{\sigma\in I'}, X)$ for $T(\pi)$ on $S(G;\Gamma_1(N))$ shall be investigated. The coefficients $c_i \in K\langle \xi_\sigma | \sigma \in I'\rangle$ $(i \geq 1)$ of $P((\xi_\sigma)_{\sigma\in I'}, X)$ can be regarded as analytic functions on $\mathcal{W}_{(n,v)}$. We then have the following

Proposition 2.5. *Assume the condition* (ram). *We take two elements* $(s_\sigma)_{\sigma\in I'}$, $(s'_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$. *We assume that there exists an integer* $m \geq 0$ *such that*

$$|s_\sigma - s'_\sigma| \leq p^{-m\cdot \max_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}}$$

for all $\sigma \in I'$. *Then we have*

$$|c_i((s_\sigma)_{\sigma\in I'}) - c_i((s'_\sigma)_{\sigma\in I'})| \leq p^{-(m+\lambda')\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}}$$

for all $i \geq 1$, *where we put* $\lambda' := \min_{1\leq i\leq s}\{1 - \frac{e(\mathfrak{p}_i)}{p-1}\}$.

Proof. Since $S(G;\Gamma_1(N))$ can be regarded as a direct summand of S^h via the isomorphism (10) in Definition 1.3 (1), it is enough to show the statement for the coefficients c'_i of the characteristic power series $P'((\xi_\sigma)_{\sigma\in I'}, X)$ of $T(\pi)$ on S^h. Note that both $S_{(s,t)}$ and $S_{(s',t')}$ can be generated by the same orthonormal basis $\mathfrak{M}$ over $K_{(s,t)}$ and $K_{(s',t')}$, respectively. For $M = (\prod_{\tau\in I\setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau}) \prod_{\sigma\in I'} x_\sigma^{m_\sigma} \in \mathfrak{M}$ and $\gamma = \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \gamma_2 \in \Gamma_1(N) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \Gamma_1(N)$ with $\gamma_1, \gamma_2 \in \Gamma_1(N)$, we see that

$$(17)\ [\gamma]_{(s,t)} \cdot M = \prod_{\tau\in I\setminus I'} \det(\gamma_\tau)^{v_\tau} (\prod_{\sigma\in I'} j_\sigma(\gamma_\sigma)^{s_\sigma} \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_\sigma}) \times((\prod_{\tau\in I\setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau})^t \gamma_\tau^\iota) \prod_{\sigma\in I'} (\gamma_\sigma \cdot x_\sigma)^{m_\sigma} \quad \text{and}$$

$$(18)\ [\gamma]_{(s',t')} \cdot M = \prod_{\tau\in I\setminus I'} \det(\gamma_\tau)^{v_\tau} (\prod_{\sigma\in I'} j_\sigma(\gamma_\sigma)^{s'_\sigma} \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t'_\sigma}) \times((\prod_{\tau\in I\setminus I'} X_\tau^{a_\tau} Y_\tau^{b_\tau})^t \gamma_\tau^\iota) \prod_{\sigma\in I'} (\gamma_\sigma \cdot x_\sigma)^{m_\sigma}.$$

By the assumption that $|s_\sigma - s'_\sigma| \leq p^{-\frac{m}{e(\mathfrak{p}_i)}}$ for each $\sigma \in I'$ with i such that $\sigma \in I_i$, we can write in $\mathbb{C}_p$

$$s'_\sigma = s_\sigma + (\pi_i^\sigma)^m u_\sigma \quad \text{and} \quad t'_\sigma = t_\sigma - \frac{u_\sigma}{2}(\pi_i^\sigma)^m$$

with some $u_\sigma \in \mathcal{O}_{\mathbb{C}_p}$ by Definition 1.2. Then we have

$$(19) \qquad j_\sigma(\gamma_\sigma)^{s'_\sigma} = j_\sigma(\gamma_\sigma)^{s_\sigma}(j_\sigma(\gamma_\sigma)^{(\pi_i^\sigma)^m})^{u_\sigma} \quad \text{and}$$
$$\det(\gamma_{1\sigma}\gamma_{2\sigma})^{t'_\sigma} = \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_\sigma}(\det(\gamma_{1\sigma}\gamma_{2\sigma})^{(\pi_i^\sigma)^m})^{\frac{u_\sigma}{2}}.$$

Noting that $j_\sigma(\gamma_\sigma)$ and $\det(\gamma_{1\sigma}\gamma_{2\sigma})$ are of the form $1 + \pi_i^\sigma a$ with some a with norm $|a| \leq 1$, by (17), (18) and (19) and the formula (7) in Section 1.2, we can calculate that for each $\sigma \in I'$ with i such that $\sigma \in I_i$, $|j_\sigma(\gamma_\sigma)^{s'_\sigma} - j_\sigma(\gamma_\sigma)^{s_\sigma}|$ and $|\det(\gamma_{1\sigma}\gamma_{2\sigma})^{t'_\sigma} - \det(\gamma_{1\sigma}\gamma_{2\sigma})^{t_\sigma}|$ are at most $|\pi_i^\sigma|^{m+\lambda'}$, because we can see easily that

$$|\frac{(\pi_i^\sigma)^{km}(\pi_i^\sigma)^k}{k!}u'_\sigma(u'_\sigma - 1)\cdots(u'_\sigma - k + 1)| \leq |\pi_i^\sigma|^{m+\lambda'} \quad (k \geq 1,\ m \geq 0)$$

under the condition (ram). Here the symbol u'_σ stands for both u_σ and $\frac{u_\sigma}{2}$. By Proposition 2.2 and the isomorphism (11) in Definition 1.3, this implies that the absolute values of all components in the difference of the representation matrices of $T(\pi)$ on $S^h_{(s,t)}$ and the one on $S^h_{(s',t')}$ calculated before are at most $p^{-(m+\lambda')\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}}$. This implies that

$$|c_i((s_\sigma)_{\sigma\in I'}) - c_i((s'_\sigma)_{\sigma\in I'})| \leq p^{-(m+\lambda')\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}}.$$

for all $i \geq 1$. □

Let $(s_\sigma)_{\sigma\in I'}, (s'_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$. By Proposition 2.4, we see that

$$N_{(s,t)}(x),\ N_{(s',t')}(x) \geq \mu(x).$$

We put

$$\nu(x) := \frac{2\lambda g'}{(g'+1)(g'+2)^2}(\frac{g'!}{h_n})^{\frac{1}{g'}}(x^{\frac{1}{g'}} - (h_n(g'+1))^{\frac{1}{g'}})$$

for $x \in \mathbb{R}_{\geq 0}$. Then ν is a strictly monotone increasing function, and we have

$$\nu(0) < 0 \quad \text{and} \quad \lim_{x\to\infty} \nu(x) = \infty.$$

Moreover, the inverse function

$$\nu^{-1}(x) = h_n(\frac{(g'+1)(g'+2)^2}{2\lambda g'(g'!)^{\frac{1}{g'}}}x + (g'+1)^{\frac{1}{g'}})^{g'}$$

of ν is also a monotone increasing function on $\mathbb{R}_{\geq 0}$ and $\nu^{-1}(x) \geq 0$ for $x \geq 0$. For $\alpha \in \mathbb{Q}_{\geq 0}$, we put

$$m(\alpha) := (\frac{\max_{1\leq i\leq s}\{e(\mathfrak{p}_i)\}}{\min_{1\leq i\leq s}\{e(\mathfrak{p}_i)\}})[\alpha\nu^{-1}(\alpha)].$$

By Proposition 2.5, we then see that if $|s_\sigma - s'_\sigma| \leq p^{-m(\alpha)}$ for all $\sigma \in I'$, then

$$|c_i((s_\sigma)_{\sigma\in I'}) - c_i((s'_\sigma)_{\sigma\in I'})| \leq p^{-\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}((\max_{1\leq i\leq s}\{e(\mathfrak{p}_i)\})[\alpha\nu^{-1}(\alpha)]+\lambda')}$$

for all $i \geq 1$. Since we can replace $\mathbb{Z}_p$ (resp. $m_v(\alpha)+1$) by $\mathcal{O}_{\mathbb{C}_p}$ (resp. $\min_{1\leq i\leq s}\{\frac{1}{e(\mathfrak{p}_i)}\}((\max_{1\leq i\leq s}\{e(\mathfrak{p}_i)\})[\alpha\nu^{-1}(\alpha)]+\lambda')$) in the statement of [14, Lemma 4.1], we have the following

Proposition 2.6. *Assume the condition* (ram). *For any* $\alpha \in \mathbb{Q}_{\geq 0}$, *we put*

$$m(\alpha) := (\frac{\max_{1\leq i\leq s}\{e(\mathfrak{p}_i)\}}{\min_{1\leq i\leq s}\{e(\mathfrak{p}_i)\}})[\alpha h_n(\frac{(g'+1)(g'+2)^2}{2\lambda g'(g'!)^{\frac{1}{g'}}}\alpha + (g'+1)^{\frac{1}{g'}})^{g'}].$$

If $(s_\sigma)_{\sigma\in I'}$, $(s'_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$ *satisfy* $|s_\sigma - s'_\sigma| \leq p^{-m(\alpha)}$ *for all* $\sigma \in I'$, *then the slope-*α*-part of the Newton polygons of* $P((s_\sigma)_{\sigma\in I'}, X)$ *and* $P((s'_\sigma)_{\sigma\in I'}, X)$ *are equal.*

By combining this proposition with [12, Corollary of Section IV.4], we obtain the following

Theorem 2.7. *Assume the condition* (ram). *Let* $\alpha \in \mathbb{Q}_{\geq 0}$ *and* $(s_\sigma)_{\sigma\in I'}$, $(s'_\sigma)_{\sigma\in I'} \in \mathcal{W}_{(n,v)}(\mathbb{C}_p)$. *If* $|s_\sigma - s'_\sigma| \leq p^{-m(\alpha)}$ *for all* $\sigma \in I'$, *then we have*

$$\dim_{\mathbb{C}_p} S_{(s,t)}(G;\Gamma_1(N))^{\alpha}_{\mathbb{C}_p} = \dim_{\mathbb{C}_p} S_{(s',t')}(G;\Gamma_1(N))^{\alpha}_{\mathbb{C}_p}.$$

Further, by Theorem 2.3, we then have immediately the following

Corollary 2.8. *Assume the condition* (ram). *If* $(n'_\sigma)_{\sigma\in I'}$, $(n''_\sigma)_{\sigma\in I'} \in W^{\mathrm{cl}}_{(n,v)}$ *satisfy the conditions that* $|n'_\sigma - n''_\sigma| \leq p^{-m(\alpha)}$ *for all* $\sigma \in I'$ *and* $\nu_{n'}$, $\nu_{n''} > \alpha$, *then we have*

$$\dim_{\mathbb{C}_p} S^{\mathrm{cl}}_{(n',v')}(G;\Gamma_1(N);\mathbb{C}_p)^{\alpha} = \dim_{\mathbb{C}_p} S^{\mathrm{cl}}_{(n'',v'')}(G;\Gamma_1(N);\mathbb{C}_p)^{\alpha}.$$

Remark 2.2. In Corollary 2.8, we need to assume the condition (ram) to apply the modified Wan's lemma with the *positive* rational number λ'. This corollary is a generalization of Coleman's result [5, Theorem B3.4] which gives a solution to a conjecture of Gouvêa and Mazur [7, Conjecture 1 in Section 5].

Remark 2.3. Kassaei [11] has constructed overconvergent $\mathcal{P}$-adic modular forms on quaternion algebras defined over any totally real field F which are unramified at $\mathcal{P}$ and exactly one infinite place, where $\mathcal{P}$ is a prime ideal of F above p whose residue field has cardinality > 3. Then he has also showed the local constancy of dimensions of the spaces of overconvergent forms ([11, Theorem 1.1]).

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